Four-level systems and a universal quantum gate

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Abstract

We discuss a possibility of implementing a universal quantum XOR gate, by using two coupled quantum dots subjected to external magnetic fields that are parallel and slightly different. We consider this system in two different field configurations. In the first case, parallel external fields with the intensity difference at each spin being proportional to the time-dependent interaction between the spins. General exact solution describing this system is presented and analyzed to adjust parameters of the field. Then we consider parallel fields with intensity difference at each spin being constant and the interaction between the spins switching on and off adiabatically. In both cases we adjust characteristics of the external fields (their intensities and duration) in order to have the parallel pulse adequate for constructing the XOR gate. In order to provide a complete theoretical description of all the cases, we derive relations between the spin interaction, the inter-dot distance, and the external field.

1 Introduction

At present there exist a belief that single quantum systems, e.g., atomic traps, and electronic devices whose operation involves only a few number of electrons, could be used to implement the so-called quantum gates for quantum computations. Quantum computations (quantum computers, or simply QC in what follows) can efficiently solve problems that are considered intractable by classical computers, e.g., the factoring of primes [1] and the simulation of others quantum systems [2]. In QC, the classical bit, that can assume the binary values zero and one, is replaced by the states of a two-level quantum system. The computation is performed by the manipulation of these quantum systems and the power of the computers depends on the number of systems involved. These two-level quantum systems are called *qubits* and these manipulations *quantum gates*. A quantum gate, in a process involving N qubits, can be represented by

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a $2N$ unitary matrix. The great peculiarity of quantum computers reside in the power of true parallel processing, since the incoming qubit can be prepared in a superposition of all possible states and, consequently, the out-coming processed qubit will be in a superposition of all possible answers. But in order to use this astonishing feature of parallel processing, we need to be capable of constructing, and of maintaining, an entanglement between the qubits, which constitutes the principal difficulty in the implementation of this technology. As for classical computers, for the QC the accomplishment of an arbitrary algorithm can be performed using just a few specific manipulations called universal quantum gates. With these universal quantum gates, a process that acts in an arbitrary number of qubits can be constructed using gates that act only in one and two qubits.

Although the first realizations of quantum computers were performed using trapped ions [3], NMR [4], and optical cavities [5], it is believed that, as for classical computers, for QC the most promising object for a possible large-scale implementation, i.e., a implementation involving a large number of qubits, will be the solid-state devices. Among these devices, one can highlight the system of two coupled semiconductor quantum dots [7]. In this system the qubits are the one-half spin states of an excess electron in each dot, and the universal quantum gates can be obtained by performing arbitrary rotation of the individual spin (one-qubit gate) and one operation capable of entangling the spins of the two electrons (two-qubit gate). Any such operation with this entangling feature can be used [8]. The rotations and the entanglement operations can be performed by applying external electromagnetic pulses to the dots.

These pulses can be applied in a serial manner, which perform sequentially the one and the two-qubit gates using a sequence of pulses, or in a parallel manner, where the two and one-quibit gates are performed at once by the application of just one adequate pulse (called a *parallel pulse*). Due to the decohering effects it is important to perform the operations in a minimal time, which, as discussed in [9], can be achieved more adequately with a parallel pulse.

The adequate parallel pulse to implement a quantum gate can be obtained if we know the Hamiltonian that describes the coupled quantum dots, but to construct this Hamiltonian, one has to know the exact relation between the external parameters of the system (such as the external fields and the physical characteristics of the dot) and the interaction between the electrons. An advance in the description of this relation was obtained in [13], where the authors have used the Heitler-London approximation to obtain an explicit expression that relates the interaction between the dots, the inter-dots distance and the external electromagnetic field. However, in this article, the two dots were subject to the same magnetic field, and, as we will see, a little difference between the fields in the dots possess a great influence in the evolution of the system and, consequently, in the implementation of the gates. In addition, after the Hamiltonian is known, we have to construct exact solutions of the evolution equation of this four-level system. In our work [11] we studied possible exact solutions of a system of two coupled spin one-half particles subject to external time-dependent magnetic fields.

In the present work we discuss one possibility of implementing an universal quantum gate, namely the *exclusive or* (XOR) gate, by using two coupled quantum dots subjected to external magnetic fields that are parallel and slightly different. First, in Sections 2 and 3, we make a brief discussion of the model with the parallel external magnetic fields of general form, and describe possible corresponding exact solutions from our previous work [11]. In section 4 we consider two cases of system of two coupled quantum dots. In the first case, parallel external fields with intensity difference at each spin being proportional to the interaction between the spins, the latter depends on time. General exact solution describing this system is presented and analyzed to adjust parameters of the field. Then we consider parallel fields with intensity difference at each spin being constant and interaction between the spins switching on and off adiabatically. In both cases we adjust characteristics of the external fields (their intensities and duration) in order to have the parallel pulse adequate for constructing the XOR gate. In order to have a complete theoretical description of all the cases, one needs to find relations between the spin interaction and external fields. This problem is solved in Section 5. To this end we generalize results of [13], derived for equal parallel external field, to the case of different fields, obtaining relations between the spin interaction, the inter-dot distance, and the external field.

2 Four-level system

In non relativistic quantum mechanics the dynamics of a fixed spin one-half particle, subject to a time-dependent external field $\mathbf{F}(t)$, can be described by the Hamiltonian $\hat{h} = (\boldsymbol{\sigma} \cdot \mathbf{F})$ [10], where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and, such as in what follows, we set $\hbar = 1$. The quantum Hamiltonian of two interacting one-half spins subjected to external fields $\mathbf{G}$ and $\mathbf{F}$, respectively, is chosen as, see e.g. [14],

$$\hat{H}(\mathbf{G}, \mathbf{F}, J) = (\boldsymbol{\rho} \cdot \mathbf{G}) + (\boldsymbol{\Sigma} \cdot \mathbf{F}) + \frac{J}{2}(\boldsymbol{\Sigma} \cdot \boldsymbol{\rho}) \,, \tag{1}$$

where the function $J = J(t)$ defines the spin interaction, $\mathbf{G} = (G_1(t), G_2(t), G_3(t))$ and $\mathbf{F} = (F_1(t), F_2(t), F_3(t))$ time-dependent external fields for each particle, and the 4×4 Dirac-matrices $\boldsymbol{\rho}$ and $\boldsymbol{\Sigma}$ are in the standard representation

$$\boldsymbol{\Sigma} = I \otimes \boldsymbol{\sigma} \,, \; \boldsymbol{\rho} = \boldsymbol{\sigma} \otimes I \,, (\boldsymbol{\Sigma} \cdot \boldsymbol{\rho}) = \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \otimes \sigma_i \,, \tag{2}$$

where I is the 2×2 identity. In matrix notation we have $\boldsymbol{\Sigma}$ =diag $(\boldsymbol{\sigma}, \boldsymbol{\sigma})$ and

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \; \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \; \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The interaction of the spins in (1) is know as the *Heisenberg interaction* and describe, for example, the interactions in the well known Hubbard model [6]. In

particular, under certain conditions, this interaction can be used to describe the coupling between two quantum-dots [7]. In this case, it is possible to control not only the external fields, but also the spin interaction. When these quantum dots are used to implement a quantum gate, the operation of the gate is performed by varying the external fields and the spin interaction during a certain time τ. To this end, in order to justify the Heisenberg interaction, the following conditions (see [7]) must hold:

1. All higher one-particle states of a dot can be ignored, this requires the difference between energy levels ΔE to be much greater than kT;

2. The time τ cannot be too small to avoid transitions to higher energy levels, so that the time scale should be bigger than $\hbar/\Delta E$;

3. The decoherence time of the physical system should be much bigger than τ.

The use of these quantum dots, to the implement quantum algorithms, requires controlling the individual spins, which can be done by the fields $\mathbf{B}$ and $\mathbf{G}$, and any interaction between the spins capable of creating a entangled state starting from a original product state [8]. A system described by the Hamiltonian (1) satisfies these conditions. Namely, when the fields $\mathbf{B}$ and $\mathbf{G}$ are zero, the evolution operation $R_t(\mathbf{G},\mathbf{F},J)$ of the Hamiltonian (1) can be written as [11]

$$\begin{aligned} &R_t(0,0,J) = \exp\left[i\Phi(t)/2\right]\left[\mathbb{I}\cos\Phi(t) - iA\sin\Phi(t)\right], \\ &A = \frac{1}{2}\left[\mathbb{I} + (\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})\right] \ , \ \Phi(t) = \int_{t_0}^{t} J(\tau)\, d\tau, \end{aligned} \tag{3}$$

where $\mathbb{I}$ is the 4×4 unity matrix. From the above expression we see that, when $\Phi = \pi/4$, the evolution operator acts as the universal gate known as square root of swap ($U_{\mathrm{sw}}^{1/2}$). The interaction function J can experimentally be controlled in many different ways, e.g., by applying electrical or magnetic fields to the dots [12]. So we can construct any quantum gate by a sequence of pulses that turning on and off the external fields and the interaction. For example, the XOR gate can be constructed as [13]

$$U_{\mathrm{XOR}} = \exp\left(i\frac{\pi}{4}\rho_3\right)\exp\left(-i\frac{\pi}{4}\Sigma_3\right)U_{\mathrm{sw}}^{1/2}\exp\left(i\frac{\pi}{2}\rho_3\right)U_{\mathrm{sw}}^{1/2}\ . \tag{4}$$

On the other hand, the construction of gates by a sequence of pulses is not appropriate, because the duration of all the sequence can be too long, violating condition (3) above, or the pulses need to vary too fast, violating the condition (2). So it is important to be able to implement the gates at once, applying just one adequate field. The sequence of pulses are called serial pulses, while this single pulse is called parallel pulse (for other arguments in favor of parallel pulses see [9]). For example, in the case of the XOR gate (4), to use a parallel

pulse one needs to find a field whose evolution operation, at a given instant τ of time, has the form

$$R_\tau\left(\mathbf{G},\mathbf{F},J\right)=U_{\mathrm{XOR}}=\exp\left[-i\frac{\pi}{4}\left(\Sigma_3\rho_3+\Sigma_3+\rho_3\right)\right]\ . \tag{5}$$

In order to find this parallel pulse, in a general case, we need to know exact solutions of the Schrödinger equation with the Hamiltonian (1) for different kinds of external fields and interactions. A large number of these exact solutions are present in our previous work [11], and here we will use some of these results to describe the implementation of a U_{XOR} gate. The procedure developed here can easily be extended to others gates using others fields present in [11].

3 Parallel External Fields

Let each spin in our system be subject to different time-dependent external fields along the z direction,

$$\mathbf{G}=\left(0,0,B_1\right)\ ,\ \mathbf{F}=\left(0,0,B_2\right)\ ,\ B_{1,2}=B_{1,2}\left(t\right)\ . \tag{6}$$

Therefore the Hamiltonian (1) assumes the form

$$\hat{H}=\frac{1}{2}\left[\left(\Sigma_3+\rho_3\right)B_+-\left(\Sigma_3-\rho_3\right)B_--J\right]+AJ\ ,\ B_\pm\left(t\right)=B_1\left(t\right)\pm B_2\left(t\right)\ , \tag{7}$$

with the constant 4×4 orthogonal matrix A given in (3). For the first and fourth components v_1 and v_4 of the four-spinor Ψ, solution of the Schrödinger equation ($i\dot{\Psi}=\hat{H}\Psi$) with the above Hamiltonian, we get

$$v_1=C_1\exp\left[-i\int_0^t\left(\frac{J}{2}+B_+\right)\,d\tau\right]\ ,\ v_4=C_4\exp\left[-i\int_0^t\left(\frac{J}{2}-B_+\right)\,d\tau\right]\ , \tag{8}$$

where $C_{1,4}$ are complex constants. Besides, for the components $v_{2,3}$ of Ψ, we obtain the expression

$$i\dot{\psi}'=\left[\left(\boldsymbol{\sigma}\cdot\mathbf{K}\right)-\frac{J}{2}\right]\psi'\ ,\ \psi'=\left(\begin{array}{c}v_2\\v_3\end{array}\right)\ , \tag{9}$$

$$\mathbf{K}\left(t\right)=\left(J\left(t\right),0,B_-\left(t\right)\right)\ . \tag{10}$$

Making the transformation

$$\psi'\left(t\right)=\exp\left[\frac{i}{2}\int_0^t J\left(\tau\right)\,d\tau\right]\psi\left(t\right)\ , \tag{11}$$

the two-component spinor ψ obeys the equation for a single particle with spin one-half in an effective external field $\mathbf{K}\left(t\right)$, i.e., $i\dot{\psi}=\left(\boldsymbol{\sigma}\cdot\mathbf{K}\right)\psi$ [11]. Consequently, in this case, the four-level system problem reduces to finding solutions of a two-level system. Writing the solution of this two-level-system as

$\psi(t) = \hat{u}_t \psi(0)$, using the 2×2 evolution operator $\hat{u}_t(J, B_-)$, we can write the evolution operator of the four-level-system governed by the Hamiltonian (7) as

$$R_t(\mathbf{G}, \mathbf{F}, J) = \exp\left(-\frac{i}{2}\left[(\Sigma_3 + \rho_3)\Gamma(t) + \Sigma_3 \rho_3 \Phi(t)\right]\right) M(t) \ ,$$
$$\Gamma(t) = \int_0^t B_+(\tau)\, d\tau \ , \ \Phi(t) = \int_0^t J(t)\, d\tau \ , \tag{12}$$

with the matrix $M(t)$ given by components of the single particle evolution operator $\hat{u}_t$ as

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & u_{12} & 0 \\ 0 & u_{21} & u_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

Although the interaction function J depends on the applied external fields, as we will see in the Section 5, when $B_- << B_+$, we can make $J = J(B_+)$ and consider the vector $\mathbf{K}$ in (10) as composed of two independent functions $J(t)$ and $B_-(t)$. Besides, the interaction function can be controlled by electric fields, whose interference in the spin states via the spin-orbit coupling, in many practical application, can be neglect. Then, if necessary, we can consider the functions J, B_- and B_+ independently. In addition, although the obtained expressions depend on the sum and on the difference between the fields at the spins, only the average value of this sum is relevant, so that the explicit form of its time variation can be arbitrary.

4 The Exclusive OR gate

Here we establish the parameters of two kinds of parallel external fields that can act as a XOR gate, that is, fields in the form (6) whose evolution operator, at a given instant of time, has the form (5). The same analysis presented here can also be carried out for the 26 different families of external fields presented in [10], which gives a wide range of fields to choose from that are more appropriate to adjust to experimental setups. All the development depends on the ability to vary independently the difference between the fields B_- and the interaction J, but, as we see in Section 5, there are various circumstances when this requirement can be satisfied.

4.1 Interaction proportional to the external fields' difference

If the interaction J is proportional to the difference B_- between the fields, we can write

$$J(t) = q(t) \sin\lambda \, , \ B_-(t) = q(t) \cos\lambda \, ,$$

where $q(t)$ is an arbitrary function of time, and λ is a constant. Then, the general solution of equation (9) can be written as $\psi(t) = \hat{u}_t \psi(0)$ with the evolution operator $\hat{u}_t$ given by

$$\hat{u}_t = \cos\omega - i\left(\sigma_1 \sin\lambda + \sigma_3 \cos\lambda\right)\sin\omega\,,\ \omega(t) = \int_0^t q(\tau)\ d\tau\ .$$

Selecting the instants T when $\cos\omega = 1$,

$$\omega(T) = \int_0^T q(\tau)\ d\tau = 2n\pi\ ,\ n \in \mathbb{N}^*\ , \tag{13}$$

the evolution operator (12) of this problem assumes the form

$$R_T = \exp(i3n\pi)\exp\left[-\frac{i}{2}\left(\Gamma\left[\Sigma_3 + \rho_3\right] + 2n\pi\Sigma_3\rho_3 \sin\lambda\right)\right]\ .$$

So to obtain the XOR gate (5), up to a phase, at every instant T (13), we have to set

$$\Gamma(T) = 2n\pi \sin\lambda = \frac{\pi}{2} \bmod (2\pi)\ ,$$

or, in a more explicit form,

$$\sin\lambda = \frac{4m+1}{4n}\ ,\ m \in \mathbb{N}\ ,\ m < n\ .$$

When the interaction J and the fields' difference B_- are constants, so q is a constant, we obtain

$$\Gamma(T) = JT \bmod (2\pi)\ ,\quad T = \frac{2n\pi}{\sqrt{J^2 + B_-^2}}\ . \tag{14}$$

For the special case of a constant sum of fields B_+, the above expression can be compared with the XOR gate obtained in [9]. To have a idea about the physical values involved in the expression (14), for a typical GaAs quantum-dot [13], for an interaction about 50 μeV and a B_- field of the 10 mT we obtain a XOR gate in a time of 10 ps.

4.2 Adiabatic pulse

The interaction J can be controlled by varying the field B_+, or by varying the electric potential between the dots. However, as described in condition (2) of section 2, these variations can not be so fast, so as to prevent the excitation of higher energy-levels. This problem can be prevented by using an adiabatic variation of the interaction, which can be obtained by a time-dependence in the form of $\operatorname{sech}(\omega t)$ [9]. A lot of solutions of this form can be found in [10]. For example, for a variation in the form

$$J(t) = a/\cosh\omega t\,,\ B_- = c\ , \tag{15}$$

where a, c are constants and $\omega << \Delta E/\hbar$ (see condition (2) in section 2), the general solution of equation (9) can be written as $\psi(t) = \hat{u}_t \psi(0)$ with the evolution operator $\hat{u}_t$ given by

$$\hat{u}_t(t) = \frac{1}{|G_2^0|^2 + |G_1^0|^2} \begin{pmatrix} G_1^{0\,*} & G_2^{0\,*} \\ -G_2^0 & G_1^0 \end{pmatrix} \begin{pmatrix} G_1(z) & -G_2^*(z) \\ G_2(z) & G_1^*(z) \end{pmatrix} , \tag{16}$$

where the $*$ indicates complex conjugation and

$$\begin{aligned} &G_1(z) = (2c - i\omega) z^{-\nu} (1-z)^{\nu} F(\lambda, -\lambda; \gamma; z) \ , \\ &G_2(z) = 2az^{-\nu+1/2} (1-z)^{\nu+1/2} F(1+\lambda, 1-\lambda, \gamma+1; z) \ , \\ &z(t) = \frac{1}{2}(1 - \tanh \omega t) \ , \ G_i^0 = G_i(1/2) \ , \\ &\nu = -\frac{ic}{2\omega} \ , \ \gamma = 1/2 - 2\nu \ , \ \lambda = \frac{|a|}{\omega} \ . \end{aligned}$$

In these expressions, $F(\alpha, \beta, \gamma, z)$ is the Gauss hypergeometric function.

If we use the sech function to create a pulse of the interaction function, this pulse will be turned off when $t >> 1/\omega$. In this time limit we have $\lim_{\omega t \to \infty} z = e^{-2\omega t}$ and $F(\alpha, \beta, \gamma, 0) = 1$. If in this limit, we choose

$$G_2^0 = aF\left(1+\lambda, 1-\lambda, \gamma+1; \frac{1}{2}\right) = 0 \ , \tag{17}$$

the evolution operator (16) assumes the form

$$\hat{u}_t(\omega t \to \infty) = \exp(-i\sigma_3 tc) \ . \tag{18}$$

So, following the procedure of the preceding sections, we see that the evolution operator (12) will behave as a XOR gate (5), in the instant T, when $\Gamma(T) = \pi/2 \bmod (2\pi)$ and

$$\exp(\omega T) = \tan\left(\frac{\pi}{4a}[\omega(1+4m) + a]\right) \ , \ T = \frac{n\pi}{c} \ , \ n, m \in \mathbb{N}^* \ .$$

The condition (17) can be obtained in various ways, for example, in the special case when $c = 0$ (and consequently $\lim_{\omega t \to \infty} \hat{u} = I$ in (18) and we can abandon the condition $T = n\pi/c$) we can use the relation [16],

$$F\left(1+\lambda, 1-\lambda; \frac{3}{2}; \frac{1}{2}\right) = \frac{1}{\lambda} \sin\left(\frac{\lambda\pi}{2}\right) \ . \tag{19}$$

So, we obtain the desired condition whenever $|a| = 2\omega m, \ m \in \mathbb{N}^*$.

5 Relation between the interaction function and the external fields

To calculate the parameters of the parallel pulses obtained, one needs to know the explicit dependence between the external fields and the interaction function.

However, for different magnetic fields applied at the spins, the non-homogeneity of the field make it very complicated to calculate this dependence, since, in this case, the Zeeman terms ($\boldsymbol{\rho}\cdot\mathbf{G}$ and $\boldsymbol{\Sigma}\cdot\mathbf{F}$) will no more be independent of the space coordinates. Despite this fact, since only a small difference of the fields is important, the homogeneous field approximation, as well as the *Heitler-London* approximation, can still be used to describe the interaction function. In the article [13], the authors study two quantum dots, coupled with the Heisenberg interaction (1), and use the Heitler-London approximation to obtain an expression for the interaction function $J(V, a, B, E)$ as a function of the gate voltage between the coupled quantum dots (V), the inter-dot distance (a), an external magnetic field (B) and an external electric field (E). The knowledge of the interaction as a function of various parameters allows to compensate the changes in J with the variation of one given parameter by controlling other parameter. So one can, for example, vary the magnetic field B and maintain J constant by changing the inter-dot potential V. This procedure allows treating some parameters of the Hamiltonian of the system as independent. However, the analysis developed in [13] concerns only the problem of two dots subject to the same magnetic field. Since the work developed here depends on the difference between the fields in each dot, we will repeat the process given in the cited article, but with different magnetic fields in each dot. The model consists of two identical bi-dimensional dots with an inter-dot separation of $2a$, each dot with one electron of charge e and subject, respectively, to the magnetic field B_1 and B_2 in the z direction. The field B_i is considered homogeneous inside the dot i. In the Heitler-London approximation, we start with a combination of the orbital ground states of bi-dimensional single dots and combine this state in a symmetric ($|\Psi_+\rangle$) and an anti-symmetric ($|\Psi_-\rangle$) states for the double dot problem as

$$|\Psi_\pm\rangle = \frac{|-+\rangle \pm |+-\rangle}{\sqrt{2 \pm S^2}}$$

where $\varphi_\pm(\mathbf{r}) = \langle \mathbf{r} | \pm \rangle$ denotes the one-particle orbital centered at $\mathbf{r} = (\pm a, 0)$ ($2a$ is the inter-dot separation) and $S = \langle + | - \rangle$ the overlap between the two orbitals. The interaction function, or the exchange energy, is the difference between the energies, $J = \langle \Psi_- | H_{orb} | \Psi_- \rangle - \langle \Psi_+ | H_{orb} | \Psi_+ \rangle$ where the double-dot orbital Hamiltonian is given by

$$\begin{aligned}
&H_{orb} = h^0(\mathbf{r}_1) + h^0(\mathbf{r}_2) + C \ , \ C = \frac{e^2}{\kappa |\mathbf{r}_1 - \mathbf{r}_2|} \ , \\
&h^0(\mathbf{r}_i) = \frac{1}{2m}\left[\mathbf{p}_i - \frac{e}{c}\mathbf{A}(\mathbf{r}_i)\right]^2 + V(\mathbf{r}_i) \ , \ i = 1, 2 \ , \\
&\mathbf{A}(\mathbf{r}_i) = \frac{B_i}{2}(-y_i, x_i, 0) \ , \ V(x, y) = \frac{m\omega_0^2}{2}\left[\frac{1}{4a^2}\left(x^2 - a^2\right)^2 + y^2\right] \ . \qquad (20)
\end{aligned}$$

In this expression c is the speed of the light, κ is the dielectric constant of the medium, m is the effective mass of the electron (e.g., $m = 0,067 m_e$ in GaAs), B_i is the magnetic field at dot i and the harmonic potential well $V(\mathbf{r})$ of frequency ω_0 is motivated by experimental results [15]. The matrix elements needed to

calculate J can be obtained by adding and subtracting in (20) the harmonic potentials $V_{\pm}$ centered at $x_i = (-1)^i a$ for the i-th electron,

$$
\begin{aligned}
&H_{orb} = h^0_- (\mathbf{r}_1) + h^0_+ (\mathbf{r}_2) + W + C \ , \\
&h^0_{\pm} (\mathbf{r}_i) = \frac{1}{2m} \left[\mathbf{p}_i - \frac{e}{c} \mathbf{A} (\mathbf{r}_i) \right]^2 + V_{\pm} (\mathbf{r}_i) \ , \\
&V_{\pm} (\mathbf{r}_i) = \frac{m\omega_0^2}{2} \left[(x_i \pm a)^2 + y_i^2 \right] \ , \ W = \sum_{i=1}^{2} V (\mathbf{r}_i) - \left[V_- (\mathbf{r}_1) + V_+ (\mathbf{r}_2) \right] \ , \quad (21)
\end{aligned}
$$

and using the ground states functions $\varphi_{\pm}$ centered at $\mathbf{r} = (\pm a, 0)$,

$$
\begin{aligned}
&\varphi_{\pm} (\mathbf{r}) = \sqrt{\frac{m\omega_{\pm}}{\pi\hbar}} \exp \left(\pm \frac{ima\omega_{L\pm}}{\hbar} y \right) \exp \left(- \frac{m\omega_{\pm} \left[(x \mp a)^2 + y^2 \right]}{2\hbar} \right) \ , \\
&\omega_{\pm} = \sqrt{\omega_0 + \omega_{L_{\pm}}^2} \ , \ \omega_{L_{-(+)}} = eB_{1(2)}/2mc \ , \quad (22)
\end{aligned}
$$

which are eigenfunctions of $h^0_{\pm}$ with energy $\omega_{\pm}$,

$$
h^0_{-(+)} \left(\mathbf{r}_{1(2)} \right) \varphi_{-(+)} \left(\mathbf{r}_{1(2)} \right) = \omega_{-(+)} \varphi_{-(+)} \left(\mathbf{r}_{1(2)} \right) \ .
$$

With this, the expression for the interaction function becomes

$$
\begin{aligned}
&J = \frac{2S^2}{(1 - S^4)} \left[L - \frac{\hbar\omega_0}{4} \frac{\left(b_+^2 - b_-^2 \right) (b_- - b_+)}{b_+ b_-} \right] \ , \\
&L = \left(\langle 12| \, C + W \, |12\rangle - \frac{\mathrm{Re} \, \langle 12| \, C + W \, |21\rangle}{S^2} \right) \ , \ b_{\pm} = \omega_{\pm}/\omega_0 \ , \quad (23)
\end{aligned}
$$

using the expression (22) for the functions $\varphi_{\pm} (\mathbf{r}_i) = \langle \mathbf{r}_i \, | \pm \rangle$ we get

$$
\begin{aligned}
&\frac{S^2}{(1 - S^4)} = \frac{\left(1 - \Delta^2 \right)}{\left(2 \sinh (2M) + \Delta \exp (-2M) \left(2 - \Delta^3 \right) \right)} \\
&M = \frac{2d^2}{(b_+ + b_-)} \left[b_- b_+ + \frac{\left(\omega_{L_+} + \omega_{L_-} \right)^2}{4\omega_0^2} \right] \ , \ \Delta = \frac{(b_- - b_+)}{(b_- + b_+)} \ , \ d = \frac{a}{a_0} \ ,
\end{aligned}
$$

where $a_0 = \sqrt{\hbar/m\omega_0}$ is the effective Bohr radius of the dot. For the matrix elements of W in (23) we have

$$
\begin{aligned}
&\langle 12| \, W \, |12\rangle - \frac{\langle 12| \, W \, |21\rangle}{S^2} = \\
&\frac{\hbar\omega_0}{2} \left\{ \frac{3}{2d^2 \left(b_- + b_+ \right)^2} \left[\frac{\left(1 + \Delta^2 \right)}{\left(1 - \Delta^2 \right)^2} - 1 \right] - 3 \frac{\left(\Delta^2 - 1 \right)}{(b_- + b_+)} - \frac{d^2}{2} \left(\Delta^4 - 6\Delta^2 - 3 \right) \right\}
\end{aligned}
$$

and for the matrix elements of the electric interaction C between the electrons

$$\langle 12|\, C\, |12\rangle - \frac{\mathrm{Re}\,\langle 12|\, C\, |21\rangle}{S^2} =$$

$$\frac{e^2}{a_0\kappa}\sqrt{\frac{\pi}{2}\bar{b}}\left\{\sqrt{(1-\Delta^2)}\exp\left[-d^2\left(1-\Delta^2\right)\bar{b}\right]\mathrm{I}_0\left[d^2\left(1-\Delta^2\right)\bar{b}\right] - \exp\left[\frac{d^2}{2}K\right]\mathrm{I}_0\left[\frac{d^2}{2}K\right]\right\}\ ,$$

$$K = \bar{b}\left(1+\Delta^2\right) - \frac{1}{\bar{b}} + \sqrt{\left[\left(1-\Delta^2\right)\bar{b}\right]^2 - 2\left(1+\Delta^2\right) + \frac{1}{\bar{b}^2}}\ ,\ \bar{b} = \frac{(b_- + b_+)}{2}\ ,$$

where I_0 is the zeroth-order Bessel function. The quantity Δ is related to the difference and to sum of the fields at the dots by the relation

$$\Delta = \frac{1}{2}\frac{B_+B_-}{\left[\left(2\hbar\frac{\omega_0}{\mu_B}\right)^2 + B_+^2 + B_-^2 + \sqrt{\left[\left(2\hbar\frac{\omega_0}{\mu_B}\right)^2 + B_+^2 + B_-^2\right]^2 - \left(2B_+B_-\right)^2}\right]}\ .$$

In the denominator of the above expression, usually only the first term is relevant, e.g., for a typical GaAs dot we have $\hbar\omega_0/\mu_B = 50$ T. Besides, when $B_+ >> B_-$ we have $\Delta \propto B_-/B_+$ and for the same field at both dots we obtain the result given in [9].

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References

[1] P.W. Shor, *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science* (IEEE Computer Society, Los Alamitos, 1994)

[2] R.P. Feynman, *Simulating physics with computers*, Int. J. Theor. Phys. **21**, 467 (1982)

[3] C. Monroe et al., *Demonstration of a Fundamental Quantum Logic Gate*, Phys. Rev. Lett. **75**, 4714 (1995)

[4] I.L. Chuang, N. Gershenfeld, and K. Kubinec, *Experimental Implementation of Fast Quantum Searching*, Phys. Rev. Lett. **80**, 3408 (1998)

[5] Q.A. Turchette et al., *Measurement of Conditional Phase Shifts for Quantum Logic*, Phys. Rev. Lett. **75**, 4710 (1995)

[6] N.W. Aschcroft, and N.D. Mermim, *Solid State Physics* (Saunders, Philadelphia, 1976)

[7] D. Loss, and D.P. DiVincenzo, *Quantum computation with quantum dots*, Phys. Rev. A 57, 120 (1998)

[8] Bremner M.J. *et al.*, *Practical Scheme for Quantum Computation with Any Two-Qubit Entangling Gate*, Phys. Rev. Lett. **89**, 247902 (2002)

[9] G. Burkard, D. Loss, D.P. DiVincenzo, and J.A. Smolin, *Physical optimization of quantum error correction circuits*, Phys. Rev. B **60**, 404 (1999)

[10] Bagrov V.G., Gitman D.M., Baldiotti M.C., and Levin A., *Spin equation and its solutions,* Ann. Phys. (Leipzig) **14** [11-12] (2005) pp.764-789

[11] V.G. Bagrov, M.C. Baldiotti, D.M. Gitman, and A.D. Levin, *Two interacting spins in external fields*, Ann. Phys. (Leipzig) **16**, No 4, 274 (2007)

[12] J.R. Petta et al., *Coherent Manipulation o Coupled Electron Spins in Semiconductor Quantum Dots*, Science **309**, 2180 (2005)

[13] G. Burkard, D. Loss, and D.P. DiVincenzo, *Coupled quantum dots as a quantum gate*, Phys. Rev. B **59**, 1829 (1999)

[14] Townsend J.S., *A Modern Approach to Quantum Mechanics* (McGraw-Hill, New York 1992)

[15] S. Turucha et al., *Shell Filling and Spin Effects in a Few Electron Quantum Dot*, Phys. Rev. Lett. **77**, 3613 (1996)

[16] Gradshtein I.S., Ryzhik N.M., *Tables of Integrals, Sums, Series and Products*, (Nauka, Moscow 1971)