Instituto de Física
Universidade de São Paulo

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Instituto de Física, Universidade de São Paulo, CP 66.318
05315-970, São Paulo, SP, Brasil

A. D. Levin , D. M. Gitman, and R. A. Castro

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UNIVERSIDADE DE SÃO PAULO
Instituto de Física
Cidade Universitária
Caixa Postal 66.318
05315-970 - São Paulo - Brasil

Entanglement of two-qubit photon beam by magnetic field

A. D. Levin,[1,*] D. M. Gitman,[2,†] and R. A. Castro[1,‡]

[1] *Institute of Physics, University of São Paulo, Brazil*

[2] *Institute of Physics, University of São Paulo, Brazil; Tomsk State University, Russia*

We have studied the possibility of affecting the entanglement measure of 2-qubit system consisting of two photons with different fixed frequencies but with two arbitrary linear polarizations, moving in the same direction, by the help of an applied external magnetic field. The interaction between the magnetic field and the photons in our model is achieved through intermediate electrons that interact with both the photons and the magnetic field. The possibility of exact theoretical analysis of this scheme is based on known exact solutions that describe the interaction of an electron subjected to an external magnetic field (or a medium of electrons not interacting with each other) with a quantized field of two photons. We adapt these exact solutions to the case under consideration. Using explicit wave functions for the resulting electromagnetic field, we calculate the entanglement measure of the photon beam as a function of the applied magnetic field and parameters of the electron medium.

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1. INTRODUCTION

Entanglement is a pure quantum property which is associated with a quantum non-separability of parts of a composite system. Entangled states became a power tool for studying both principal questions in quantum theory and quantum computation and information theory [1–4]. However, we believe that the complete understanding of the nature of quantum entanglement still requires a detailed consideration of a variety of relatively simple cases, not only in nonrelativistic quantum mechanics, but in QFT as well, see e.g. [5]. Here models with exact solutions could be very useful. In this article, we are going to use exact solutions of a relativistic quantum mechanical problem to study the question of how to prepare entangled states with a given entanglement measure. Namely, we study one way to affect the entanglement measure of a 2-qubit system, consisting of two photons moving in the same direction with different frequencies and each one with two possible linear polarizations with the help of an applied external magnetic field. The interaction between the magnetic field and the photons in our model is achieved through intermediate electrons that interact both with the photons and the magnetic field. An experimental realization of this theoretical scheme could be the following. Let us suppose that a beam consisting of the two photons propagating from the signal sender to a recipient crosses a region filed with free electrons subjected to an action of the magnetic field. Thus, there appear a possibility of creating an indirect interaction between the external magnetic field and the photon beam. Leaving the region filled with the electrons the photons will, in the general case, be registered in an entangled state if their initial state was separable, or, if the initial state was already entangled, they will be registered in an entangled state with a modified initial entanglement measure. The theoretical support for this scheme is based on exact solutions of quantum equations of motion that describe an interaction of an electron subjected to an external magnetic field (or a medium of electrons not interacting with each other) with a quantized field of two photons with different frequencies and arbitrary linear polarizations. These exact solutions were studied in Refs. [6–10]. In sect. 2 we apply results of this study to the case under consideration. In sect. 3, using wave functions obtained in sect. 2, we calculate the entanglement measure of the photon beam as a function of the applied magnetic field and parameters of the electron medium. These calculations are done in the lowest order in the small parameter which appears in the problem naturally as a product of the fine-structure constant and density of the electron medium to illustrate the proposed idea. Formulas for the entanglement measure which we use in our problem are placed in the Appendix. It should be noted that a preliminary consideration of a similar problem was presented in [11].

*Electronic address: alevin@if.usp.br
†Electronic address: gitman@if.usp.br
‡Electronic address: rialcap@usp.br

2. TWO-QUBIT PHOTON BEAM INTERACTING WITH AN ELECTRON PLACED IN CONSTANT UNIFORM MAGNETIC FIELD

Consider a system of photons (in what follows photon beam) moving in the same direction $\mathbf{n}$ and interacting with a Dirac electron. At the same time whole the system is placed in an external constant and uniform magnetic field $\mathbf{B} = B\mathbf{n}$ parallel to the photon beam. In fact, this external field affects directly only the electron, but then, due to electron-photon interaction, affects photons as well. The quantum motion of such a system was studied in detail in Refs. [6–10]. Solving the problem in the volume box $V = L^3$, one obtains, in fact, quantum states for a photon beam interacting with a free electron gas with a given particle density $\rho = V^{-1}$.

Below we describe a class of solutions of this kind that correspond to a photon beam that consists of two photons with different frequencies, each one with two possible polarizations. The system under consideration is based on the following Hamiltonian

$$\hat{H}_{e,\gamma} = \hat{H}_\gamma + \gamma^0 \left(\boldsymbol{\gamma}\hat{\mathbf{P}}\right) + m\gamma^0\,,\ \hat{\mathbf{P}} = \hat{\mathbf{p}} + e\left[\hat{\mathbf{A}}(\mathbf{r}) + \mathbf{A}^{\mathrm{ext}}(\mathbf{r})\right],\ \mathbf{r} = (x, y, z)\,. \tag{1}$$

Here $\hat{H}_\gamma$ is the Hamiltonian of the two free transversal photons, that move in $\mathbf{n}$ direction; $\gamma^\mu = \left(\gamma^0, \boldsymbol{\gamma}\ \right)$ are Dirac gamma matrices [12]; $\hat{\mathbf{A}}(\mathbf{r})$ is the operator-valued vector potential of the photons in the Coulomb gauge, $\hat{A}_0 = 0$, $\operatorname{div}\hat{\mathbf{A}}(\mathbf{r}) = 0$; $\mathbf{r}$ are electron coordinates; $\hat{\mathbf{p}} = -i\boldsymbol{\nabla}$ is the electron momentum operator, and $\mathbf{A}^{\mathrm{ext}}(\mathbf{r})$ is the vector potential of the magnetic field in the Landau gauge ($A_x^{\mathrm{ext}} = -By,\ A_0^{\mathrm{ext}} = A_y^{\mathrm{ext}} = A_z^{\mathrm{ext}} = 0$), $B > 0$ magnitude of the magnetic field, $e > 0$ is the absolute value of the electron charge, and m is the electron mass. Following original work, we represent solutions in the Heavyside system of units[1]. Provided that $\mathbf{n}$ is chosen along the axis z, $\mathbf{n} = (0, 0, 1)$, momenta of the photons from the beam are

$$\mathbf{k}_s = 2\pi L^{-1}\,(0, 0, m_s) = \kappa_s \mathbf{n},\ s = 1, 2,\ \kappa_s = \kappa_0 m_s,\ \kappa_0 = 2\pi L^{-1},\ m_s \in \mathbb{N}\,, \tag{2}$$

so that

$$\hat{H}_\gamma = \sum_{s=1,2;\lambda} \kappa_s a^+_{s,\lambda} a_{s,\lambda},\ \hat{\mathbf{A}}(\mathbf{r}) = \sum_{s=1,2;\lambda} \frac{1}{e}\sqrt{\frac{\varepsilon}{2\kappa_s}}\left[a_{s,\lambda} e^{i\kappa_s z} + a^+_{s,\lambda} e^{-i\kappa_s z}\right]\mathbf{e}_\lambda\,.$$

Here $V = L^3$ is the quantization box volume, $\mathbf{e}_\lambda$, $\lambda = 1, 2$ are real polarization vectors, $(\mathbf{e}_\lambda \mathbf{e}_{\lambda'}) = \delta_{\lambda,\lambda'}$, $(\mathbf{n}\mathbf{e}_\lambda) = 0$ and $\varepsilon = e^2/L^3$. The photon creation and annihilation operators $a^+_{s,\lambda}$ and $a_{s,\lambda}$ are labeled by s and by λ and obey the Bose type commutation relations. The only nonzero relations are

$$a_{s',\lambda'} a^+_{s,\lambda} - a^+_{s,\lambda} a_{s',\lambda'} = \delta_{s,s'}\delta_{\lambda,\lambda'}\,,\ s, s' = 1, 2, \quad \lambda, \lambda' = 1, 2.$$

The quantity ε characterizes a strength of the interaction between the charge and the plane-wave field. If we interpret $\rho = V^{-1} = L^{-3}$ as the electron density, then $\varepsilon = e^2\rho$. The dimensionality of ε is $[\varepsilon] = l^{-3}$ (where l is the dimensionality of length). Being written with $\hbar$ and c restored, it has the form:

$$\varepsilon = \alpha\rho = \frac{\alpha\kappa_0^3}{8\pi^3}\,,\quad \alpha = \frac{e^2}{\hbar c} = 1/137, \tag{3}$$

where α is the fine-structure constant.

Motion of an electron in the magnetic field can be represented as an oscillator motion, described by new Bose creation, a_0^+, and annihilation, a_0, operators,

$$\sqrt{2}a_0^+ = \eta - \partial_\eta,\ \sqrt{2}a_0 = \eta + \partial_\eta,\quad \eta = \frac{eBy - p_x}{\sqrt{eB}}. \tag{4}$$

The operators a_0 and a_0^+ commute with every photon operator $a_{k,\lambda}$ and $a^+_{k,\lambda}$.

Using a canonical transformation, one can diagonalize the total Hamiltonian (1) such that it is reduced to two terms that describe two subsystems – a subsystem of a quasielectron and a subsystem of quasiphotons – that do not

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[1] where $\hbar = c = 1$, and the Coulomb law takes the form $F = q_1 q_2/4\pi r^2$, also $m_G = \frac{\hbar}{c} m_H$, $t_G = \frac{1}{c} t_H$, and $e_G = \sqrt{\frac{c\hbar}{4\pi}} e_H$, $B_G = \sqrt{4\pi c\hbar} B_H$.

interact between themselves,

$$\hat{H} = \tilde{H}_\gamma + \tilde{H}_e, \quad \tilde{H}_e = r_0 c_0^+ c_0 + \frac{m^2}{2(np)} - \frac{\omega}{2},$$
$$\tilde{H}_\gamma = \sum_{s=0,1,2;\lambda} r_{s\lambda} c_{s,\lambda}^+ c_{s,\lambda} + \tilde{H}_{\gamma 0}, \quad \tilde{H}_{\gamma 0} = -\sum_{s,k=0,1,2;\lambda,\lambda'} r_{k\lambda'} |v_{s\lambda,k\lambda'}|^2 + \frac{\epsilon\,(\kappa_1+\kappa_2)}{2\kappa_1\kappa_2}, \tag{5}$$

where

$$\epsilon = \frac{\varepsilon}{(np)} \geq 0, \quad \omega = \frac{eB}{(np)} \geq 0, \ (np) = p_0 - p_z > 0, \tag{6}$$

p_0 is the electron energy and p_z is z-projection of the electron momentum, such that for the electron states $(np) > 0$; the quantities $r_{k\lambda}$ are positive roots of the equation

$$\sum_{s=1,2} \frac{\epsilon}{r_{k\lambda}^2 - \kappa_s^2} = 1 + \frac{(-1)^{\lambda-1}\omega}{r_{k\lambda}}, \quad r_{0\lambda} = r_0 \delta_{\lambda 1}, \tag{7}$$

The matrices $v_{s\lambda,k\lambda'}$ are involved in the above mentioned canonical transformation, which being written in matrix form reads

$$a = uc - vc^+, \ a^+ = c^+u^+ - cv^+; \quad c = u^+a + v^T a^+, \ c^+ = a^+u + av^*;$$
$$uu^+ - vv^+ = 1, \ vu^T - uv^T = 0\,. \tag{8}$$

This linear uniform canonical transformation [13] relates initial creation and annihilation operators $a_{k,\lambda}$ and $a_{k,\lambda}^+$, $k = 0,1,2$, $a_{0,\lambda} = a_0\delta_{\lambda 1}$, to new creation and annihilation operators $c_{k,\lambda}$ and $c_{k,\lambda}^+$, $k = 0,1,2$, $c_{0,\lambda} = c_0\delta_{\lambda 1}$. The free photon operators $a_{s,\lambda}^+$ and $a_{s,\lambda}$, $s = 1,2$, $\lambda = 1,2$, are transformed to new quasiphoton operators $c_{s,\lambda}^+$ and $c_{s,\lambda}$, $s = 1,2$, $\lambda = 1,2$, and the electron creation and annihilation operators a_0^+ and a_0 are transformed to the corresponding quasielectron operators c_0^+ and c_0.

For our purposes it is necessary to write here explicitly only the matrices $u_{s\lambda,k\lambda'}$ and $v_{s\lambda,k\lambda'}$ that correspond to the transformation of the photon operators, i.e., matrices with the indices $s,k = 1,2$, and $\lambda = 1,2$. These matrices have the form

$$u_{s\lambda,k\lambda'} = \left[\left(\sqrt{\frac{r_{k\lambda'}}{\kappa_s}} + \sqrt{\frac{\kappa_s}{r_{k\lambda'}}}\right) \frac{(-1)^{\lambda'-1}\delta_{\lambda,1} - i\delta_{\lambda,2}}{2(r_{k\lambda'}^2 - \kappa_s^2)}\right] q_{k\lambda'},$$
$$v_{s\lambda,k\lambda'} = \left[\left(\sqrt{\frac{r_{k\lambda'}}{\kappa_s}} - \sqrt{\frac{\kappa_s}{r_{k\lambda'}}}\right) \frac{(-1)^{\lambda'-1}\delta_{\lambda,1} - i\delta_{\lambda,2}}{2(r_{k\lambda'}^2 - \kappa_s^2)}\right] q_{k\lambda'},$$
$$q_{k\lambda} = \left[\frac{(-1)^\lambda \omega}{r_{k\lambda}^3 \epsilon} + 2\sum_{s=1,2} (r_{k\lambda}^2 - \kappa_s^2)^{-2}\right]^{-1/2}. \tag{9}$$

Stationary states of the system are $\Psi = \Psi_\gamma \otimes \Psi_e$, where Ψ_γ is the state vector of the quasiphotons,

$$\Psi_\gamma = \prod_{\lambda_1=1,2} \frac{(c_{1,\lambda_1}^+)^{N_{1,\lambda_1}}}{\sqrt{N_{1,\lambda_1}!}} |0_1\rangle_c \otimes \prod_{\lambda_2=1,2} \frac{(c_{2,\lambda_2}^+)^{N_{2,\lambda_2}}}{\sqrt{N_{2,\lambda_2}!}} |0_2\rangle_c\,, \tag{10}$$

where $c_{s\lambda}|0_s\rangle_c = 0$, $s = 1,2$, $\forall\lambda$, and Ψ_e is the state vector of the quasielectron, the explicit form of which is not important for our purposes.

3. ENTANGLEMENT IN TWO-QUBIT PHOTON BEAM

In this article, to illustrate the proposed idea, we consider the case in which the parameter ϵ is small. That is why, in what follow, we calculate all the quantities neglecting terms smaller than ϵ as $\epsilon \to 0$.

In this approximation, for $k = 1,2$, we obtain

$$r_{k\lambda} = \kappa_k + \frac{\left[2\kappa_k^2 - \left(\kappa_1^2 + \kappa_2^2\right)\right]\epsilon}{(-1)^{\lambda-1} 2\omega\left[2\kappa_k^2 - (\kappa_1^2+\kappa_2^2)\right] + \kappa_k\left[5\kappa_k^2 - 3\left(\kappa_1^2+\kappa_2^2\right)\right] + \frac{\kappa_1^2\kappa_2^2}{\kappa_k}}. \tag{11}$$

3.1. Photons with antiparallel polarizations

Consider now states (10) with only two quasi-photons, one of the first kind, and another of the second kind, and with anti-parallel polarizations, which we take as $\lambda_1 = 1$ and $\lambda_2 = 2$. Such a state vector corresponds to $N_{1,1} = N_{2,2} = 1$, $N_{1,2} = N_{2,1} = 0$ and has the form

$$\Psi_\gamma(\uparrow,\downarrow) = c^+_{1,1}c^+_{2,2}\left|0\right\rangle_c , \quad \left|0\right\rangle_c = \left|0_1\right\rangle_c \otimes \left|0_2\right\rangle_c \ . \tag{12}$$

From the point of view of quasi-photons this is a separable state. However, if an observer analyses this state using tools that register free photons (to be consistent we have to suppose that in the region where such measurements is performed the magnetic field and the electron density are zero) he will observe an entangled two free photon state. The wave function of such a state can be obtained from the one eq. (12) by expressing all the quasi-photon operators and vacuum vectors in terms of the corresponding free photon quantities. In this consideration, we disregard all the terms quadratic and more than quadratic in the small parameter ϵ, see (6). One can easily verify that in such an approximation, one can simply replace the quasi-photon vacua $\left|0_s\right\rangle_c$ by the corresponding free photon vacua $\left|0\right\rangle_s$, $a_{s,\lambda}\left|0_s\right\rangle = 0$, so that

$$\Psi_\gamma(\uparrow,\downarrow) = c^+_{1,1}c^+_{2,2}\left|0\right\rangle_c \simeq c^+_{1,1}c^+_{2,2}\left|0\right\rangle , \quad \left|0\right\rangle = \left|0_1\right\rangle \otimes \left|0_2\right\rangle . \tag{13}$$

The operator $c^+_{1,1}c^+_{2,2}$ has to be expressed in terms of the free photon operators using the canonical transformations (8) and (9). Using the explicit form of the matrices u and v from eqs. (9) one can see that in the approximation under consideration the last expression for the state vector $\Psi_\gamma(\uparrow,\downarrow)$ (13) takes the form

$$\Psi_\gamma(\uparrow,\downarrow) \simeq \sum_{s,s',\lambda,\lambda'} u_{s\lambda}\tilde{u}_{s'\lambda'}a^+_{s,\lambda}a^+_{s',\lambda'}\left|0\right\rangle , \quad u_{s\lambda} = u_{s\lambda,11}, \ \tilde{u}_{s\lambda} = u_{s\lambda,22}. \tag{14}$$

Then one can see that in the approximation under consideration we have to neglect terms of the form $u_{1\lambda}\tilde{u}_{1\lambda'}a^+_{1,\lambda}a^+_{1,\lambda'}$ and $u_{2\lambda}\tilde{u}_{2\lambda'}a^+_{2,\lambda}a^+_{2,\lambda'}$ in the right hand side of eq. (14). Thus, we obtain

$$\Psi_\gamma(\uparrow,\downarrow) \simeq \sum_{\lambda,\lambda'=1,2} \vartheta_{\lambda\lambda'}a^+_{1,\lambda}a^+_{2,\lambda'}\left|0\right\rangle , \ \vartheta_{\lambda\lambda'} = u_{1\lambda}\tilde{u}_{2\lambda'} + \tilde{u}_{1\lambda}u, \tag{15}$$

The vectors $a^+_{1,\lambda}a^+_{2,\lambda'}\left|0\right\rangle$ can be represented as elements of the computational basis $\left|ll'\right\rangle , \ l,l' = 0,1$ (35),

$$\left|ll'\right\rangle = a^+_{1,l+1}\left|0_1\right\rangle \otimes a^+_{2,l',1}\left|0_2\right\rangle , \ l,l' = 0,1. \tag{16}$$

Then

$$\begin{aligned}
&\Psi_\gamma(\uparrow,\downarrow) \simeq \sum_{\lambda,\lambda'=1,2} \vartheta_{\lambda\lambda'}\left|(\lambda-1)(\lambda'-1)\right\rangle = \vartheta_{11}\left|00\right\rangle + \vartheta_{22}\left|11\right\rangle + \vartheta_{12}\left|01\right\rangle + \vartheta_{21}\left|10\right\rangle , \\
&\vartheta_{11} = u_{11,11}u_{21,22} + u_{21,11}u_{11,22}, \ \vartheta_{12} = u_{11,11}u_{22,22} + u_{22,11}u_{11,22}, \\
&\vartheta_{21} = u_{12,11}u_{21,22} + u_{21,11}u_{12,22}, \ \vartheta_{22} = u_{12,11}u_{22,22} + u_{22,11}u_{12,22}.
\end{aligned} \tag{17}$$

To be able to use results presented in the Appendix to calculate the entanglement measure, we have to identify the state (17) with the pure 2-qubit state of the general form given by eq. (36). In the case under consideration, we obtain

$$\begin{aligned}
&\upsilon_1 = \vartheta_{11} = u_{11,11}u_{21,22} + u_{21,11}u_{11,22}, \ \upsilon_2 = \vartheta_{12} = u_{11,11}u_{22,22} + u_{22,11}u_{11,22}, \\
&\upsilon_3 = \vartheta_{21} = u_{12,11}u_{21,22} + u_{21,11}u_{12,22}, \ \upsilon_4 = \vartheta_{22} = u_{12,11}u_{22,22} + u_{22,11}u_{12,22} \ .
\end{aligned} \tag{18}$$

Using the explicit form of the matrices u from eq. (9) and square roots $r_{k\lambda}$ from eq. (**??**), we calculate the quantities υ_i. They are

$$\upsilon_1 = -(a+b), \quad \upsilon_2 = i(a-b), \quad \upsilon_3 = -i(a-b), \quad \upsilon_4 = \upsilon_1, \tag{19}$$

where

$$\begin{aligned}
a &= \frac{\left(\sqrt{\frac{r_{11}}{\kappa_2}} + \sqrt{\frac{\kappa_2}{r_{11}}}\right)\left(\sqrt{\frac{r_{22}}{\kappa_1}} + \sqrt{\frac{\kappa_1}{r_{22}}}\right)}{4\left(r_{11}^2 - \kappa_2^2\right)\left(r_{22}^2 - \kappa_1^2\right)\sqrt{\frac{2}{\left(r_{11}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{11}^2-\kappa_2^2\right)^2} - \frac{\omega}{r_{11}^3\epsilon}}\sqrt{\frac{2}{\left(r_{22}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{22}^2-\kappa_2^2\right)^2} + \frac{\omega}{r_{22}^3\epsilon}}}, \\
b &= \frac{\left(\sqrt{\frac{r_{11}}{\kappa_1}} + \sqrt{\frac{\kappa_1}{r_{11}}}\right)\left(\sqrt{\frac{r_{22}}{\kappa_2}} + \sqrt{\frac{\kappa_2}{r_{22}}}\right)}{4\left(r_{11}^2 - \kappa_1^2\right)\left(r_{22}^2 - \kappa_2^2\right)\sqrt{\frac{2}{\left(r_{11}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{11}^2-\kappa_2^2\right)^2} - \frac{\omega}{r_{11}^3\epsilon}}\sqrt{\frac{2}{\left(r_{22}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{22}^2-\kappa_2^2\right)^2} + \frac{\omega}{r_{22}^3\epsilon}}}.
\end{aligned} \tag{20}$$

Then we use eqs. (39), (41), and (42) to obtain the quantity y,

$$y = 4\left|\left(a^2 - b^2\right)\right| = \left|1 - 2\omega\epsilon\left(\frac{\kappa_{11}^{(1)}}{\kappa_1} - \frac{\kappa_{22}^{(1)}}{\kappa_2}\right)\right| = 1 - \epsilon\Phi, \quad 0 \leq \epsilon\Phi < 1,$$
$$\Phi = \omega\frac{\left(\kappa_2\left(\omega - \kappa_2\right)^2 - \kappa_1^3 - 2\omega\kappa_1^2 - \omega^2\kappa_1\right)}{2\kappa_1\kappa_2\left(\omega + \kappa_1\right)^2\left(\omega - \kappa_2\right)^2}. \tag{21}$$

The asymptotic behavior of the enanglement measure $E\left(\Psi_\gamma\left(\uparrow,\downarrow\right)\right)$ as $\epsilon \to 0$ reads

$$E\left(\Psi_\gamma\left(\uparrow,\downarrow\right)\right) = \frac{\Phi}{2\ln 2}\left[\epsilon\left(1 - \ln\left(\Phi/2\right)\right) - \epsilon\ln\epsilon\right]. \tag{22}$$

One can verify that the enanglement measure $E\left(\Psi_\gamma\left(\downarrow,\uparrow\right)\right)$ has the same form (22).

We have calculated the entangelment measure for different values of cyclotron frequencies $\omega = 0 \div 0.5THz$ achievable in a laboratory. For this study we selected photon angular frequencies starting with red light $\kappa_1 = 2500THz$ and calculate the $E\left(\Psi_\gamma\left(\uparrow,\downarrow\right)\right)$ as a function of $\Delta\kappa = \kappa_2 - \kappa_1$ ranging from red to ultraviolet. The result is shown in figure 1 as a surface plot, where the color gradient represents the values of entangelment measure.

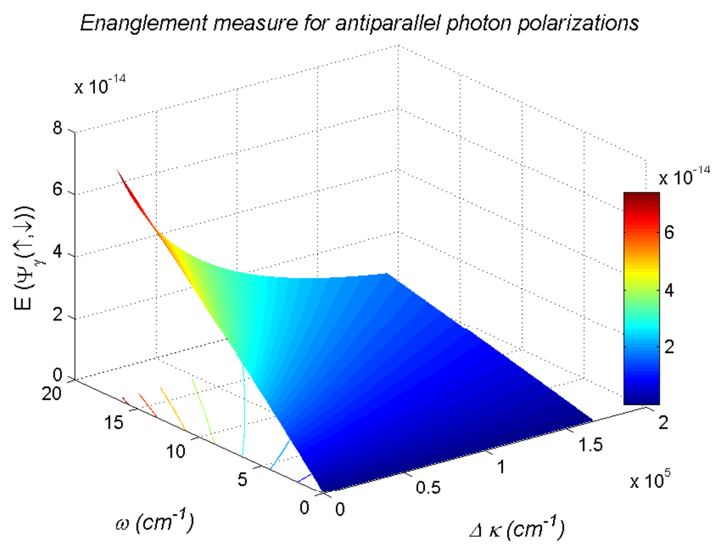

FIG. 1: Eenanglement measure $E\left(\Psi_\gamma\left(\downarrow,\uparrow\right)\right)$ as afunction of ω and $\Delta\kappa$, with $\epsilon = 0.1$.

3.2. Photons with parallel polarizations aligned along the magnetic field

Let us reconsider states (10) with only two quasiphotons, one of the first kind, and another one of the second kind and with parallel polarizations $\lambda_1 = 1$ and $\lambda_2 = 1$. Such a state vector corresponds to $N_{1,1} = N_{2,1} = 1$, $N_{1,2} = N_{2,2} = 0$ and has the form

$$\Psi_\gamma\left(\uparrow,\uparrow\right) = c_{1,1}^{+}c_{2,1}^{+}\left|0\right\rangle_c, \quad \left|0\right\rangle_c = \left|0_1\right\rangle_c \otimes \left|0_2\right\rangle_c . \tag{23}$$

Using the same arguments that were used in the case of antiparallel polarizations, we obtain

$$\Psi_\gamma\left(\uparrow,\uparrow\right) \simeq \vartheta_{11}\left|00\right\rangle + \vartheta_{22}\left|11\right\rangle + \vartheta_{12}\left|01\right\rangle + \vartheta_{21}\left|10\right\rangle,$$

where

$$\vartheta_{11} = u_{11,11}u_{21,21} + u_{21,11}u_{11,21}, \quad \vartheta_{12} = u_{11,11}u_{22,21} + u_{22,11}u_{11,21},$$
$$\vartheta_{21} = u_{12,11}u_{21,21} + u_{21,11}u_{12,21}, \quad \vartheta_{22} = u_{12,11}u_{22,21} + u_{22,11}u_{12,21}. \tag{24}$$

Using the explicit form of the matrices u from eq. (9) and square roots $r_{k\lambda}$ from eq. (**??**), we calculate the quantities υ_i. They are

$$\upsilon_1 = a + b,\ \upsilon_2 = -i\upsilon_1,\ \upsilon_3 = -i\upsilon_1,\ \upsilon_4 = -\upsilon_1, \tag{25}$$

where

$$a = \frac{\left(\sqrt{\frac{r_{11}}{\kappa_2}} + \sqrt{\frac{\kappa_2}{r_{11}}}\right)\left(\sqrt{\frac{r_{21}}{\kappa_1}} + \sqrt{\frac{\kappa_1}{r_{21}}}\right)}{4\left(r_{11}^2 - \kappa_2^2\right)\left(r_{21}^2 - \kappa_1^2\right)\sqrt{\frac{2}{\left(r_{11}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{11}^2-\kappa_2^2\right)^2} - \frac{\omega}{r_{11}^3\epsilon}}\sqrt{\frac{2}{\left(r_{21}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{21}^2-\kappa_2^2\right)^2} - \frac{\omega}{r_{21}^3\epsilon}}},$$

$$b = \frac{\left(\sqrt{\frac{r_{11}}{\kappa_1}} + \sqrt{\frac{\kappa_1}{r_{11}}}\right)\left(\sqrt{\frac{r_{21}}{\kappa_2}} + \sqrt{\frac{\kappa_2}{r_{21}}}\right)}{4\left(r_{11}^2 - \kappa_1^2\right)\left(r_{21}^2 - \kappa_2^2\right)\sqrt{\frac{2}{\left(r_{11}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{11}^2-\kappa_2^2\right)^2} - \frac{\omega}{r_{11}^3\epsilon}}\sqrt{\frac{2}{\left(r_{21}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{21}^2-\kappa_2^2\right)^2} - \frac{\omega}{r_{21}^3\epsilon}}}. \tag{26}$$

Then we use eqs. (39), (41), and (42) to obtain the quantity y and the entanglement measure of the state $\Psi_\gamma(\uparrow,\uparrow)$:

$$y = \sqrt{4\left|\upsilon_1 i\upsilon_1 + (-i\upsilon_1)(-\upsilon_1)\right|^2} = 4\left|\upsilon_1^2\right| = 1,\ E(\Psi_\gamma(\uparrow,\uparrow)) = 0. \tag{27}$$

3.3. Photons with parallel polarizations aligned against the magnetic field

Now we consider states (10) with only two quasi-photons, one of the first kind, and another one of the second kind and with parallel polarizations $\lambda_1 = 2$, $\lambda_2 = 2$. Such a state vector corresponds to $N_{1,2} = N_{2,2} = 1$, $N_{1,1} = N_{2,1} = 0$ and has the form

$$\Psi_\gamma(\downarrow,\downarrow) = c_{1,2}^+ c_{2,2}^+ \left|0\right\rangle_c, \qquad \left|0\right\rangle_c = \left|0_1\right\rangle_c \otimes \left|0_2\right\rangle_c\ . \tag{28}$$

Using the same arguments that were used in the previous cases, we obtain

$$\Psi_\gamma(\downarrow,\downarrow) \simeq \vartheta_{11}\left|00\right\rangle + \vartheta_{22}\left|11\right\rangle + \vartheta_{12}\left|01\right\rangle + \vartheta_{21}\left|10\right\rangle, \tag{29}$$

where

$$\begin{aligned}&\vartheta_{11} = u_{11,12}u_{21,22} + u_{21,12}u_{11,22}, \quad \vartheta_{12} = u_{11,12}u_{22,22} + u_{22,12}u_{11,22},\\ &\vartheta_{21} = u_{12,12}u_{21,22} + u_{21,12}u_{12,22}, \quad \vartheta_{22} = u_{12,12}u_{22,22} + u_{22,12}u_{12,22}\ .\end{aligned} \tag{30}$$

Using the explicit form of the matrices u from eq. (9) and square roots $r_{k\lambda}$ from eq. (**??**), we calculate the quantities υ_i. They are

$$\upsilon_1 = (a+b), \upsilon_2 = i(a+b), \upsilon_3 = i(a+b), \upsilon_4 = -\upsilon_1, \tag{31}$$

where

$$a = \frac{\left(\sqrt{\frac{r_{12}}{\kappa_2}} + \sqrt{\frac{\kappa_2}{r_{12}}}\right)\left(\sqrt{\frac{r_{22}}{\kappa_1}} + \sqrt{\frac{\kappa_1}{r_{22}}}\right)}{4\left(r_{12}^2 - \kappa_2^2\right)\left(r_{22}^2 - \kappa_1^2\right)\sqrt{\frac{2}{\left(r_{12}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{12}^2-\kappa_2^2\right)^2} + \frac{\omega}{r_{12}^3\epsilon}}\sqrt{\frac{2}{\left(r_{22}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{22}^2-\kappa_2^2\right)^2} + \frac{\omega}{r_{22}^3\epsilon}}},$$

$$b = \frac{\left(\sqrt{\frac{r_{12}}{\kappa_1}} + \sqrt{\frac{\kappa_1}{r_{12}}}\right)\left(\sqrt{\frac{r_{22}}{\kappa_2}} + \sqrt{\frac{\kappa_2}{r_{22}}}\right)}{4\left(r_{12}^2 - \kappa_1^2\right)\left(r_{22}^2 - \kappa_2^2\right)\sqrt{\frac{2}{\left(r_{12}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{12}^2-\kappa_2^2\right)^2} + \frac{\omega}{r_{12}^3\epsilon}}\sqrt{\frac{2}{\left(r_{22}^2-\kappa_1^2\right)^2} + \frac{2}{\left(r_{22}^2-\kappa_2^2\right)^2} + \frac{\omega}{r_{22}^3\epsilon}}}. \tag{32}$$

With the help of eqs. (39), (41), and (42) we obtain the quantity y,

$$\begin{aligned}&y = 4\left|(a+b)^2\right| = \left|1 - 2\epsilon\omega\left(\frac{\kappa_{12}^{(1)}}{\kappa_1} + \frac{\kappa_{22}^{(1)}}{\kappa_2}\right)\right| = 1 - \epsilon\Phi, \quad 0 \le \epsilon\Phi < 1,\\ &\Phi = \omega\frac{\left(\omega^2\kappa_2 + \kappa_1\left(\omega^2 - 4\omega\kappa_2 + \kappa_2^2\right) + \kappa_1^2\kappa_2\right)}{2\kappa_1\kappa_2\left(\omega-\kappa_1\right)^2\left(\omega-\kappa_2\right)^2}.\end{aligned} \tag{33}$$

Then the asymptotic behavior of the entanglement measure $E\left(\Psi_\gamma\left(\downarrow,\downarrow\right)\right)$ as $\epsilon\rightarrow 0$ reads:

$$E\left(\Psi_\gamma\left(\downarrow,\downarrow\right)\right)=\frac{\Phi}{2\ln 2}\left[\epsilon\left(1-\ln\left(\Phi/2\right)\right)-\epsilon\ln\epsilon\right]. \tag{34}$$

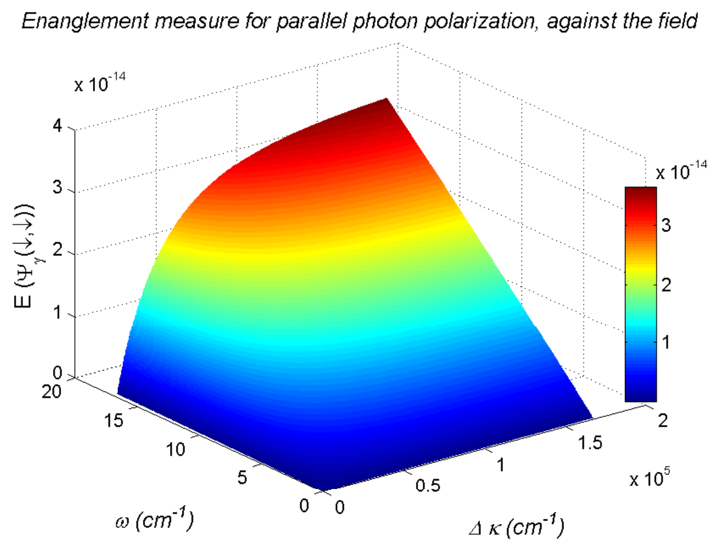

FIG. 2: Enanglement measure $E\left(\Psi_\gamma\left(\downarrow,\downarrow\right)\right)$ as a function of ω and $\Delta\kappa$, with $\epsilon=0.1$.

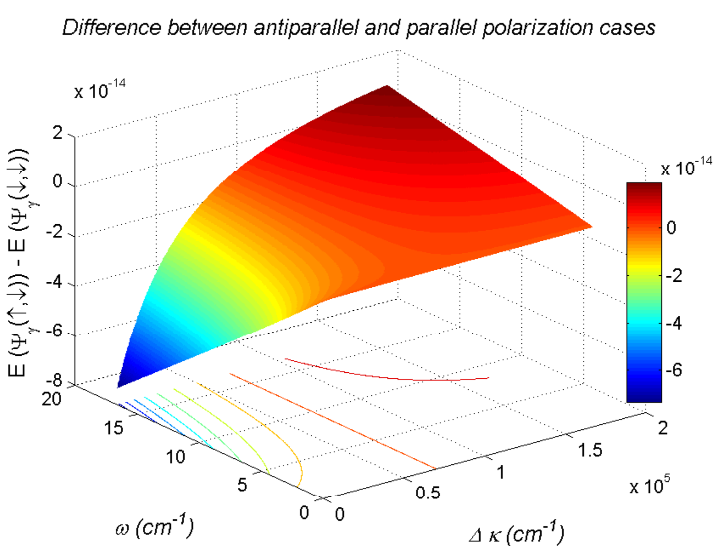

FIG. 3: Difference $E\left(\Psi_\gamma\left(\downarrow,\uparrow\right)\right)-E\left(\Psi_\gamma\left(\downarrow,\downarrow\right)\right)$ - as a function of ω and $\Delta\kappa$, with $\epsilon=0.1$.

4. CONCLUDING REMARKS

Considering an adequate quantum mechanical model, we have demonstrated that a 2-qubit system, consisting of two photons moving in the same direction with different frequencies and each one with two possible linear polarizations, can be controllably entangled with the help of an applied external magnetic field via an intermediate interaction with the electron medium. Then, we succeeded to express the corresponding entanglement measure via problem parameters, such as photon frequencies, magnitude of the magnetic field, and parameters of the electron medium. We have discovered that, in the general case, the entanglement measure depends on the magnitude of the applied magnetic field and, thus, can be controlled by the latter. As a rule, entanglement increases with increasing magnetic field (with increasing cyclotron frequency). In this relation, it should be noted that we did not consider resonance cases where cyclotron frequency approaches photon frequencies. Obviously, the entanglement depends on the parameters that specify the electron medium such as the electron density and electron energy and momentum. We did not study this dependence in this work; these characteristics were fixed by choosing a natural, small parameter in our calculations. The obtained results allow us to see how the entanglement measure depends on the fixed parameters that characterize the system under consideration, i.e. on the choice of initial states of the photons and on photon frequencies. Thus we have the following observations: if both photon polarizations coincide and coincide with the direction of the magnetic field, then no entanglement occurs. The entanglement takes place if at least one photon polarizations is aligned against the magnetic field. In this respect, we have a direct analogy with the Pauli interaction between spin and a magnetic field. However, it seams that this interaction depends also on the photon frequency and the resulting entanglement effect depends on both photon frequencies, or on the first photon frequency and the difference between the frequencies. In case that was initially chosen, i.e. parallel photon polarizations against the magnetic field, entanglement increases as this difference grows. When polarizations of both photons are opposite to the magnetic field, the entanglement effect depends on the combination of the magnetic field magnitude and the difference in photon frequencies.

We understand that our study is based on exact solutions of the model problem - electron interacting with a quantized field of two photons and with a constant uniform magnetic field. First of all, the constant uniform magnetic field is an idealization, which cannot be realized experimentally. However, such an idealization allows exact solutions and is often used in QED. Sometimes, one can verify that local space-time processes do not depend essentially on the asymptotic behavior of the external field. More realistic results could be obtained if magnetic fields vanishing on space-time infinity were used. In calculating the entanglement measure, we have also used the first order approximation in the natural parameter $\epsilon = \frac{\rho}{137(np)}$, supposing that it is small, $\epsilon \sim 0.1$. In fact, this imposes restrictions on the electron density ρ and electron energy and momentum, $(np) = p_0 - p_z$. However, this approximation was enough for our semi qualitative preliminary study of the problem. All the above-mentioned approximations would have to be carefully estimated in order to extract real numbers for possible experimental realization of the controlled entanglement of photons by a magnetic field.

Quantum entanglement in 2-qubit systems

Let us consider a 2-qubit quantum system, i.e., a composite quantum system that includes two 1-qubit systems A_1 and A_2, each of them having 2-dimensional Hilbert space $\mathcal{H}_{A_a} = \mathbb{C}^2$, $a = 1, 2$. In the spaces $\mathcal{H}_{A_a}$ we use the orthonormal basis $|\vartheta_1\rangle = |0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, $|\vartheta_2\rangle = |1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$. The composite system has 4-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} = \mathbb{C}^2 \otimes \mathbb{C}^2$. In the latter space, we use the so-called computational orthonormalized basis

$$
\begin{aligned}
&|\vartheta_1\rangle = |00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T, \ |\vartheta_2\rangle = |01\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T, \\
&|\vartheta_3\rangle = |10\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, \ |\vartheta_4\rangle = |11\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T,
\end{aligned} \tag{35}
$$

where $|ab\rangle = |a\rangle \otimes |b\rangle$. A pure 2-qubit state $|\Psi\rangle$ of the general form reads

$$
|\Psi\rangle = \sum_{i=1}^{4} v_i \, |\vartheta_i\rangle \, , \ \sum_{i=1}^{4} |v_i|^2 = 1. \tag{36}
$$

Its density operator $\hat{R}$ has the form

$$
\hat{R} = |\Psi\rangle\langle\Psi| = [v_1 \, |00\rangle + v_2 \, |01\rangle + v_3 \, |10\rangle + v_4 \, |11\rangle] \, [\langle 00| v_1^* + \langle 01| v_2^* + \langle 10| v_3^* + \langle 11| v_4^*] \, , \tag{37}
$$

Calculating the reduced density operator $\hat{\rho}^{(1)}$ of the subsystem A_1, we obtain

$$
\begin{aligned}
&\hat{\rho}^{(1)} = \mathrm{tr}_2 \hat{R} = \langle 0|\hat{R}\,|0\rangle + \langle 1|\hat{R}\,|1\rangle = |v_1|^2 \, |0\rangle \, \langle 0| + v_1 v_3^* \, |0\rangle \, \langle 1| \\
&+ |v_2|^2 \, |0\rangle \, \langle 0| + v_2 v_4^* \, |0\rangle \, \langle 1| + v_3 v_1^* \, |1\rangle \, \langle 0| + |v_3|^2 \, |1\rangle \, \langle 1| + v_4 v_2^* \, |1\rangle \, \langle 0| + |v_4|^2 \, |1\rangle \, \langle 1|.
\end{aligned} \tag{38}
$$

Taking into account that

$$|0\rangle\langle 0| = (I+\sigma_3)/2,\quad |1\rangle\langle 1| = (I-\sigma_3)/2,\quad |0\rangle\langle 1| = (\sigma_1+i\sigma_2)/2,\quad |0\rangle\langle 1| = (\sigma_1-i\sigma_2)/2,$$

we obtain matrix elements $\rho_{ab}^{(1)},\ a,b=1,2$ of the operator $\hat{\rho}^{(1)}$,

$$\rho_{11}^{(1)} = |\upsilon_1|^2+|\upsilon_2|^2,\quad \rho_{12}^{(1)} = \upsilon_1\upsilon_3^*+\upsilon_2\upsilon_4^*,\quad \rho_{21}^{(1)} = \upsilon_3\upsilon_1^*+\upsilon_4\upsilon_2^*,\quad \rho_{22}^{(1)} = |\upsilon_3|^2+|\upsilon_4|^2. \tag{39}$$

The entanglement measure $E(\Psi)$ in the system under consideration can be calculated as the information entropy (the von Neumann entropy with Boltzmann constant the is equal to $1/\ln 2$) of the reduced density operator $\hat{\rho}^{(1)}$ of subsystems A_1 (the same results we obtain calculating the von Neumann entropy of the reduced operator $\hat{\rho}^{(2)} = \mathrm{tr}_1\hat{R}$ of the subsystem A_2) [14],

$$E^{(1)}(\Psi) = -\mathrm{tr}\left(\hat{\rho}^{(1)}\log_2\hat{\rho}^{(1)}\right). \tag{40}$$

It should be noted that other quantitative characteristics of the entanglement measure exist, see Refs. [15, 16] and [17–24].

The quantity $E(\Psi) = -\mathrm{tr}(\hat{\rho}\log_2\hat{\rho})$ we calculate for an arbitrary 2×2 matrix $\hat{\rho} = (\rho_{ab})$. First we find eigenvalues λ_a, $a=1,2$, of $\hat{\rho}$,

$$\lambda_a = \frac{1}{2}\left[\rho_{11}+\rho_{22}+(-1)^a y\right],\quad y = \sqrt[+]{(\rho_{11}-\rho_{22})^2+4|\rho_{12}|^2}. \tag{41}$$

Using these eigenvalues, we find

$$\begin{aligned} E(\Psi) &= -\sum_{a=1,2}\lambda_a\log_2\lambda_a = -\frac{1}{\ln 4}\left[(1-y)\ln\left(\frac{1-y}{2}\right)+(1+y)\ln\left(\frac{1+y}{2}\right)\right] \\ &= -\left[z\log_2 z+(1-z)\log_2(1-z)\right],\quad z = \frac{1+y}{2}. \end{aligned} \tag{42}$$

One can see that expression (42) has the form derived in Ref. [15, 16]) with $z = \frac{1}{2}\left(1+\sqrt{1-C^2}\right)$, where the quantity C is called the concurrence. According to Ref. [15, 16]) it can be calculated as $C = \left|\sum_k \alpha_k^2\right|$, where α_k are decomposition coefficients of the state vector $|\Psi\rangle = \sum_{k=1}^{4}\alpha_k|e_k\rangle$ of a 2-qubit system with respect to the so-called magic basis $|e_k\rangle$. Our result (42) allows one to calculate the entanglement measure $E(\Psi)$ using decomposition coefficients of the state vector in the computational basis.

Applying this general formula (42) to the case under consideration, where eqs. (39) hold, we obtain

$$\begin{aligned} E^{(1)}(\Psi) &= -\left[z^{(1)}\log_2 z^{(1)}+\left(1-z^{(1)}\right)\log_2(1-z^{(1)})\right], \\ z^{(1)} &= \frac{1}{2}\left[1+\sqrt{\left(|\upsilon_1|^2+|\upsilon_2|^2-|\upsilon_3|^2-|\upsilon_4|^2\right)^2+4\left|\upsilon_1\upsilon_3^*+\upsilon_2\upsilon_4^*\right|^2}\right]. \end{aligned}$$

Using the normalization condition (36) one can see that $E^{(2)}(\Psi) = -\mathrm{tr}\left(\hat{\rho}^{(2)}\log_2\hat{\rho}^{(2)}\right) = E^{(1)}(\Psi)$.

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