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Extended Electron

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Abstract: In this paper we study the Ermakov-Lewis Invariants for seven non-linear Schrödinger equations by using the quantum mechanical formalism of the de Broglie-Bohm.

Keywords: De Broglie-Bohm Quantum Mechanics; Ermakov-Lewis Invariants; Non-Linear Schrödinger Equations

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2.7 The Schrödinger-Nassar Equation for an Extended Electron

In 1892,[23] Hendrik Antoon Lorentz and, in 1905,[24] Max Abraham argued that when an electron (with velocity v and charge e), is accelerated, there are additional forces acting due to the electron's own electromagnetic field. However, the so-called *Lorentz-Abraham Equation* for a point-charge electron:

$$m\ dv/dt\ =\ (2\ e^2)/(3\ c^2)\ d^2v/dt^2\ +\ F_{ext}, \quad (2.7.1)$$

it was found to be unsatisfactory because, for $F_{ext}\ =\ 0$, it admits runaway solutions. These solutions clearly violate the law of inertia.

Since the seminal works of Lorentz and Abraham, many papers and textbooks have given great consideration to the proper equation of motion of an electron.[26−33] The problematic runaway solutions were circumverted by Sommerfeld[27] and Page[28] adopting an *electron extended model* (EEM). In the nonrelativistic case of a sphere with uniform surface charge, such an electron obeys in good approximation the difference-differential equantion:[29−31]

$$m\ dv/dt\ =\ (e^2)/(3\ L^2\ c)\ [v(t\ -\ 2\ L/c)\ -v(t)]\ +\ F_{ext}. \quad (2.7.2)$$

This EEM model is finite and causal if the diameter L of the electron is larger than the classical electron radius $r_e\ =\ e^2/(m\ c^2)$. We will analyze here only the case of the sphere with uniform surface charge; the case of a volume charge is considerably more complicated and adds nothing to the understanding of the problem.

The dynamics of charges is a key example of the importance of obeying the validity limits of a physical theory. If classical equations can no longer be trusted at distances of the order (or below) the Compton wavelength, what is the *Schrödinger Equation* that can replace equation (2.7.2)? Within QED, workers have not been able to derive an equation of motion and it is unclear whether QED can actually produce an equation of motion at all.

By using the Quantum Mechanical of the de Broglie-Bohm,[7] Nassar[34] propose an answer to this problem in the nonrelativistic regime. So, the *Schrödinger-Nassar Equation for an Extended Electron* ($SNEEE$), is given by:

$$\mathrm{i}\ \hbar\ \frac{\partial\Psi(x,\ t)}{\partial t}\ =\ -\ \frac{\hbar^2}{2\ m}\ \frac{\partial^2\ \Psi(x,\ t)}{\partial x^2}\ +$$

$$+\ \Big(\ V(x,\ t)\ +\ i\ \hbar\ \gamma\ \ell n\ \Big[\ \frac{\psi(x,\ t\ -\ 2\ L/c)\ \psi^*\ (x,\ t)}{\psi^*\ (x,\ t\ -\ 2\ L/c)\ \psi\ (x,\ t)}\ \Big]\ \Big)\ \times\ \Psi(x,\ t)\ , \qquad (2.7.3)$$

where:

$$\gamma\ =\ \frac{e^2}{6\ m\ L^2\ c}, \qquad (2.7.4)$$

is a constant, beeing $\Psi(x,\ t)$ and $\Psi(x,\ t\ -\ 2\ L/c)$ wave functions, and $V(x,\ t)$ is the time dependent potential of the physical system in study.

Using the same operational protocol of the item (2.1) and considering the eqs. (2.1) and (2.1.18a-f), we obtain:[7]

$$\frac{\partial\rho(x,\ t)}{\partial t}\ +\ \frac{\partial[\rho(x,\ t)\ v_{qu}(x,\ t)]}{\partial x}\ =\ 0\ , \qquad (2.7.5)$$

where [see (2.1.23)]:

$$\rho(x,\ t)\ =\ [2\ \pi\ a^2(t)]^{-\ 1/2}\ \times\ exp\left(-\ \frac{[x\ -\ q(t)]^2}{2\ {}^2(t)}\right)\ , \qquad (2.7.6)$$

and:

$$\frac{\partial v_{qu}(x,\ t)}{\partial t}\ +\ v_{qu}(x\ t)\ \frac{\partial v_{qu}(x,\ t)}{\partial x}\ +\ \omega^2(t)\ x\ =$$

$$=\ -\ \frac{2\ \gamma}{m}\ [\ v_{que}(x,\ t\ -\ 2L/c)\ -\ v_{qu}(x,\ t)\ ]\ -\ \frac{1}{m}\ \frac{\partial}{\partial x}\ V_{qu}(x,\ t)], \qquad (2.7.7)$$

where [see (2.1.8f)]:

$$V_{qu}(x,\ t)\ =\ -\ \frac{\hbar^2}{2\ m}\ \frac{1}{\sqrt{\rho}}\ \frac{\partial^2 \sqrt{\rho}}{\partial x^2}\ , \qquad (2.7.8)$$

and:[7]

<u>Gradient of the wave function extended</u>: $\frac{\hbar}{m}\ \frac{\partial S(x,\ t\ -\ 2L/c)}{\partial x}\ \longleftrightarrow$

<u>Quantum velocity extended</u>: $v_{que}(x,\ t\ -\ 2L/c) \equiv v_{que}\ ,$ (2.7.9a,b)

Observe that the eq. (2.7.5) preserves the *Continuity Equation* and the eq. (2.7.6) has the aspect of the *Navier-Stokes Equation.*[8]

So, taking the eqs. (2.7.5,7), and using the same operational protocol of the item 2.1, results:[9]

$$v_{qu}(x,\ t)\ =\ \frac{\dot{a}(t)}{a(t)}\ [x\ -\ q(t)]\ +\ \dot{q}(t)\ . \qquad (2.7.10)$$

Expanding $S(x,\ t\ -\ 2\ L/c)$ around of $q(t)$ up to *second Taylor order* and using the eqs. (2.7.9a,b), results:[35]

$$v_{qu}(x,\ t\ -\ 2\ L/c)\ =\ \frac{\dot{a}(t\ -\ 2\ L/c)}{a(t)}\ \times\ [x\ -\ q(t)]\ +\ \dot{q}(t\ -\ 2\ L/c)\ . \qquad (2.7.11)$$

Using the eqs. (2.7.6,10) results:[35]

$$\frac{1}{m}\ \frac{\partial V_{qu}(x,\ t)}{\partial x}\ =\ \frac{\hbar^2}{4\ m^2\ a^4(t)}\ \times\ [x\ -\ q(t)] \qquad (2.7.12)$$

$$\frac{\partial v_{qu}(x,\ t)}{\partial t}\ =\ \left(\frac{\ddot{a}(t)}{a(t)}\ -\ \frac{\dot{a}^2(t)}{a^2(t)}\right)\ \times\ [x\ -\ q(t)]\ -\ \dot{q}(t)\ \frac{\dot{a}(t)}{a(t)}\ +\ \ddot{q}(t)\ , \qquad (2.7.13)$$

and:

$$\frac{\partial v_{qu}(x,\ t)}{\partial x}\ =\ \frac{\dot{a}(t)}{a(t)}\ . \qquad (2.7.14)$$

Substituing the eqs. (2.7.10-14) into the eq. (2.7.7), adding and subtracting the term $\omega^2(t)\ q(t)$, we have:

$$\Big(\frac{\ddot{a}(t)}{a(t)} \ - \ \frac{\dot{a}^2(t)}{a^2(t)} \Big) \times [x \ - \ q(t)] \ - \ \dot{q}(t)\ \frac{\dot{a}(t)}{a(t)} \ + \ \ddot{q}(t) \ +$$

$$+ \Big(\frac{\dot{a}(t)}{a(t)} \times [x \ - \ q(t)] \ + \ \dot{q}(t) \Big) \frac{\dot{a}(t)}{a(t)} \ + \ \omega^2(t) \times [x \ - \ q(t)] \ + \ \omega^2(t)\ q(t) \ =$$

$$= \ - \ \frac{2\ \gamma}{m} \Big(\frac{\dot{a}(t \ - \ 2\ L/c)}{a(t)} \times [x \ - \ q(t)] \ + \ \dot{q}(t \ - \ 2\ L/c) \ -$$

$$- \ \frac{\dot{a}(t)}{a(t)} \times [x \ - \ q(t)] \ + \ \dot{q}(t) \Big) \ - \ \frac{\hbar^2}{4\ m^2\ a^4(t)}\ [x \ - \ q(t)] \ \ \rightarrow$$

$$\Big(\frac{\ddot{a}(t)}{a(t)} \ - \ 2\ \gamma \Big[\frac{\dot{a}(t \ - \ 2\ L/c)}{a(t)} \ - \ \frac{\dot{a}(t)}{a(t)} \Big] \ - \ \frac{\hbar^2}{4\ m^2\ a^4(t)} \ + \ \omega^2(t) \Big) \times [x \ - \ q(t)] \ +$$

$$+ \ \ddot{q}(t) \ + \ 2\ \gamma\ [\ \dot{q}(t \ - \ 2\ L/c) \ - \ \dot{q}\ ] \ + \ \omega^2(t)\ q(t) \ = \ 0\ . \qquad (2.7.15)$$

To satisfay the eq. (2.7.15), the following conditions must be obeyed:

$$\frac{\ddot{a}(t)}{a(t)} \ - \ 2\ \gamma\ [\ \frac{\dot{a}(t \ - \ 2\ L/c)}{a(t)} \ - \ \frac{\dot{a}(t)}{a(t)}\ ] \ - \ \frac{\hbar^2}{4\ m^2\ a^4(t)} \ + \ \omega^2(t) \ = \ 0 \ \ \rightarrow$$

$$\ddot{a}(t) \ - \ 2\ \gamma\ [\ \dot{a}(t \ - \ 2\ L/c) \ - \ \dot{a}(t)\ ] \ + \ \omega^2(t)\ a(t) \ = \ \frac{\hbar^2}{4\ m^2\ a^3(t)}\ , \qquad (2.7.16)$$

and:

$$\ddot{q}(t) \ + \ 2\ \gamma\ [\ \dot{q}(t \ - \ 2\ L/c) \ - \ \dot{q}(t)\ ] \ + \ \omega^2(t)\ q(t) \ = \ 0\ . \qquad (2.7.17)$$

Putting [see eq. (2.1.33)]:

$$a(t) \ = \ (\ \frac{\hbar^2}{4\ m^2}\ )^{1/4}\ \alpha(t)\ , \qquad (2.7.18)$$

and:

$$a(t \ - \ 2\ L/c) \ = \ (\ \frac{\hbar^2}{4\ m^2}\ )^{1/4}\ \alpha(t \ - \ 2\ L/c). \qquad (2.7.19)$$

Substituting the eqs. (2.7.18,19) into the eq. (2.7.16), results:

$$\ddot{\alpha}(t) \ - \ 2\ \gamma\ [\ \dot{\alpha}(t \ - \ 2\ L/c) \ - \ \dot{\alpha}(t)\ ] \ + \ \omega^2(t)\ \alpha(t) \ = \ \frac{1}{\alpha^3(t)}\ . \qquad (2.7.20)$$

Finally, eliminating the factor $\omega^2(t)$ into the eqs. (2.7.17,20), we get:

$$\ddot\alpha(t) - 2\,\gamma\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,] - \Big(\,\frac{\ddot q}{q(t)} + 2\,\gamma\,[\,\frac{\dot q(t - 2\,L/c)}{q(t)} - \frac{\dot q(t)}{q(t)}\,]\,\Big)\,\alpha(t) = \frac{1}{\alpha^3(t)} \rightarrow$$

$$\ddot\alpha(t)\,q(t) - 2\,\gamma\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,]\,q(t) -$$

$$-\Big(\,\ddot q + 2\,\gamma\,[\,\dot q(t - 2\,L/c) - \dot q(t)\,]\,\Big)\,\alpha(t) = \frac{q(t)}{\alpha^3(t)} \rightarrow$$

$$\ddot\alpha(t)\,q(t) - \ddot q(t)\,\alpha(t) -$$

$$-2\,\gamma\,\Big(\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,]\,q(t) + [\,\dot q(t - 2\,L/c) - \dot q(t)\,]\,\alpha(t)\,\Big) = \frac{q(t)}{\alpha^3(t)} \rightarrow$$

$$\frac{d}{dt}\,[\ddot\alpha(t)\,q(t) - \ddot q(t)\,\alpha(t)] -$$

$$-2\,\gamma\,\Big(\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,]\,q(t) + [\,\dot q(t - 2\,L/c) - \dot q(t)\,]\,\alpha(t)\,\Big) = \frac{q(t)}{\alpha^3(t)} \rightarrow$$

$$[\dot\alpha(t)\,q(t) - \dot q(t)\,\alpha(t)]\,\frac{d}{dt}\,[\ddot\alpha(t)\,q(t) - \ddot q(t)\,\alpha(t)] -$$

$$-2\,\gamma\,\Big(\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,]\,q(t) + [\,\dot q(t - 2\,L/c) - \dot q(t)\,]\,\alpha(t)\,\Big) \times$$

$$\times\,[\dot\alpha(t)\,q(t) - \dot q(t)\,\alpha(t)] = \frac{q(t)}{\alpha^3(t)} \times [\dot\alpha(t)\,q(t) - \dot q(t)\,\alpha(t)] = -\,\frac{d}{dt}\,(\,\frac{1}{2}\,[\,\frac{q(t)}{\alpha(t)}\,]^2\,) \rightarrow$$

$$\frac{d}{dt}\,\Big(\,\frac{1}{2}\,[\,\dot\alpha(t)\,q(t) - \dot q(t)\,]^{1/2}\,\Big) + \frac{d}{dt}\,\Big(\,\frac{1}{2}\,[\,\frac{q(t)}{\alpha(t)}\,]^2\,\Big) =$$

$$= 2\,\gamma\,\Big(\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,]\,q(t) + [\,\dot q(t - 2\,L/c) - \dot q(t)\,]\,\alpha(t)\,\Big) \times$$

$$\times\,[\dot\alpha(t)\,q(t) - \dot q(t)\,\alpha(t)] =$$

$$= -\,2\,\alpha^2(t)\,\gamma\,\Big(\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,]\,q(t) +$$

$$+\,[\,\dot q(t - 2\,L/c) - \dot q(t)\,]\,\Big)\,\frac{d}{dt}\,[\,\frac{q(t)}{\alpha(t)}\,] \rightarrow$$

$$\frac{dI}{dt} = -\,2\,\alpha^2(t)\,\gamma\,\Big(\,[\,\dot\alpha(t - 2\,L/c) - \dot\alpha(t)\,]\,q(t) +$$

$$+\,[\,\dot q(t - 2\,L/c) - \dot q(t)\,]\,\Big)\,\frac{d}{dt}\,[\,\frac{q(t)}{\alpha(t)}\,]\,, \qquad (2.7.21)$$

where [see eq. (1.1)]:

$$I \ = \ \tfrac{1}{2} \left[\ (\dot{\alpha} \ q \ - \ \dot{q} \ \alpha)^2 \ + \ (\tfrac{q}{\alpha})^2 \ \right] , \qquad (2.7.22)$$

which represents the *Ermakov-Lewis Invariant* (ELI) of the $TDHO$.[35]

In conclusion, the eq. (2.7.21) we have shown that the *The Schrödinger-Nassar Equation for an Extended Electron* has not an ELI for the $TDHO$.

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