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Abstract: In this paper we study the Feynman Propagator, the Ermakov-Lewis invariant and the Bohmian Trajectories for the Logarithmic Nonlinear Schrödinger-Nassar Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm.

Keywords: De Broglie-Bohm Quantum Mechanics; Feynman Propagator, Ermakov-Lewis Invariant, Bohmian Trajectory; Logarithmic Nonlinear Schrödinger-Nassar Equation.

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1. Introduction: The de Broglie-Bohm Quantum Mechanics and the Quantum Bohmian Trajectory

In 1948, R. P. Feynman (Feynman, 1948) formulated the following principle of minimum action for the quantum mechanics:

The transition amplitude between the states $| a >$ and $| b >$ of a quantum-mechanical system is given by the sum of the elementary contributions, one for each trajectory passing by $| a >$ at the time t_a and by $| b >$ at the time t_b. Each one of these contributions have the same modulus, but its phase is the classical action $S_{c\ell}$ for each trajectory.

This principle is represented by the following expression known as the "Feynman propagator":

$$K(b,a)=\int_a^b \exp[\frac{i}{\hbar}S(a,.b)]Dx(t) \ , \qquad (1.1)$$

where $S(b, a)$ is the *classical action* given by:

$$S(b,a)=\int_{t_a}^{t_b} L(x,\dot{x},t)dt \ , \qquad (1.2)$$

$L(x, \dot{x}, t)$ is the Lagrangean and $D\,x(t)$ is the Feynman's Measurement. It indicates that we must perform the integration taking into account all the ways connecting the states $|\,a>$ and $|\,b>$.

The eq. (1.1) which defines $K(b, a)$ is called *path integral* or *Feynman integral* and the Schrödinger wavefunction $\Psi(x, t)$ of any physical system is given by (we indicate the initial position and initial time by x_o and t_o, respectively): (Feynman and Hibbs, 1965)

$$\Psi(x,t)=\int_{-\infty}^{+\infty} K(x,x_0;t,t_0)\Psi(x_0,t_0)dx_0 \ , \qquad (1.3)$$

with the *quantum causality condition*: (Bernstein, 1985)

$$\lim_{t,t_0 \to 0} K(x,x_0;t,t_0)=\delta(x-x_0) \ . \qquad (1.4)$$

2. The Logarithmic Nonlinear Schrödinger-Nassar Equation

In 2013, A. B. Nassar (Nassar, 2013) proposed a logarithmic nonlinear Schrödinger equation to represent time dependent physical systems. In this article, let us considerer this same equation with a potential energy $V(x, t)$. Then, we have:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} +$$

$$+\left\{V(x,t)-i\hbar\kappa\left[\ell n|\Psi(x,t)|^2 - <\ell n|\Psi(x,t)|^2>\right]\right\}\times\Psi(x,t) \ , \qquad (2.1)$$

where $\Psi(x, t)$ is a wave function which describes a given system and κ caracterizes the resolution of the measurement. The last term in the eq. (2.1) arises from the requirement that the integration of this equation with respect to the variable x under the condition that for a particle the expectation value of the energy $< E(t) >$ defined as:

$$<E(t)>=\int_{-\infty}^{+\infty} \Psi^*(x,t)E(t)\Psi(x,t)dx \, , \qquad (2.2)$$

must be equal to the expectation values of the kinetic and potential energies.

2.1.1. The Wave Packet of the Logarithmic Nonlinear Schrödinger-Nassar Equation

Writting the wave function $\Psi(x, t)$ in the polar form defined by the Madelung-Bohm transformation: (Madelung, 1926, Bohm, 1952)

$$\Psi(x, t) = \varphi(x, t) \times exp\,[i\,S(x, t)]\,, \qquad (2.1.1.1)$$

where $\varphi(x, t)$ will be defined in what follows.

Calculating the derivatives temporal and spatial of (2.1.1.1), we get: (Bassalo et al., 2002)

$$\frac{\partial\Psi}{\partial t} = \exp(iS)\left(\frac{\partial\varphi}{\partial t} + i\varphi\frac{\partial S}{\partial t}\right)\,, \qquad (2.1.1.2a)$$

$$\frac{\partial^2\Psi}{\partial x^2} = \exp(iS)\left[\frac{\partial^2\varphi}{\partial x^2} + 2i\frac{\partial S}{\partial x}\frac{\partial\varphi}{\partial x} + i\varphi\frac{\partial^2 S}{\partial x^2} - \varphi\left(\frac{\partial S}{\partial x}\right)^2\right]\,. \qquad (2.1.1.2b)$$

Now, substituting the eqs. (2.1.1.2a,b) into the eq. (2.1) and remembering that *exp* [*i S*] is common factor, we have:

$$i\hbar\left(\frac{\partial\varphi}{\partial t} + i\varphi\frac{\partial S}{\partial t}\right) =$$

$$= -\frac{\hbar^2}{2m}\left[\frac{\partial^2\varphi}{\partial x^2} + 2i\frac{\partial S}{\partial x}\frac{\partial\varphi}{\partial x} + i\varphi\frac{\partial^2 S}{\partial x^2} - \varphi\left(\frac{\partial S}{\partial x}\right)^2\right] +$$

$$+ \left\{V(x,t) - i\hbar\kappa\left[\ell n(\varphi)^2 - <\ell n(\varphi)^2>\right]\right\} \times \varphi\,. \quad (2.1.1.3)$$

Multiplying the eq. (2.1.1.3) by $\frac{1}{\varphi}$ and separating the real and imaginary parts, results:

a) <u>imaginary part</u>

$$\frac{\hbar}{\varphi}\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\left(2\frac{1}{\varphi}\frac{\partial S}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\right) - \hbar\kappa\left[\ell n(\varphi)^2 - <\ell n(\varphi)^2>\right]\,, \qquad (2.1.1.4)$$

b) <u>real part</u>

$$-\hbar\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2} - \left(\frac{\partial S}{\partial x}\right)^2\right] + V(x,t)\,. \qquad (2.1.1.5)$$

2.1.2. Dynamics of the Logarithmic Nonlinear Schrödinger-Nassar Equation

Now, let us see the correlation between the expressions (2.1.1.4-5) and the traditional equations of the Ideal Fluid Dynamics (See books on the Fluid Mechanics, for instance: Streeter and Debler, 1986, Coimbra, 1967, Landau and Lifshitz, 1969, Bassalo, 1973, Cattani, 1990/2005): a) *Continuity Equation*, b) *Euler's equation*. To do this let us perform the following correspondences:

Quantum density probability: $|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) \quad \leftrightarrow$

Quantum mass density: $\rho(x,t) = \varphi^2(x,t) \quad \leftrightarrow \quad \sqrt{\rho(x,t)} = \varphi(x,t)$, (2.1.2.1a,b)

Gradient of the wave function: $\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x} \quad \leftrightarrow$

Quantum velocity: $v_{qu}(x, t) \equiv v_{qu}$, (2.1.2.1c,d)

Bohm quantum potential:

$$V_{qu}(x,t) \equiv V_{qu} = -\frac{\hbar^2}{2m}\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2} = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\ . \quad (2.1.2.1\ \text{e,f})$$

Putting the relations (2.1.2.1a-d) into the equation (2.1.1.4) and considering that $\partial(\ell n\, x)/\partial y = (1/x)\,(\partial x/\partial y)$ and $\ell n(x^m) = m\, \ell n\, x$, we get:

$$\frac{1}{\varphi}\frac{\partial\varphi}{\partial t} = -\frac{\hbar}{2m}\left(2\frac{1}{\varphi}\frac{\partial S}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\right) - \kappa\left[\ell n(\varphi)^2 - <\ell n(\varphi)^2>\right] \rightarrow$$

$$\frac{\partial[\ell n(\varphi)]}{\partial t} = -\frac{\hbar}{m}\frac{\partial S}{\partial x}\frac{\partial[\ell n(\varphi)]}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right) - \kappa\left[\ell n(\varphi)^2 - <\ell n(\varphi)^2>\right] \rightarrow$$

$$\frac{\partial(\ell n\sqrt{\rho})}{\partial t} = -\frac{\hbar}{m}\frac{\partial S}{\partial x}\frac{\partial(\ell n\sqrt{\rho})}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right) - \kappa[\ell n(\rho) - <\ell n(\rho)> \quad \rightarrow$$

$$\frac{1}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial t} = -v_{qu}\frac{1}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial x} - \frac{1}{2}\frac{\partial v_{qu}}{\partial x} - \kappa[\ell n(\rho) - <\ell n(\rho)>] \quad \rightarrow$$

$$\frac{\partial\rho}{\partial t} = -v_{qu}\frac{\partial\rho}{\partial x} - \rho\frac{\partial v_{qu}}{\partial x} - 2\kappa[\ell n(\rho) - <\ell n(\rho)>]\rho \quad \rightarrow$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_{qu})}{\partial x}-2\kappa[\ell n(\rho)-<\ell n(\rho)>]\rho \ , \qquad (2.1.2.2)$$

expression that indicates <u>decoherence</u> of the physical system represented by the Logarithmic Nonlinear Schrödinger-Nassar Equation (*LNLS−NE*) [eq. (2.1)]; then the *Continuity Equation* its not preserved.

Now, let us obtained another dynamic equation of the *LNLS−NE*. So, differentiating the eq. (2.1.1.5) with respect x and using the eqs. (2.1.2.1a-e), we obtain:

$$-\hbar\frac{\partial^2 S}{\partial x\partial t}=-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2}-\left(\frac{\partial S}{\partial x}\right)^2\right]+\frac{\partial V(x,t)}{\partial x}\rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right)=$$

$$=-\frac{1}{m}\frac{\partial}{\partial x}\left(-\frac{\hbar^2}{2m}\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2}\right)-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right)^2-\frac{1}{m}\frac{\partial V(x,t)}{\partial x}=$$

$$=-\frac{1}{m}\frac{\partial V_{qu}(x,t)}{\partial x}-\frac{1}{2}\frac{\partial}{\partial x}\left[v_{qu}(x,t)\right]^2-\frac{1}{m}\frac{\partial V(x,t)}{\partial x}\rightarrow$$

$$\frac{\partial v_{qu}(x,t)}{\partial t}+v_{qu}(x,t)\frac{\partial v_{qu}(x,t)}{\partial x}=-\frac{1}{m}\frac{\partial}{\partial x}\left[V(x,t)+V_{qu}(x,t)\right] . \qquad (2.1.2.3)$$

We must observe that the eq. (2.1.2.3) is an equation similar to the *Euler Equation* which governs the motion of a fluid particle.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*: (See books on the Fluid mechanics, for instance)

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_{qu}\frac{\partial}{\partial x} \ , \qquad (2.1.2.4a)$$

and that:

$$v_{qu}(x,t)\Big|_{x=x(t)}=\frac{dx}{dt} \ , \qquad (2.1.2.4b)$$

the eq. (2.1.2.3) could be written as:

$$m\frac{d^2x}{dt^2}=-\frac{\partial V_{qu}(x,t)}{\partial x}\rightarrow$$

$$m\frac{d^2x}{dt^2} = F_{qu}(x,t)\Big|_{x=x(t)} \qquad (2.1.2.5)$$

We note that the eq. (2.1.2.5) has a form of the *Second Newton Law*, being the first term of the second member is the *classical newtonian force* and the second is the *quantum bohmnian force*.

2.1.3 The Quantum Wave Packet for the Logarithmic Nonlinear Schrödinger-Nassar Equation Linearized along a Classical Trajetory

In order to find the quantum wave packet for the Logarithmic Nonlinear Schrödinger-Nassar Equation (*LNLS−NE*) linearized along a classical trajetory, let us the considerer the *ansatz*: (Nassar, 2013)

$$\rho(x,t) = [2\pi\delta^2(t)]^{-1/2} \times \exp\{-\frac{[x-q(t)]^2}{2\delta^2(t)}\}, \qquad (2.1.3.1a)$$

or [use the eq. (2.1.2.1a,b)]:

$$\varphi(x,t) = [2\pi\delta^2(t)]^{-1/4} \times \exp\{-\frac{[x-q(t)]^2}{4\delta^2(t)}\}, \qquad (2.1.3.1b)$$

where $\delta(t)$ and $q(t) = \langle x \rangle$ are auxiliary functions of time, to will be determined in what follows, representing the *width* and the *center of mass of wave packet*, respectively.

Taking the eq. (2.1.3.1a), let us calculated the expressions [remember that $\ell n\,(ab) = \ell n\, a + \ell n\, b$ and $\ell n\, e^{\alpha} = \alpha$]:

$$\ell n[\rho(x,t)] = \ell n\left([2\pi\delta^2(t)]^{-1/2} \times \exp\{-\frac{[x-q(t)]^2}{2\delta^2(t)}\}\right) =$$

$$= \ell n[2\pi\delta^2(t)]^{-1/2} - \frac{[x-q(t)]^2}{2\delta^2(t)} . \qquad (2.1.3.2)$$

$$< \ell n[\rho(x,t)] > = < \ell n\left([2\pi\delta^2(t)]^{-1/2} \times \exp\{-\frac{[x-q(t)]^2}{2\delta^2(t)}\}\right) > =$$

$$= \ell n[2\pi\delta^2(t)]^{-1/2} - < \frac{[x-q(t)]^2}{2\delta^2(t)} > . \qquad (2.1.3.3)$$

Considering that:

$$\int_{-\infty}^{+\infty} z^2 \exp(-z^2)dz = \frac{\sqrt{\pi}}{2} , \qquad (2.1.3.4)$$

and:

$$< f(x,t) >= \int_{-\infty}^{+\infty} \rho(x,t) f(x,t) dx = g(t) , \quad (2..1.3.5)$$

we have: (Bassalo et al., 2010)

$$\ell n[\rho(x,t)] - < \ell n[\rho(x,t)] >= -\frac{\delta^2(t)}{2\rho(x,t)} \frac{\partial^2 \rho(x,t)}{\partial x^2} . \quad (2.1.3.6)$$

Insering the eq. (2.1.3.6) into eq. (2.1.2.2), results:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_{qu})}{\partial x} = -2k\rho(-\frac{\delta^2}{2\rho} \frac{\partial^2 \rho}{\partial x^2}) \quad \rightarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_{qu})}{\partial x} - \frac{\partial}{\partial x}(D \frac{\partial \rho}{\partial x}) = 0 , \quad (2.1.3.7a)$$

where:

$$D = \kappa \delta^2 . \quad (2.1.3.7b)$$

Defining (Nassar, 1986a):

$$\vartheta_{qu} = v_{qu} - \frac{D}{\rho} \frac{\partial \rho}{\partial x} , \quad (2.1.3.8)$$

then the eq. (2.1.3.7a) will be the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \vartheta)}{\partial x} = 0 . \quad (2.1.3.9)$$

Differenting the eq. (2.1.3.1a) in the variables *t* and *x* [remembering that *x* and *t* are independent variables], results: (Bassalo et al., 2010)

$$\frac{\partial \rho}{\partial t} = \rho \left\{ -\frac{\dot{\delta}(t)}{\delta(t)} + \frac{\dot{q}(t)}{\delta^2(t)}[x-q(t)] + \frac{\dot{a}(t)}{\delta^3(t)}[x-q(t)]^2 \right\}, \qquad (2.1.3.10)$$

$$\frac{\partial \rho}{\partial x} = -\rho \frac{[x-q(t)]}{\delta^2(t)}. \qquad (2.1.3.11)$$

Now, substituting the eqs. (2.1.3.10,11) into (2.1.3.9) and integrating the result, we obtain: (Bassalo et al., 2010)

$$\vartheta_{qu}(x,t) = \frac{\dot{\delta}(t)}{\delta(t)}[x-q(t)] + \dot{q}(t). \qquad (2.1.3.12)$$

Using the eqs. (2.1.3.7b,8,12), we have:

$$v_{qu}(x,t) = [\frac{\dot{\delta}(t)}{\delta(t)} - k] \times [x-q(t)] + \dot{q}(t). \qquad (2.1.3.13)$$

To obtain the quantum wave packet [$\Psi(x\ ,\ t)$] of the *LNLS*−*NE* given by eq. (2.1), let us expand the functions S(x, t) and V_{qu}(x, t) around of $q(t)$ up to *second Taylor order*. In this way, we have:

$$S(x,t) = S[q(t),t] + S'[q(t),t] \times [x-q(t)] + \frac{S''[q(t),t]}{2} \times [x-q(t)]^2 \qquad (2.1.3.14)$$

$$V(x,t) = V[q(t),t] + V'[q(t),t] \times [x-q(t)] + \frac{V''[q(t),t]}{2} \times [x-q(t)]^2 \qquad (2.1.3.15)$$

$$V_{qu}(x,t) = V_{qu}[q(t),t] + V'_{qu}[q(t),t] \times [x-q(t)] +$$

$$+ \frac{V''_{qu}[q(t),t]}{2} \times [x-q(t)]^2 \qquad (2.1.3.16)$$

where (’) and " means, respectively, $\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial q^2}$.

Differentiating the eq. (2.1.3.14) in the variable x, multiplying the result by ħ/m, using the eqs. (2.1.2.1c,d) and (2.1.3.13), results:

$$\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x} = \frac{\hbar}{m}\{S'[q(t),t] + S''[q(t),t] \times [x - q(t)]\} =$$

$$= v_{qu}(x,t) = [\frac{\dot{\delta}(t)}{\delta(t)} - k] \times [x - q(t)] + \dot{q}(t) \qquad \rightarrow$$

$$S'[q(t),t] = \frac{m\dot{q}(t)}{\hbar}, \quad S''[q(t),t] = \frac{m}{\hbar}[\frac{\dot{q}(t)}{\hbar} - \kappa]. \qquad (2.1.3.17a,b)$$

Substituting the eq. (2.1.3.17a,b) into the eq. (2.1.3.14), we have:

$$S(x,t) = S_0(t) + \frac{m\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\hbar}[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times [x - q(t)]^2, \quad (2.1.3.18a)$$

where:

$$S_0(t) \equiv S[q(t),t], \qquad (2.1.3.18b)$$

are the *quantum action*.

Differentiating the eq. (2.1.3.18a) in relation to the time t, we obtain (remembering that $\partial x/\partial t = 0$):

$$\frac{\partial S}{\partial t} = \dot{S}_0(t) + \frac{\partial}{\partial t}\left\{\frac{m\dot{q}(t)}{\hbar} \times [x - q(t)]\right\} + \frac{\partial}{\partial t}\left\{\frac{m}{2\hbar}[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times [x - q(t)]^2\right\} \qquad \rightarrow$$

$$\frac{\partial S}{\partial t} = \dot{S}_0(t) + \frac{m\ddot{q}(t)}{\hbar} \times [x - q(t)] - \frac{m\dot{q}^2(t)}{\hbar} +$$

$$+ \frac{m}{2\hbar}[\frac{\ddot{\delta}(t)}{\delta(t)} - \frac{\dot{\delta}^2(t)}{\delta^2(t)}] \times [x - q(t)]^2 - \frac{m\dot{q}(t)}{\hbar}[\frac{\dot{\delta}(t)}{\delta(t)} - k] \times [x - q(t)]. \qquad (2.1.3.19)$$

Considering the eqs. (2.1.2.1a) and (2.1.3.1b), let us write V_{qu} given by eq. (2.1.2.1e) in terms of potencies of [x - q(t)]. Before, we calculate the following derivations:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}\left([2\pi\delta^2(t)]^{-1/4} \times \exp\left\{-\frac{[x - q(t)]^2}{4\delta^2(t)}\right\}\right) =$$

$$= [2\pi\delta^2(t)]^{-1/4} \times \exp\left\{-\frac{[x - q(t)]^2}{4\delta^2(t)}\right\} \times \frac{\partial}{\partial x}\left\{-\frac{[x - q(t)]^2}{4\delta^2(t)}\right\} \rightarrow$$

$$\frac{\partial \varphi}{\partial x} = - \ [2\pi\delta^2(t)]^{-1/4} \times \exp\left\{-\frac{[x-q(t)]^2}{4\delta^2(t)}\right\} \times \left\{\frac{[x-q(t)]}{2\delta^2(t)}\right\} = -\varphi \times \left\{\frac{[x-q(t)]}{2\delta^2(t)}\right\},$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left\{-\varphi \times \frac{[x-q(t)]}{2\delta^2(t)}\right\} =$$

$$= -\varphi \times \frac{\partial}{\partial x}\left\{\frac{[x-q(t)]}{2\delta^2(t)}\right\} - \frac{[x-q(t)]}{2\delta^2(t)} \times \frac{\partial \varphi}{\partial x} =$$

$$= \varphi \times \frac{1}{2\delta^2(t)} + \frac{[x-q(t)]^2}{4\delta^4(t)} \times \varphi \rightarrow$$

$$\frac{1}{\varphi}\frac{\partial^2 \varphi}{\partial x^2} = \frac{[x-q(t)]^2}{4\delta^4(t)} - \frac{1}{2\delta^2(t)} . \qquad (2.1.3.20)$$

Substituting the eq. (2.1.3.20) in the equation (2.1.2.1e), taking into account the eqs. (2.1.3.16), and considering the identity of polynomial, results:

$$V_{qu}(x,t) = -\frac{\hbar^2}{2m}\left\{\frac{[x-q(t)]^2}{4\delta^4(t)} - \frac{1}{2\delta^2(t)}\right\} =$$

$$= V_{qu}[q(t),t] + V'_{qu}[q(t),t] \times [x-q(t)] + \frac{1}{2}V''_{qu}[q(t),t] \times [x-q(t)]^2 \rightarrow$$

$$V_{qu}[q(t),t] = \frac{\hbar^2}{4m\delta^2(t)}, \qquad V'_{qu}[q(t),t] = 0, \qquad V''_{qu}[q(t),t] = -\frac{\hbar^2}{4m\delta^4(t)} \rightarrow$$

$$V_{qu}[q(t),t] = \frac{\hbar^2}{4m\delta^2(t)} - \frac{\hbar^2}{8m\delta^4(t)} \times [x-q(t)]^2 . \qquad (2.1.3.21)$$

Using the eqs. (2.1.1.5) and (2.1.2.1c,e), we have:

$$-\hbar\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\varphi}\frac{\partial^2 \varphi}{\partial x^2} + \frac{m}{2}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right)^2 + V(x,t) \rightarrow$$

$$\hbar\frac{\partial S}{\partial t} + \frac{m}{2}v_{qu}^2 + V(x,t) + V_{qu}(x,t) = 0 . \qquad (2.1.3.22)$$

Inserting the eqs. (2.1.3.13,15,19,21) into eq. (2.1.3.22), we obtain:

$$\hbar\{\dot{S}_0(t) + \frac{m\ddot{q}(t)}{\hbar} \times [x-q(t)] - \frac{m\dot{q}^2(t)}{\hbar} + \frac{m}{2\hbar}[\frac{\ddot{\delta}(t)}{\delta(t)} - \frac{\dot{\delta}^2(t)}{\delta^2(t)}] \times [x-q(t)]^2 \ -$$

$$- \frac{m\dot{q}(t)}{\hbar}[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times [x - q(t)]\} + \frac{m}{2}\{[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times [x - q(t)] + \dot{q}(t)\}^2 \ +$$

$$+ \ V[q(t),t] + V'[q(t),t] \times [x - q(t)] + \frac{V''[q(t),t]}{2} \times [x - q(t)]^2 \ +$$

$$+ \frac{\hbar^2}{4m\delta^2(t)} - \frac{\hbar^2}{8m\delta^4(t)} \times [x - q(t)]^2 . \qquad (2.1.3.23)$$

Since $(x - q)^o = 1$, we can gather together the eq. (2.1.3.22) in potencies of the $(x - q)$, obtaining:

$$\{\hbar\dot{S}_0(t) - \frac{1}{2}m\dot{q}^2(t) + V[q(t),t] + \frac{\hbar^2}{4m\delta^2(t)}\} \times [x - q(t)]^0 \ +$$

$$+ \ m\ddot{q}(t) + V''[q(t),t] \times [x - q(t)]^1 \ +$$

$$+ \ \{\frac{m}{2}[\frac{\ddot{\delta}(t)}{\delta(t)} - 2\kappa\frac{\dot{\delta}(t)}{\delta(t)} + \kappa^2] \ +$$

$$+ \ \frac{1}{2}V''[q(t),t] - \frac{\hbar^2}{8m\delta^4(t)}\} \times [x - q(t)]^2 = 0. \qquad (2.1.3.24)$$

As the above relation is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_0(t) = \frac{1}{\hbar}\{\frac{1}{2}m\dot{q}^2(t) + V[q(t),t] + \frac{\hbar^2}{4m\delta^2(t)}\}, \qquad (2.1.3.25)$$

$$\ddot{\delta}(t) - 2\kappa\dot{\delta}(t) + \kappa^2\delta(t) + \frac{1}{m}V''[q(t),t] = \frac{\hbar^2}{4m^2\delta^3(t)} \ , \qquad (2.1.3.26)$$

$$\ddot{q}(t) + \frac{1}{m}V'[q(t),t] = 0. \qquad (2.1.3.27)$$

Assuming that the following initial conditions are obeyed:

$$q(0) = x_0 \ , \ \ \dot{q}(0) = v_0 \ , \ \ \delta(0) = \delta_0 \, , \ \ \dot{\delta}(0) = b_0 \ , \qquad (2.1.3.28a\text{-}d)$$

and that [see eqs.(2.1.2.1c,d) and (2.1.3.17b)]:

$$S_0(0) = \frac{mv_0 x_0}{\hbar} \ , \qquad (2.1.3.29)$$

the integration of the expression (2.1.3.25) will be given by:

$$S_0(t) = \frac{1}{\hbar}\int_0^t dt' \{\frac{1}{2}m\dot{q}^2(t') - V[q(t'),t'] - \frac{\hbar^2}{4m\delta^2(t')}\} + \frac{mv_0 x_0}{\hbar} \ . \qquad (2.1.3.30)$$

Taking into account the expressions (2.1.3.18a,b) in the equation (2.1.3.30) results:

$$S(x,t) = \frac{1}{\hbar}\int_0^t dt' \{\frac{m\dot{q}^2(t')}{2} - V[q(t'),t'] - \frac{\hbar^2}{4m\delta^2(t')}\} \ +$$

$$+ \ \frac{mv_0 x_0}{\hbar} + \frac{m\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\hbar}[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times [x - q(t)]^2 \ . \qquad (2.1.3.31)$$

This result obtained above permit us, finally, to obtain the wave packet for the *LNLS*–*NE*. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b) and (2.1.3.31), we get:

$$\Psi(x,t) = [2\pi\delta^2(t)]^{-1/4} \times \exp\{(\frac{im}{2\hbar}[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] - \frac{1}{4\delta^2(t)}) \times [x - q(t)]^2\} \times$$

$$\times \exp\{\frac{im\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{imv_0 x_0}{\hbar}\} \times$$

$$\times \exp\{\frac{i}{\hbar}\int_0^t dt' [\frac{m\dot{q}^2(t)}{2} - V[q(t'),t'] - \frac{\hbar^2}{4m\delta^2(t')}]\} . \qquad (2.1.3.32)$$

2.1.4. Calculation of the Feynman Propagator for the Logarithmic Nonlinear Schrödinger-Nassar Equation Linearized along a Classical Trajetory

The looked for Feynman-de Broglie-Bohm propagator for the Logarithmic Nonlinear Schrödinger-Nassar Equation (*LNLS*–*NE*) linearized along a classical trajectory, will be calculated using the eqs. (1.3) and (2.1.3.24). However, in the eq. (1.3), we will put with no loss of generality, t_o = 0. Thus: (Bassalo et al., 2002)

$$\Psi(x,t) = \int_{-\infty}^{+\infty} K(x,x_0;t,t_0)\Psi(x_0,0)dx_0 \ . \qquad (2.1.4.1)$$

Initially let us define the normalized quantity:

$$\Phi(v_0,x,t) = \left(2\pi\delta_0^2\right)^{1/4}\Psi(v_0,x,t) \ , \qquad (2.1.4.2)$$

which satisfies the following *completeness relation*: (Bernstein,1985)

$$\int_{-\infty}^{+\infty} dv_0 \Phi^*(v_0, x, t)\Phi(v_0, x', t) = \frac{2\pi\hbar}{m}\delta(x - x') \ . \qquad (2.1.4.3)$$

Considering the eqs. (2.1.1.1), (2.1.2.1a,b) and (2.1.4.2,3), we get:

$$\Psi^*(x,t) \times \Psi(x,t) = \varphi^2(x,t) = \rho(x,t) \ , \qquad (2.1.4.4)$$

$$\Phi^*(v_0, x, t)\Psi(v_0, x, t) \ =$$

$$= \ (2\pi\delta_0^2)^{1/4}\,\Psi^*(v_0, x, t)\Psi(v_0, x, t) = (2\pi\delta_0^2)^{1/4}\,\rho(v_0, x, t) \qquad \rightarrow$$

$$\rho(v_0, x, t) = (2\pi\delta_0^2)^{-1/4}\,\Phi^*(v_0, x, t)\Psi(v_0, x, t) \ . \qquad (2.1.4.5)$$

On the other side, substituting the eq. (2.1.4.5) into eq. (2.1.3.9), integrating the result and using the expressions (2.1.3.1a) and (2.1.4.2) results [remembering that $\frac{\partial}{\partial x}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}$ and $\Psi^* \, \Psi(\pm\infty) \rightarrow 0$, and the integration for parts]:

$$\frac{\partial(\Phi^*\Psi)}{\partial t} + \frac{\partial(\Phi^*\Psi\,\vartheta_{qu})}{\partial x} \ = \ 0 \qquad \rightarrow$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}(\Phi^*\Psi)dx + \int_{-\infty}^{+\infty}\frac{\partial(\Phi\,\Psi\vartheta_{qu})}{\partial x}dx \quad =$$

$$= \ \frac{\partial}{\partial t}\int_{-\infty}^{+\infty}(\Phi^*\Psi)dx + (\Phi^*\Psi\,\vartheta_{qu}) \Big|_{-\infty}^{+\infty} \ = \ \frac{\partial}{\partial t}\int_{-\infty}^{+\infty}(\Phi^*\Psi)dx + (2\pi\delta_0^2)^{1/4}(\Psi^*\Psi\,\vartheta_{qu}) \Big|_{-\infty}^{+\infty} \qquad \rightarrow$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}(\Phi^*\Psi)dx \ = 0 \ . \qquad (2.1.4.6)$$

The eq. (2.1.4.6) shows that the integration is time independent. Consequently:

$$\int_{-\infty}^{+\infty} dx' \Phi^*(v_0, x', t)\Psi(x', t) = \int_{-\infty}^{+\infty} dx_0 \Phi^*(v_0, x_0, t)\Psi(x_0, t) \ . \qquad (2.1.4.7)$$

Multiplying the eq. (2.1.4.7) by $\Phi(v_o, x, t)$ and integrating over v_o and using the eq. (2.1.4.3), we will obtain [remembering that $\int_{-\infty}^{+\infty} dx' f(x')\delta(x'-x) = f(x)$]:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dv_0 dx' \Phi(v_0, x, t)\Phi^*(v_0, x', t)\Psi(x', t) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dv_0 dx_0 \Phi(v_0, x, t)\Phi^*(v_0, x_0, 0)\Psi(x_0, 0) \to$$

$$\int_{-\infty}^{+\infty} dx' \left(\frac{2\pi\hbar}{m}\right)\delta(x' - x)\Psi(x', t) = \left(\frac{2\pi\hbar}{m}\right)\Psi(x, t) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dv_0 dx_0 \Phi(v_0, x, t)\Phi^*(v_0, x_0, 0)\Psi(x_0, 0) \quad \to$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty}\left\{\left(\frac{2\pi\hbar}{m}\right)\int_{-\infty}^{+\infty} dv_0 \Phi(v_0, x, t)\Phi^*(v_0, x_0, 0)\right\} \times \Psi(dx_0, 0)dx_0 \ . \ (2.1.4.8)$$

Comparing the eqs. (2.1.4.1,8), we have:

$$K(x, x_0, t) = \frac{m}{2\pi\hbar}\int_{-\infty}^{+\infty} dv_0 \Phi(v_0, x, t)\Phi^*(v_0, x_0, 0) \ . \qquad (2.1.4.9)$$

Substituting the eqs. (2.1.3.31) and (2.1.4.2) in the equation (2.1.4.9), we obtain the Feynman Propagator for the *LNLS−NE* linearized along a classical trajetory, that we were looking for, that is [remembering that $\Phi^*(v_0, x_0, t) = \exp\left(-\frac{imv_0 x_0}{\hbar}\right)$]:

$$K(x, x_0, t) = \frac{m}{2\pi\hbar}\int_{-\infty}^{+\infty} dv_0 \sqrt{\frac{\delta_0}{\delta(t)}} \times$$

$$\times \exp\left\{\left(\frac{im}{2\hbar}\left[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa\right] - \frac{1}{4\delta^2(t)}\right) \times [x - q(t)]^2\right\} \times$$

$$\times \exp\left\{i\frac{m\dot{q}(t)}{\hbar} \times [x - q(t)]\right\} \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_0^t dt' \left(\frac{m\dot{q}^2(t')}{2} - V[q(t'), t'] - \frac{\hbar^2}{4m\delta^2(t')}\right)\right\}, \qquad (2.1.4.10)$$

where $q(t)$ and $\delta(t)$ are solutions of the differential equations given by the eqs.(2.1.3.25,26).

Finally, it is important to note that putting $\kappa = 0$ and $V[q(t'), t'] = 0$ into eq. (2.1.4.10) and eqs. (2.1.3.25,26) we obtain the free Feynman propagator. (Feynman and Hibbs, 1965, Bassalo et al., 2002)

3. Ermakov-Lewis Invariants

Many years ago, in 1967, H. R. Lewis (Lewis, 1967) has shown that there is a conserved quantity, that will be indicated by *I*, associated with the time dependent harmonic oscillator (*TDHO*) with frequency ω(*t*), given by:

$$\mathrm{I} = \frac{1}{2}(\dot{\alpha}q - \dot{q}\alpha)^2 + (\frac{q}{\alpha})^2 , \qquad (3.1)$$

where *q* and α obey, respectively the equations:

$$\ddot{q} + \omega^2(t)q \ = 0 \ , \quad \ddot{\alpha} + \omega^2(t)\alpha = \frac{1}{\alpha^3} \ . \qquad (3.2,3)$$

On the other hand, as the above expressions have also been obtained by V. P. Ermakov (Ermakov, 1880), the invariants determination of time dependent physical systems is also known as the *Ermakov-Lewis problem*. So, considerable efforts have been devoted to solve this problem and its generalizations, in the last forty years, and in many works have been published on these subjects (Nassar, 1986b-d).

3.1. The Ermakov-Lewis Invariants for the Logarithmic Nonlinear Schrödinger-Nassar Equation

Now, let us investigate the existence (or not) of these invariants for the Logarithmic Nonlinear Schrödinger-Nassar Equation (*LNLS−NE*) with the potential *V*(*x*, *t*) given by:

$$V(x,t) = \frac{1}{2}m\omega^2(t)x^2 , \qquad (3.1.1)$$

which is the Time Dependent Harmonic Oscillator Potential (*TDHOP*).

Taking the eq. (2.1.2.3) and considering the eq. (3.1.1), results:

$$\frac{\partial v_{qu}(x,t)}{\partial t} + v_{qu}(x,t)\frac{\partial v_{qu}(x,t)}{\partial x} + \omega^2(t)x = -\frac{1}{m}\frac{\partial V_{qu}(x,t)}{\partial x} \ . \qquad (3.1.2)$$

In order to integrate the eq. (3.1.2) let us assume that the expected value of ***quantum force***[$\frac{\partial V_{qu}(x,t)}{\partial x}$] is equal to zero for all times *t*, that is:

$$< \frac{\partial V_{qu}(x,t)}{\partial x} > . \to 0 \quad \leftrightarrow \quad \frac{\partial V_{qu}}{\partial x} \Big|_{x=q(t)} , \; <x> = q(t) \, . \qquad (3.1.3a\text{-}c)$$

In this way, using the eq. (2.1.2.1f), we can write the eq. (3.1.2) into two parts:

$$\frac{\partial v_{qu}(x,t)}{\partial t} + v_{qu}(x,t)\frac{\partial v_{qu}(x,t)}{\partial x} + \omega^2(t)x = k(t)\times[x - q(t)], \quad (3.1.4)$$

$$\frac{\partial}{\partial x}\left[\frac{\hbar^2}{2m^2}\frac{1}{\sqrt{\rho(x,t)}}\frac{\partial^2\sqrt{\rho(x,t)}}{\partial x^2}\right] = k(t)\times[x - q(t)]. \qquad (3.1.5)$$

Performing the differentiations indicated in the eq. (3.1.5) we get:

$$\frac{\hbar^2}{4m^2}\{\frac{1}{\rho(x,t)}\frac{\partial^3\rho(x,t)}{\partial x^3} - \frac{2}{\rho^2(x,t)}\frac{\partial\rho(x,t)}{\partial x}\frac{\partial^2\rho(x,t)}{\partial x^2} +$$

$$+ \frac{1}{\rho^3(x,t)}[\frac{\partial\rho(x,t)}{\partial x}]^3\} = k(t)\times[x - q(t)]. \qquad (3.1.6)$$

To integrate the eq. (3.1.6) it is necessary to known the initial condition for $\rho(x,t)$. (Bassalo et al., 2002) Let us assume that for t = 0 the physical system is represented by a normalized Gaussian wave packet, centered at $q(0)$, that is:

$$\rho(x,0) = [2\pi\delta^2(0)]^{-1/2}\exp\{-\frac{[x - q(0)]^2}{2\delta^2(0)}\} = \frac{1}{\sqrt{A}}\exp(-\frac{B^2}{C}), \qquad (3.1.7)$$

where:

$$A = 2\pi\delta^2(0), B = x - q(0), C = 2\delta^2(0) \; . \qquad (3.1.8\text{-}10)$$

Since the eq. (3.1.7) is a particular solution of the eq. (3.1.6), we must have:

$$\frac{\hbar^2}{4m^2}\{\frac{1}{\rho(x,0)}\frac{\partial^3\rho(x,0)}{\partial x^3} - \frac{2}{\rho^2(x,0)}\frac{\partial\rho(x,0)}{\partial x}\frac{\partial^2\rho(x,0)}{\partial x^2} +$$

$$+\ \frac{1}{\rho^3(x,0)}[\frac{\partial\rho(x,0)}{\partial x}]^3\} = k(0)\times[x-q(0)]. \qquad (3.1.11)$$

Making the differentiation indicated in the eq. (3.1.11), results: (Bassalo et al., 2002)

$$\frac{\hbar^2}{4m^2\delta^4(0)}\times[x-q(0)] = \kappa(0)\times[x-q(0)] \rightarrow$$

$$\kappa(0) = \frac{\hbar^2}{4m^2\delta^4(0)}\ , \qquad (3.1.12)$$

and:

$$\delta^4(0) = \frac{\hbar^2}{4m^2k(0)}. \qquad (3.1.13)$$

Comparing the eqs. (3.1.12,13) with the eqs. (3.1.6,7), by analogy we get:

$$k(t) = \frac{\hbar^2}{4m^2\delta^4(t)}\ , \qquad (3.1.14)$$

$$\rho(x,t) = [2\pi\delta^2(t)]^{-1/2}\exp\{-\frac{[x-q(t)]^2}{2\delta^2(t)}\}, \qquad (3.1.15)$$

and:

$$\delta^4(t) = \frac{\hbar^2}{4mk(t)}\ . \qquad (3.1.16)$$

Using the eq. (2.1.3.13) we calculate the following differentiations (remembering that *t* and *x* as independent variables):

$$\frac{\partial v_{qu}(x,t)}{\partial t} = \frac{\partial}{\partial t}\{[\frac{\dot{\delta}(t)}{\delta(t)}-\kappa]\times[x-q(t)]+\dot{q}(t)\}\ =$$

$$= [\frac{\ddot{\delta}(t)}{\delta(t)}-\frac{\dot{\delta}^2(t)}{\delta^2(t)}]\times[x-q(t)]-[\frac{\dot{\delta}(t)}{\delta(t)}-\kappa]\times\dot{q}(t)+\ddot{q}(t), \quad (3.1.17)$$

$$\frac{\partial v_{qu}(x,t)}{\partial x} = \frac{\partial}{\partial x}\{[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times [x - q(t)] + \dot{q}(t)\} = \frac{\dot{\delta}(t)}{\delta(t)} - \kappa\,. \qquad (3.1.18)$$

Putting the eqs. (2.1.3.13) and (3.1.17,18) into the eq. (3.1.4), considering the eq. (3.1.14), adding and subtracting the term $\omega^2(t)\ q(t)$, results: (Bassalo et al., 2002)

$$[\frac{\ddot{\delta}(t)}{\delta(t)} - \frac{\dot{\delta}^2(t)}{\delta^2(t)}] \times [x - q(t)] - [\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times \dot{q}(t) + \ddot{q}(t) +$$

$$+\ \{[\frac{\dot{\delta}(t)}{\delta(t)} - \kappa] \times [x - q(t)] + \dot{q}(t)\} \times [\frac{\dot{\delta}(t)}{\delta(t)} - \kappa]\ +$$

$$+ \omega^2(t) \times [x - q(t)] + \omega^2(t) \times q(t) = \frac{\hbar^2}{4m^2\delta^4(t)} = 0 \rightarrow$$

$$[\frac{\ddot{\delta}(t)}{\delta(t)} - 2\frac{\dot{\delta}(t)}{\delta(t)}\kappa + \kappa^2 + \omega^2(t) - \frac{\hbar^2}{4m^2\delta^4(t)}] \times [x - q(t)]^1\ +$$

$$+\ [\ddot{q}(t) + \omega^2(t)q(t)] \times [x - q(t)]^0\ = 0\,. \qquad (3.1.19)$$

To satisfy eq. (3.1.19), the following conditions must be obeyed:

$$\frac{\ddot{\delta}(t)}{\delta(t)} - 2\frac{\dot{\delta}(t)}{\delta(t)}\kappa + \kappa^2 + \omega^2(t) - \frac{\hbar^2}{4m^2\delta^4(t)}\ = 0 \quad \rightarrow$$

$$\ddot{\delta}(t) - 2\delta(t)\kappa + [\kappa^2 + \omega^2(t)]\delta(t) = \frac{\hbar^2}{4m^2\delta^3(t)}, \quad (3.1.20)$$

$$\ddot{q}(t) + \omega^2(t)q(t)\ = 0\,. \qquad (3.1.21)$$

Putting:

$$\delta(t) = \left(\frac{\hbar^2}{4m^2}\right)^{1/4} \alpha(t)\ , \qquad (3.1.22)$$

we obtain:

$$\dot{\delta}(t) = (\frac{\hbar^2}{4m^2})^{1/4}\dot{\alpha}(t), \qquad \ddot{\delta}(t) = (\frac{\hbar^2}{4m^2})^{1/4}\ddot{\alpha}(t) \ . \qquad (3.1.23,24)$$

Inserting the eqs. (3.1.22-24) into the eq. (3.1.20) and multiplyng the result by α, we get: (Bassalo et al., 2002)

$$\ddot{\alpha} - 2\dot{\alpha}\kappa + (\omega^2 + \kappa^2)\alpha = \frac{1}{\alpha^3} \ . \qquad (3.1.25)$$

Finally, eliminating the factor ω^2 into the eqs. (3.1.20,21), we get:

$$\ddot{\alpha} - 2\dot{\alpha}\kappa + (\omega^2 + \kappa^2)\alpha = \frac{1}{\alpha^3} \quad \rightarrow \quad \ddot{\alpha}q - \ddot{q}\alpha - 2\dot{\alpha}\kappa q + \kappa^2\alpha q = \frac{q}{\alpha^3} \quad \rightarrow$$

$$\frac{d}{dt}(\dot{\alpha}q - \dot{q}\alpha) = \frac{q}{\alpha^3} + 2\dot{\alpha}\kappa q - \kappa^2\alpha q \quad \rightarrow$$

$$(\dot{\alpha}q - \dot{q}\alpha)\frac{d}{dt}(\dot{\alpha}q - \dot{q}\alpha) = (\dot{\alpha}q - \dot{q}\alpha)\frac{q}{\alpha^3} + (\dot{\alpha}q - \dot{q}\alpha)(2\alpha\kappa q - \kappa^2\alpha q) \quad \rightarrow$$

$$\frac{d}{dt}[\frac{1}{2}(\dot{\alpha}q - \dot{q}\alpha)^2] + \frac{d}{dt}[\frac{1}{2}(\frac{q}{\alpha})^2] = -(2\dot{\alpha}\kappa q - \kappa^2\alpha q)\alpha^2\frac{d}{dt}(\frac{q}{\alpha}) \quad \rightarrow$$

$$\frac{d}{dt}\{[\frac{1}{2}(\dot{\alpha}q - \dot{q}\alpha)^2 + (\frac{q}{\alpha})^2]\} = -(2\dot{\alpha}\kappa - \kappa^2\alpha)q\alpha^2\frac{d}{dt}(\frac{q}{\alpha}) \quad \rightarrow$$

$$\frac{dI}{dt} = -(2\dot{\alpha}\kappa - \kappa^2\alpha)q\alpha^2\frac{d}{dt}(\frac{q}{\alpha}), \qquad (3.1.36)$$

where [see eq. (3.1)]:

$$\mathrm{I} = \frac{1}{2}(\dot{\alpha}q - \dot{q}\alpha)^2 + (\frac{q}{\alpha})^2, \quad (3.1.37)$$

which represents the *Ermakov-Lewis Invariant* (*ELI*) of the *TDHO*. (Bassalo et al., 2002)

In conclusion, the eq. (3.1.36) we have shown that the *LNLS−NE* <u>has not</u> an *ELI* for the *TDHO*.

4. The Bohmian Trajectories for the Logarithmic Nonlinear Schrödinger-Nassar Equation

The associated Bohmian Trajectories (Nassar, 2013, Sanz, 2000, Pan, 2010, Holland, 2005, Wyatt, 2005) for the Logarithmic Nonlinear Schrödinger-Nassar Equation (*LNLS−NE*) of an evolving *ith* particle of the ensemble with an initial position x_{0i} can be calculated by considering that:

$$\dot{x}_i(t) = v_{qu}[x_i(t), t]. \qquad (4.1)$$

Then substituting the eq. (4.1) into eq. (2.1.3.13), results:

$$\dot{x}_i(t) = [\frac{\dot{\delta}(t)}{\delta(t)} - k] \times [x_i(t) - q(t)] + \dot{q}(t) \quad \rightarrow$$

$$\dot{x}_i(t) - \dot{q}(t) = [\frac{\dot{\delta}(t)}{\delta(t)} - k] \times [x_i(t) - q(t)] \quad \rightarrow \quad \frac{\dot{x}_t(t) - \dot{q}(t)}{x_i(t) - q(t)} = \frac{\dot{\delta}(t)}{\delta(t)} - \kappa \quad \rightarrow$$

$$\int_0^t \frac{d}{dt}\{\ell n[x_i(t) - q(t)]\}dt = \int_0^t \frac{d}{dt}\{\ell n[\delta(t)])]\}dt - \int_0^t \kappa dt \rightarrow$$

$$\ell n\frac{[x_i(t) - q(t)]}{[x_{0i} - q_0]} = \ell n\frac{\delta(t)}{\delta_0} - \kappa t \quad \rightarrow \quad \frac{[x_i(t) - q(t)]}{[x_{0i} - q_0]} = \frac{\delta(t)}{\delta_0} - \exp(\kappa t) \quad \rightarrow$$

$$x_i(t) = q(t) + (x_{0i} - q_0) \times \frac{\delta(t)}{\delta(0)} \times \exp(-\kappa t) \ . \qquad (4.2)$$

The eqs. (3.1.20,21) show that a continuous measurement of a quantum wave packet gives specific features to its evolution: the appearance of distinct classical and quantum elements, respectively. This measurement consists of monitoring the position of the quantum systems and the result is the measured classical path $q(t)$ for t within a quantum uncertainty $\delta(t)$.

4.1 The Bohmian Trajectories for the Logarithmic Nonlinear Schrödinger-Nassar Equation in Stationary Regime

From the eqs. (3.1.20,21), we note that for $\kappa \neq 0$ a stationary regime can be reached and that the width of the wave packet can be related to the resolution of measurement as follows. Then, considering that $\delta(t) = cte$ $[\dot{\delta}(t) = 0]$ in the eqs. (3.1.20,21), we have:

$$[\kappa^2 + \omega_0^2]\delta_0 = \frac{\hbar^2}{4m^2\delta_0^3} \to \kappa^2 = \frac{1}{\tau_B^2} - \omega_0^2 \ , \qquad (4.3a)$$

where[4]:

$$\tau_B = (\frac{2m\delta_0^2}{\hbar}) = 6{,}8 \times 10^{-26} s \ , \qquad (4.3b)$$

is the *Bohmtime constant* which determines the time resolution of the quantum measurement, and:

$$\dot{q}(t) + \omega_0^2 q(t) = 0 \qquad \to \qquad q(t) = q_0 \exp(\pm i\omega_0 t) \ . \qquad (4.4)$$

The eqs. (4.3a,b) means that if an initially free wave packet is kept under a certain continuous measurement, its width (δ_0) may not spread in time. Then, the associated *Bohmnian Trajectories* (BT) [eq. (4.2)] of an evolving *ith* particle of the ensemble with an initial position x_{0i} is giving by:

$$x_i(t) = q_0 \times \exp(\pm i\omega_0 t) + (x_{0i} - q_0) \times \exp(-kt) \ . \qquad (4.5)$$

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