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Publicação IF 1690
28/07/2015

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Deterministic Chaos Theory: Basic Concepts.

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Abstract. This article was written to graduate students of Physics and Engineering. In general, **Chaos** may refer to any state of confusion or disorder and it may also refer to, for instance, mythology, philosophy and religion. In science and mathematics it is sometimes understood as irregular behavior.In this article we analyze the **Deterministic Chaos Theory**, which is a branch of mathematics and physics that deals with dynamical systems (nonlinear differential equations or mappings) with very peculiar properties. Fundamental concepts of the deterministic chaos theory are briefly analyzed. We studied in details only the chaotic motion of a damped and driven pendulum around. Some illustrative examples of conservative and dissipative chaotic motions are mentioned. Relations between chaotic, stochastic and turbulent phenomena are also commented.
Key words: chaos theory; differential equations; Poincaré sections; mapping; Lyapunov exponent.

(I) Introduction.

This paper was written for graduate students of Physics and Engineering. Are briefly analyzed essential aspects of the growing field of mathematics and physics that is been applied to study a large number of phenomena generically named "**chaotic**". These are present in many areas in science and engineering,[1-3] including astronomy, plasma physics, statistical physics, astronomy, hydrodynamics and biology. As in Greek the word "chaos" (χάος) means "confusion", random, stochastic and turbulent ptocesses are interpreted as "chaotic". However, rigorously they are different in the framework of Physics and Mathematics, as will be shown. This article written for graduate students analyzes only the basic points of chaos theory, as exactly as possible from the mathematical point of view, avoiding sometimes a rigorous approach. In Section 1 we define **"chaos"** also known as "**deterministic chaos theory**" as a consequence of peculiar properties of deterministic nonlinear ordinary differential equations (**NLODE**). These equations that describe dynamic systems have a time evolution strongly dependent on initial conditions. Chaotic motion occurs depending of initial conditions and parameters values of the nonlinear equations. In Section 2 is seen the difference between chaotic and stochastic (or random) processes. In Section 3 to give a general idea about the chaos we study in details the dissipative motion of a damped and driven pendulum introducing the Poincaré technique. In Section 4 we show that it is possible to get a good description of the chaotic process using an iterative **algebraic model** named **mapping.** With this model we study the *logistic equation* and *logistic map*. In Section 5 is presented the method created by Lyapunov, known as *Lyapunov characteristic exponent*, that is used to quantify the sensitive dependence on initial conditions for chaotic behavior. Finally, in Section 6 is briefly analyzed the open problem: "is turbulence a chaotic process"?

(1) Definition of Deterministic Systems and Chaos.

Usually in physics basics courses [1,3-6] we learn that all physical laws are described by **differential equations**. So, **integrating**, that is, solving analytically or numerically, these equations knowing the initial and boundary conditions(see Section 3) we would know the future of a physical system for all times. This is the **deterministic** view of nature. In other words, physics systems are deterministic because they obey deterministic differential equations. They can be conservative or dissipative. Remark that the deterministic development refers to the way as a system develops from one moment to the next, where the present system depends on the one just past in a well-determined way through physical laws.[1,3-6] If the initial states of deterministic systems were exactly known, future states could be theoretically predicted.

The deterministic theory survived till the 19.th beginning to be questioned after the famous visionary works of Henri Poincaré on celestial mechanics [7] performed at the end of the 19.th These works begin in 1880 when he found that can exist non-periodic orbits in the "three-body problem".

According to Poincaré [7,8] : "If we knew exactly the law of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation **approximately**. If it enabled us to predict the succeeding situation **with the same approximation**, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so: it may happen that **small differences in the initial conditions produce very great ones in the final phenomena**. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon."

In practice, as observed for many systems, knowledge about the future state is limited by the precision with which the initial state can be measured. That is, knowing the laws of nature is not enough to predict the future. There are **deterministic systems** whose time evolution has a very strong dependence on initial conditions. That is, the differential equations that govern the evolution of the system are very sensitive to initial conditions. Usually we say that "even a tiny effect, such as a butterfly flying nearby, may be enough to vary the conditions such that the future is entirely different than what it might have been, not just a tiny bit different".[1-3,9] In this way, measurements made on the state of a system at a given time may not allow us to predict the future situation even moderately far ahead, despite the fact that the governing equations are exactly known. By definition, these equations are named "**chaotic**" and that they predict a "**deterministic chaos**".

Only in recent years, with advent of computers that was allowed chaos to be studied because now it is possible to perform calculations of the time evolution of the properties of systems that include these tiny variations in the initial conditions. We begin to understand the existence of chaos when computers were readily available to calculate the long-time histories required to explain the behavior. It did not happen until the 1970s. After almost one century of investigations we learned that chaotic systems can *only be solved numerically*, and there are no simple, general ways to predict when a system will exhibit chaos.[1-3,9] We have also learned that **deterministic chaos** is always associated with **nonlinear systems; nonlinearity is a necessary condition for chaos but not a sufficient one.**

(2) Random or Stochastic Process.

According to Section 1 the deterministic model will always produce the same output from a given starting condition or initial state. On the other hand, a **random process,** sometimes called **stochastic process,** is a collection of random variables, representing the evolution of some system of random values over time.[10] Instead of describing a process which can only evolve in one way (as, for example, the solutions of an ordinary differential equation), in a stochastic process there is some indeterminacy: even if the initial condition is known, there are several (often infinitely many) directions in which the process may evolve. There is a probabilistic evolution of the initial states.

As an example, let us consider the Langevin [10,11] stochastic process. He proposed in 1908 the following **stochastic differential equation** to describe the Brownian (random) motion of a particle immersed in a fluid:[10,11]

$$m\frac{d^2\mathbf{x}}{dt^2} = -\lambda\frac{d\mathbf{x}}{dt} + \boldsymbol{\eta}\left(t\right).$$

The degree of freedom of interest here is the position $\boldsymbol{x}$ of the particle, m denotes the particle's mass. The force acting on the particle is written as a sum of a viscous force proportional to the particle's velocity (Stokes' law), and a *noise term* $\boldsymbol{\eta}(t)$ (the name given in physical contexts to terms in stochastic differential equations which are stochastic processes) representing the effect of the collisions with the molecules of the fluid. The force $\boldsymbol{\eta}(t)$ has a Gaussian probability distribution with correlation function

$$\left\langle \eta_i\left(t\right)\eta_j\left(t'\right)\right\rangle = 2\lambda k_B T\delta_{i,j}\delta\left(t-t'\right),$$

where k_B is Boltzmann's constant and T is the temperature. The δ-function form of the correlations in time means that the force at a time t is assumed to be completely uncorrelated with it at any other time. This is an approximation; the actual random force has a nonzero correlation time corresponding to the collision time of the molecules. However, Langevin´s equation is used to describe the motion of a "macroscopic" particle at a much longer time scale, and in this limit the δ-correlation and the Langevin equation become exact.

It can be difficult to tell from data whether a physical or other observed process is random or chaotic.[10,12]. In reference [3] one can see some procedures proposed to distinguish between deterministic chaos and stochastic behavior.

Finally, in Quantum Mechanics, the Schrödinger equation, which describes the continuous time evolution of a system's wave function, is **deterministic**.[13] However, the relationship between a system's wave function and the observable properties of the system appears to be **non-deterministic.**

(3) Deterministic Chaos.

According to Section 1, after ~130 years of investigations, it was verified that chaotic phenomenon is well explained when dynamic systems obey *nonlinear ordinary* differential equations or simply **NLODE** (see Appendix A). However, many papers have been published [14-16] investigating the existence of chaos for processes governed by ***partial*** differential equations [**PDE**](see Appendix B). Since this is an article is written to graduate students we avoid complex mathematical analysis required to elucidate this question we take for granted that chaos is only generated by **NLODE.** In this way, let us recall the definitions of **NLODE**. An *ordinary* differential equation is an equation containing a function of *one* independent variable and its derivatives.[17-19] The term ***ordinary*** is used in contrast with the term ***partial*** differential equation which may be with respect to *more than* one independent variable. Let x be an independent variable, $y = y(x)$ a function of x and $y^{(n)} = d^n y/dx^n$ the derivative of order n of the function y(x). An ordinary differential equation of order n can be generally written as $F(x,y,y',\ldots,y^{(n)}) = 0$. If x, y(x) and $y^{(n)}$ are linear functions and F is a linear function of these functions we say that F is an *ordinary linear* differential equation (**ODE).** When nonlinear terms are present, F is a *nonlinear ordinary* differential equation or **NLODE**. In Appendix B we show that **ODE** cannot explain the chaotic behavior.

In the N-dimensional case it is assumed that the time evolution of the dynamic of a system is described by continuous and continuous flux created by ordinary nonlinear differential equations

$$\mathbf{x}(t)/dt = \mathbf{f}_\alpha[\mathbf{x}(t)] \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_o \qquad (3.1),$$

$\mathbf{x}$, $\mathbf{f}_\alpha$ (*flow equation*) are N-vectors $\epsilon\ R^m$, m is the number of degrees of liberty of the system, $\mathbf{f}_\alpha$ is explicitly independent of time and α is a control parameter. Usually it is assumed that any **NLODE** can be integrated in the sense that they are **resolved** analytically or numerically and that the solutions obtained are **unique**. Note that rigorously in Mathematics, differential equations can be integrated [20,21] when are manifested the following features: (a) existence of many conserved quantities; (b) existence of an algebraic geometry and (c) ability to give explicit solutions. The **existence** and **uniqueness** of solutions of **NLODE** and **PDE** are commented in Appendix A and B.

To give a general idea about the chaos theory we study in details in Section (3.a) only the dissipative motion of a damped and driven pendulum. There are, however many illustrative examples of chaotic processes. We suggest the lecture of two conservative processes [22] solved with the Hamiltonian formalism. One is the motion of a particle of mass m in a double quartic non-harmonic potential (*Duffing potential*) governed by the *Duffing Hamiltonian*,

$$H(p,x,t) = p^2/2m - kx^2 + x^4 + \varepsilon x\cos(\omega_o t),$$

where the oscillating term $\varepsilon x\cos(\omega_o t)$ is a perturbative potential. A didactical approach of this case was done, for instance, by Bassalo and Cattani.[23] The second case is the conservative motion of a *double pendulum* seen, for instance, in reference [24] where are found animation pictures of the chaotic motion.

Another illustrative case is the motion of a particle with mass m submitted to a Duffing potential and to a dissipative force $\beta(dx/dt)$. That is, the motion is governed by the **NLODE** (*Duffing equation)*:[2,3]

$$md^2x/dt^2 = kx + 4x^3 - \beta(dx/dt) + \varepsilon\cos(\omega_o t).$$

Classical example is the *chaos in the solar system* (see, for instance, pag.99 of reference [3]).

(3.a) ***Chaos in damped and driven pendulum***.

In Chapters (4.1- 6), Marion [1] studies one-dimensional nonlinear motions to help the students to understand chaos. Following this author [1] we study in details the motion, found to be chaotic, of a damped and driven pendulum around its pivot point shown in Fig.1.

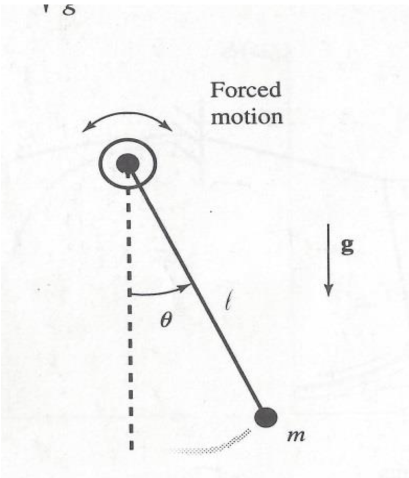

Figure 1. Damped and Driven pendulum with length ℓ.

The torque τ around the pivot point can be written as

$$\tau = I\, d^2\theta/dt^2 = -\, b\, d\theta/dt - mg\ell \sin\theta + N_d \cos(\omega_d t) \qquad (3.2),$$

where I is the moment of inertia, b the damping coefficient and N_d is the driving force of angular frequency ω_d. Dividing (3.2) by $I = m\ell^2$ results the nonlinear equation

$$d^2\theta/dt^2 = -\,(b/m\ell^2)\,(d\theta/dt) - (g/\ell)\sin\theta + (N_d/m\ell^2)\cos(\omega_d t) \qquad (3.3).$$

If we want to deal with equation (3.3) with a computer it is more convenient to use dimensionless parameters. So, let us divide (3.3) by $\omega_o^2 = g/\ell$ and define the dimensions less parameters : time $t' = t/t_o$ with $t_o = 1/\omega_o$ and driving frequency $\omega' = \omega_d/\omega_o$. The new *dimensionless* variables and parameters are

$x = \theta$ oscillating angle

$c = (b/m\ell^2\omega_o)$ damping coefficient

$F = (N_d/m\ell^2\omega_o) = (N_d/mg\ell)$ driving force strength (3.4)

$t' = t/t_o = t\sqrt{g/\ell}$ dimensionless time

$\omega = \omega_d/\omega_o = \omega_d\sqrt{\ell/g}$ driving angular frequency

Using the variables and parameters defined by (3.4) we verify that (3.3) becomes,

$$d^2x/dt'^2 = -c(dx/dt') - \sin(x) + F\cos(\omega t') \qquad (3.5).$$

Defining $y = dx/dt'$ and $z = \omega t'$, the second-order non-linear differential equation (3.5) is substituted by a system of two first order-differential equations:

$$y = dx/dt' \text{ (angular velocity)} \quad \text{and} \quad dy/dt' = -cy - \sin(x) + F\cos(z) \qquad (3.6).$$

The **NLODE** (3.5) can only be solved for x using numerical methods, given the parameters c, F and ω. This was done with a computer using a commercial software program. [1] He assumed that $c = 0.05$ and $\omega = 0.7$ and vary only the driving strength F in steps of 0.1 from 0.4 to 1.0. The motion in the phase space associated with (3.6) can be efficiently studied using the technique invented by Poincaré, named **Poincaré sections** illustrated in Figures 2 and 3**.** First is constructed a 3-dim phase space with orthogonal axis (x,y,z), where $x = \theta$, $y = dx/dt'$ and $z = \omega t'$ and second, are taken parallel planes (y,x) orthogonal to the axis z distant one of the other by a given **interval** Δz (see Fig.(2b)). These planes, or **Poincaré sections**, are used to drawn a **stroboscopic map** of the flux. This name is given because such map consists in observe the system in discrete times $t_k = 2\pi k/\omega$ $(k = 1,2,\ldots,n)$. Taking for $t = 0$ the initial values $x(0) = x_o$ and $y(0) = y_o$ we integrate numerically (3.5) up to the instant t_1 determining the point $A_1 = [x(t_1),y(t_1)]$ of the path. These values are now taken as new initial values to calculate the next point $A_2 = [x(t_2),y(t_2)]$ for t_2 and so on.. Note that the calculated path is a continuous curve. The calculated path of the pendulum in the phase space (x,y,z) pierces the planes (**stroboscopic sections**) as a function of angular speed ($y = d\theta/dt$), time ($z = \omega t'$) and the phase angle ($x = \theta$), according to Fig.(2a). The points on the intersections are labelled as A_1, A_2 and A_3, etc. This set of points A_i forms a pattern (**stroboscopic map)** when projected on the plane (y,x)(see Figs. (2.b) and 3). Poincaré realized that the simple curves represent motion with possibly analytic solutions, but the many complicated, apparently irregular, curves represent chaos.

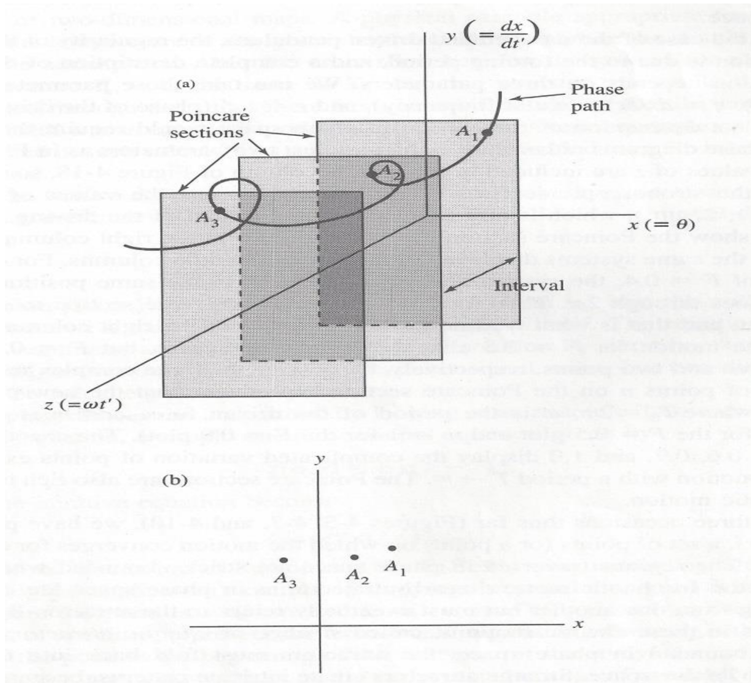

Figure 2. Technique invented by Poincaré to represent the phase space diagrams. The parallel planes are "**stroboscopic sections**" of the motion. The path pierces these planes at the points A_1, A_2, A_3,…[Fig.(2a)]. In Fig.(2b) are shown these points projected on the plane (x,y).

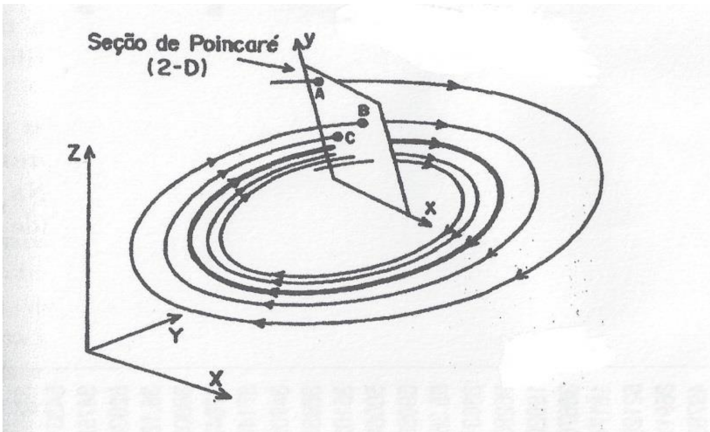

Figure 3. Illustration of the stroboscopic technique where are shown the intersections of the path with the Poincare´section.

Now let us analyze results seen in Figure 4. The *left side* displays y = dx/dt´ (angular velocity) versus time t´ when transient effects have died out. The value F = 0.4 shows a simple periodic harmonic motion (only one vibrational frequency), but the results for F = 0.5, 0.8 and 0.9, although periodic, are not so simple (few vibrational frequencies). This is also seen in the *middle column* of Fig.4 observing the phase-space plot versus the angle x**.**[6] These results indicate the beautiful and surprising behavior obtained from nonlinear dynamics: the motions are **periodic** for F = 0.4, 0.5, 0.8 and 0.9 but **chaotic** for F = 0.6, 07 and 1.0. The "black regions" are created by a very large density of different **x** values of the position occupied by the pendulum during the **chaotic** displacements.

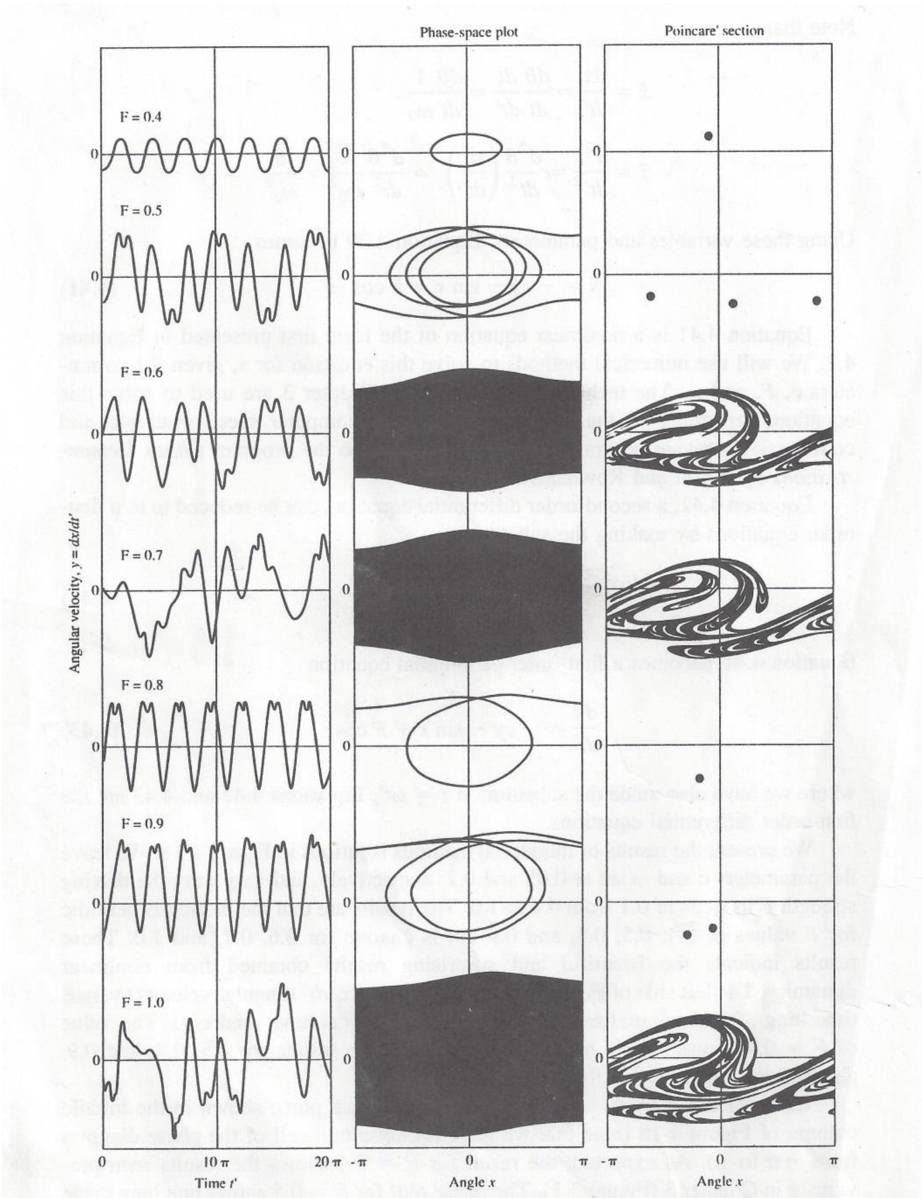

Figure 4. The damped and driven pendulum for various force values of the driving force F. The angular velocity y = (dx/dt´) versus time t´ is shown on the left. The *phase diagrams* y versus x are in the center. **Poincaré sections** are shown on the right. Note that motion is **chaotic** for driving force values 0.6, 0.7 and 1.0. The F = 0.4 shows simple harmonic motion. The results for F = 0.5, 0.8 and 0.9 although **periodic**, are hardly simple.

In the *right column* are displayed **stroboscopic maps**. For F = 0.4 and 0.8 the Poincaré sections shows only one point. There is a simple harmonic motion: the system always comes back to the same position (x,y) after z goes through 2π. For F = 0.5 and F = 0.9 there are 3 and 2 points, respectively, indicating more complex motions. The number of isolated points n shows that a new period $T = t_o n/m$ where m is an integer (m = 2 for the F = 0.5 plot and m = 1 for the F = 0.9 plot). The chaotic motions for F = 0.6, 0.7 and 1.0 present complicated variations of points expected for the chaotic motion with a period $T \rightarrow \infty$. In these cases we have aperiodic motions which is a characteristic of the deterministic chaos. Finally, we remark that only for dissipative systems there are set of points (**attractors)** or a point on which the motion converges. In chaotic motion, nearby trajectories in phase space are continually diverging from one another but must eventually return to the attractor. Due to these attractors, named **strange** or **chaotic attractors**, the motions in the phase space are necessarily bounded. The paths are continually diverging from one another but must eventually return to the attractor. The attractors create intricate patterns, folding and stretching the trajectories must occur because no trajectory intersects in the phase space, which is ruled out by deterministic dynamical motion. [6] The figures reveal a complex folded, layered structure of the attractors. Amplifying the figure we would note that the "lines" are really composed by a set of sublines. Amplifying a subline we would see another set of sublines and so on…verifying that the strange attractors usually are **fractals.**[3,25,26]

(4) Mapping.

In some cases it is very difficult to study the evolution of a nonlinear system integrating their differential equations. Sometimes it is also difficult to construct an exact nonlinear mathematical model to study physical system. In these cases it is possible to get a good description of the chaotic process using an iterative **algebraic model** named **mapping.** To understand the origin of this model let us assume that the motion of a system is described by nonlinear first-order differential equations of the form [9]

$$dx/dt = V(x) \qquad (4.1),$$

where x and V(x) are explicitly independent of time and that the motion is represented in Poincaré section Σ_R in Figure (5).

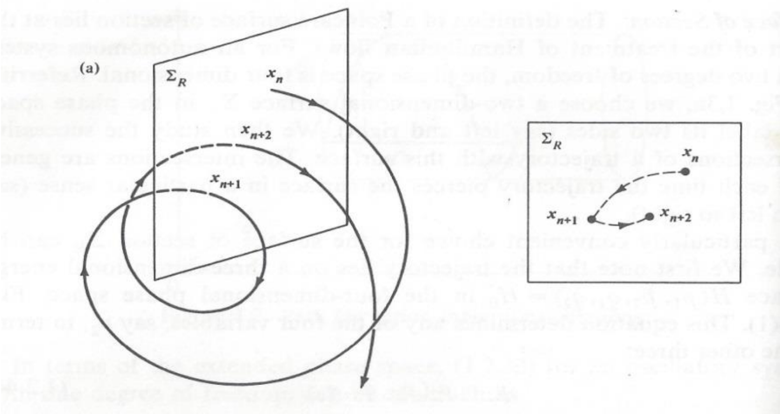

Figure 5. Trajectory of the motion piercing Poincaré section Σ_R. The right figure shows only the points x_n, x_{n+1} e x_{n+2} on Σ_R.

The Poincaré map is found by choosing a point x_n on Σ_R and integrating (4.1) to find the next intersection x_{n+1} of the orbit with Σ_R. In this way we construct the map $x_{n+1} = f(x_n)$, where the function f(x) is "invented" guided by V(x).

In a few words, denoting by n the time sequence of a system and by x the physical observable of this system we can describe the progression of a nonlinear system at a particular moment by investigating how the (n+1).th state depends on the n.th state. The evolution n → n + 1 can be written as a *difference equation* using a function $f(\alpha,x_n)$ as follows

$$x_{n+1} = f_\alpha(x_n) \qquad (4.2),$$

where α is a model-dependent *control parameter*, α and x are real numbers. The function $f_\alpha(x_n)$ generates the value x_{n+1} from x_n and the collection of points generated is said to be a **map** of the function itself. The difference equation (4.1), which is an evolution equation in the Poincaré section is considered a milestone in the field of nonlinear phenomena. Note that n must be iterated from n = 1 up to N >> 1.

(4.a) Logistic Equation and Logistic Map.

There are innumerous chaotic systems studied with the mapping approach. Famous examples are the map models for Ecological and Economic interactions: Symbiosis, Predator-Prey and Competition. [27,28] Malthus, for instance, claimed that the human population p grows obeying the law. [27]

$$dp/dt = kp \qquad (4.3).$$

Verhulst [28] argued that the population grow has inhibitory term ap^2 so that (4.3) is actually given by a nonlinear equation, called *logistic function*

$$dp/dt = kp - ap^2 \qquad (4.4),$$

which shows that the population tends asymptotically to the constant k/a.

One century later, indicating the population by x the differential equation (4.4) was substituted by the **logistic equation** [1,27,28]

$$x_{n+1} = \alpha\, x_n (1 - x_n) \qquad (4.5),$$

where $0 < \alpha < 4$ in order to assure that $0 < x_n < 1$. Note that the (4.5) must be calculated (iterated) from n = 1 up to the *cycle* n >> 1. An n *cycle* is an orbit that returns to its original position after n iterations. In reference [1] are presented **logistic maps** of $\mathbf{x_{n+1}}$ as a function of $\mathbf{x_n}$ showing that x assume **one stable** value and **only two** discrete values for α values in the interval 2.8 - 3.1, characterizing a periodic motion.

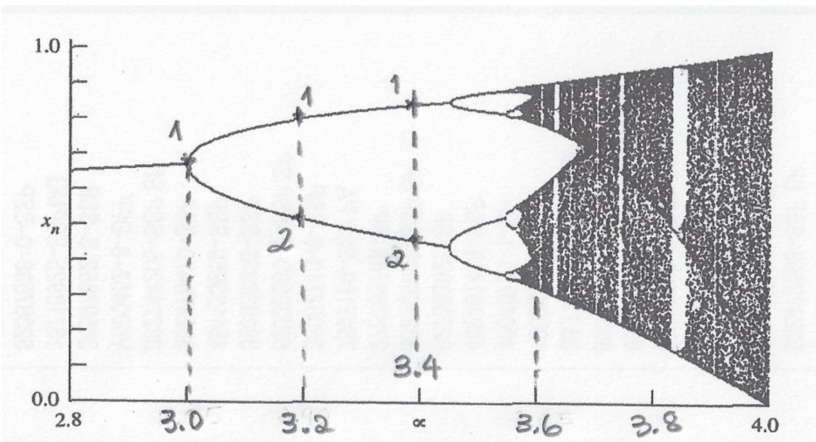

Figure 6. Bifurcation diagram x_n as function of α for **logistic equation map** (2.8 < α < 4.0).

A more general view of the evolution can be obtained plotting a **bifurcation diagram**[1,27,28](see Figure 6) where the x_n is calculated numerically after many interactions to avoid initial effects is plotted as a function of the parameter α. [1] Analyzing this figure we verify that for 2.80 < α < 3.00 there is a stable population with

$x = 0.655$ (the period is *one cycle;* $x_{n+1} = x_n$). At $\alpha = 3.1$ we see a bifurcation (because of obvious shape of the diagram) where there is a period doubling effect ($x_{n+2} = x_n$): x begins to oscillate periodically between 0.558 and 0.765. At $\alpha = 3.45$ there are two different points of bifurcation: now there appear four possible periodic oscillations. The bifurcation and period doubling continues up to an infinite number of cycles near 3.57. Chaos (black regions) occurs for many of α values between 3.57 and 4.0, but there are still windows of periodic motions ("white regions"). Detailed description of these regions can be seen, for instance, in references [29,30], where is also shown a **cobweb diagram** of the logistic map showing chaotic behavior for most values of $\alpha > 3.57$. The special case of $r = 4$ can in fact be solved exactly,[9] as can the case with $\alpha = 2$; however the general case can only be calculated numerically. For $\alpha = 4$ is $x_n = \sin^2(2^n \theta\pi)$ where the initial condition parameter θ is given by $\theta = (1/\pi) \arcsin(x_o^{1/2})$. For rational θ after a finite number of iterations x_n maps into a periodic sequence. But almost all θ are irrational, and, for irrational θ, x_n never repeats itself - it is non-periodic. This solution equation clearly demonstrates the two key features of chaos – stretching and folding: the factor 2^n shows the exponential growth of stretching, which results in sensitive dependence on initial conditions, while the squared sine function x_n keeps folded within the range {0,1}.

(5) Lyapunov Exponents.

The nonlinear terms of the differential equations amplify exponentially small differences in the initial conditions. In this way the deterministic evolution laws can create chaotic behavior, even in the absence of noise or external fluctuations. In the chaotic regime it is not possible to predict exactly the evolution of the system state during a time arbitrarily long. This is the *unpredictability* characteristic of the chaos. The temporal evolution is governed by a *continuous spectrum of frequencies* responsible for an *aperiodic behavior* (see, for instance, Fig.4). The motions present *stationary patterns*, that is, patterns that are repeated only non-periodically.[2,3]

Lyapunov created a method [1-3,25] known as *Lyapunov characteristic exponent* to quantify the sensitive dependence on initial conditions for chaotic behavior. It gives valuable information about the stability of dynamic systems. With this method it is possible to determine the minimum requirements of differential equations that are necessary to create chaos (see Appendix B). To each variable of the system is associated a Lyapunov exponent. Let us study the case of systems with only one variable[1] that assume two initial states x_o and $x_o + \varepsilon$, differing by a small amount ε. We want to investigate the possible values of x_n after n iterations from the two initial values. The difference d_n between the two x_n values after n iterations (omitting for simplicity the subscript α) is given approximately by

$$d_n = f(x_n + \varepsilon) - f(x_n) = \varepsilon \exp(n\lambda) \qquad (5.1),$$

where λ is the Lyapunov exponent that represents the coefficient of the average exponential growth per unit of time between the two states. From (5.1) we see that if λ is negative, the two orbits will eventually converge, but if positive, the nearby trajectories diverge resulting chaos. The difference d_1 between the two initial states is written as

$$d_1 = f(x_o + \varepsilon) - f(x_o) \approx \varepsilon\, (df/dx)_{xo} \qquad (5.2).$$

Now, in order to avoid confusion that sometimes is found in the chaotic literature, we remember that

$$x_{n+1} = f(x_n) = f(f(x_{n-1})) = f(f(f(x_{n-2}))) = ... = f(f(f(...f(x_o)...))) \qquad (5.2)$$

that also is written as

$$x_{n+1} = f(x_n) = f^n(x_o) \qquad (5.3),$$

where the superscript n indicates the n.th iterate of the map.

After a large number n of iterations the difference between the nearby states, using (5.1) and (5.3), will be given by

$$d_n = f(x_n + \varepsilon) - f(x_n) = f^n(x_o + \varepsilon) - f^n(x_o) = \varepsilon \exp(n\lambda) \qquad (5.4)$$

Dividing (5.3) by ε and taking the logarithm of both sides, results

$$\ln\{[f^n(x_o + \varepsilon) - f^n(x_o)/\varepsilon] = \ln[\exp(n\lambda)] = n\lambda \qquad (5.4).$$

Taking into account that ε is small we obtain from (5.4),

$$\lambda(x_o) = (1/n) \ln|df^n(x_o)/dx_o| \qquad (5.5).$$

Since $f^n(x_o)$ is obtained iterating $f(x_o)$ n times we have $f^n(x_o) = f(f(\ldots(f(x_o)\ldots))$, that is, $f^n(x_o) = f(f^{n-1}(x_o)) = f(f^{n-1}(f^{n-2}(x_o))) = \ldots$, where $x_i = f^i(x_o)$ is the result of the i.th iteration of the map f(x) from the initial condition x_o. So, using the derivative chain rule we get

$$df^n(x_o)/dx_o = \{df(x_{n-1})/dx_o\} \{ df(x_{n-2})/dx_o \}\ldots \{df(x_o)/dx_o\} \qquad (5.6).$$

Thus, for ε → ∞ and n → ∞ we get, using (5.5) and (5.6),

$$\lambda(x_o) = \lim_{n\to\infty} (1/n) \ln \left|\prod_{i=0}^{n-1} df(x_i)/dx_o\right|$$

$$= \lim_{n\to\infty} (1/n) \left\{\sum_{i=0}^{n-1} \ln\left|df(x_i)/dx_o\right|\right\} \qquad (5.7),$$

where $x_i = f^i(x_o)$. In the $\lim_{n\to\infty}$ the Lyapunov exponent becomes independent of the initial condition x_o. This occurs because when is done an infinite numbers of iterations. the attractor is entirely covered by x(t), and it does not matter the initial point x_o. As in practice n are large, but finite numbers, we calculate λ for different initial conditions and take an average of these values.

From (5.1) we verify that if λ is negative, the two orbits will eventually converge; but if λ is positive, the nearby trajectories diverge resulting chaos. From (5.4) we see that at the bifurcation λ = 0 because |df/dx| = 1 (the solution becomes unstable). When df/dx = 0 we have λ = - ∞ (the solution becomes super stable).

The λ estimation using simply the *flow equations*,[1-3] that is, without maps, are in general difficult because one has to deal with solutions of NLDE and analytic calculations. This kind of calculation for the damped and driven pendulum is seen, forinstance, in reference.[1] Using *maps* these calculations become easier. This is shown in what follows for *logistic map* and *triangular map*.

(5.a) ***Lyapunov exponents for logistic map.***

According to (5.5) or (5.7) to obtain λ are used the iterated functions $f^n(x_o)$. For the logistic map we have the logistic equation (4.5) that is, $x_{n+1} = \alpha\, x_n (1 - x_n) = f(x_n)$. As an example, the "second order" iterated function $f^2(x)$ is given by

$$f^2(x) = f(f(x)) = f(\alpha\, x(1 - x)) = \alpha(f(x)(1-f(x))) = \alpha^2 x(1-x)[1- \alpha\, x\, (1 - x)].$$

So, to get $\lambda(x_o)$ we can continue to iterate f(x) up to n >> 1 and use (5.5) or use (5.7) taking into account $f(x_i)$, with i = 1,2,…, n, remembering that $f(x_i) = f^i(x)$.

In reference [29,30] are seen **cobweb plots** (**web diagrams**) or **Verhulst diagrams** that are graphs that can be used to visualize successive iterations of the function f(x). In particular , the segments of the diagram connect the points(x,f(x)), (f(x),f(f(x))),(f(f(x)),f(f(f(x)))),…The diagram is so-named because its straight lines segments "anchored" to the functions x and f(x) resemble a *spider web*. The cobweb plot is a visual tool used to investigate the qualitative behavior of one-dimensional iterated functions such as the logistic map. With this plot it is possible to infer the long term status of an initial condition under repeated application of a map.

In Fig. 7 [25]are shown the **Lyapunov exponents** λ calculated numerically as a function of the parameter α for the **logistic map x** seen in Fig.6.

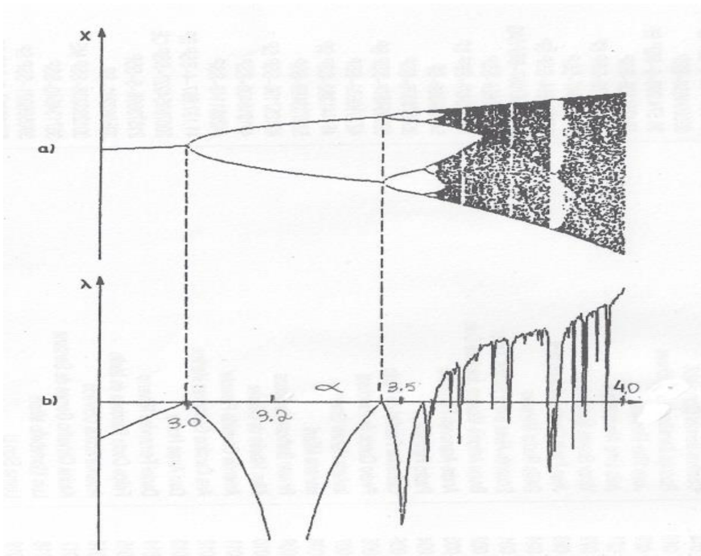

Figure 7. The Lyapunov exponents λ as a function of α for the logistic map x.

(5.b) ***Lyapunov exponents for triangular map.***

In the particular case of a **triangular map**[25,9] λ can be calculated analytically. This map, represented in Fig. 8, obey the following equations:

$$\begin{aligned} x_{n+1} &= 2\beta x_n \ , \qquad 0 < x \leq 1/2 \\ x_{n+1} &= 2\beta(1- x_n) \ , \quad 1/2 < x < 1 \ , \quad 0 < \beta \leq 1. \end{aligned} \tag{5.8}$$

Equations (5.8) can be rewritten as $x_{n+1} = f(x_n)$, where the function f(x) is given by

$$f(x) = \beta[1 - 2\, |1/2 - x|\,] \tag{5.9},$$

shown in Fig. (8.a).

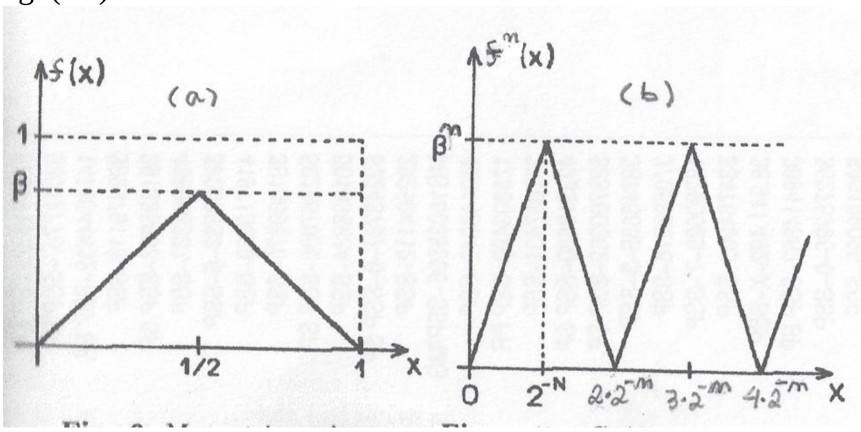

Figure 8. (a) Triangular map. (b) The f^n application to f(x).

The n.th application on 2βx of the first region $0 < x < 1/2$ give $f^n(x) = (2\beta)^n x^n$. The maximum value of $f^n(x)$ is β^n at the point $x = 2^{-n}$, shown in Fig.(8a). By symmetry the next point of minimum must be $2\ 2^{-n}$ and of maximum at $3\ 2^{-n}$ and so on…By similar arguments permit us to conclude that $f^n(x)$ for the region $1/2 < x \leq 1$ must have the behavior shown in Fig.(8b). This implies that $|f^n(x)/dx| = (2\beta)^n$ for the two regions. Taking into account (5.5) we get

$$\lambda(x_o) = (1/n) \ln|df^n(x_o)/dx_o| = \ln(2\beta) \qquad (5.10).$$

Consequently, there is chaos only for β > ½, since λ > 0.

(6) Turbulent Processes.

As seen in basic physics course[4,31] **turbulence** originated from studies of fluid motion in classical mechanics. The general equation of motion for a viscous fluid is given by the Navier-Stokes nonlinear partial differential equation (**NLPDE**),

$$\partial \mathbf{v}/\partial t + (\mathbf{v}.\mathrm{grad})\mathbf{v} = -\ \mathrm{grad}(P)/\rho - \mathrm{grad}(\Phi) + (\eta/\rho)\mathrm{lapl}(\mathbf{v}) \qquad (6.1),$$

where $\mathbf{v}(\mathbf{r},t)$ is the velocity of the fluid at point $\mathbf{r}$, P is the pressure, ρ the density of fluid, Φ(r) the gravitational potential and η the viscosity. This equation is a miracle of brevity, relating a fluid´s velocity, pressure, density and viscosity.[32] Since (6.1) is **NLPDE** it is not submitted to any general method of solution (see Appendix B).

Laminar flux occurs for very small Reynolds number $Re = vL\rho/\eta << 1$, [36,37] where v is a typical fluid velocity and L is some characteristic length in the flux. In these conditions (6.1) can be approximated by a linear partial differential equation (**LPDE**) and all elements of volume of the fluid describe well defined trajectories $\mathbf{r} = \mathbf{r}(t)$. Since there are an infinite number of elements of volume δV the resulting **LPDE** has an infinite number of degrees of freedom which is a characteristic of the **PDE** (see Appendix B). For Re >> 1 the nonlinear effects become dominant being responsible for the phenomenon called **turbulence.** In these conditions the flux becomes **disordered:** the trajectories of the fluid elements δV are irregular and develop eddies, ripples and whorls. In spite of this yet there is some sort of order found within the disorder or turbulence which could be described as self-similar or fractal.[32]

An open problem is to find a mathematical formalism able to describe this disordered state.[32-34] Turbulence in fluid dynamics is being understood in infinite dimensional phase space under the flow defined by the Navier-Stokes equation. We have seen that in the finite dimensional phase space physical systems can be described with very good precision by **LODE** and **NLODE** that can solved exactly or numerically. They can in principle reveal all detailed structures of the dynamical systems. Turbulence in fluid mechanics is generated by a **NLPDE** anchored in an infinite dimensional phase space. Is turbulence a chaotic process? Up to nowadays it is well-known that the theory of chaos in finite-dimensional dynamical systems has been well-developed. Such theory has produced important mathematical theorems and led to important applications in physics, chemistry, biology, engineering, etc. [33]

Note that, in the contrary, theory of chaos in **PDE** has not been well-developed. In terms of applications, most of important natural phenomena are described by linear and nonlinear partial differential equations (wave equations, Yang-Mills equations, Navier-Stokes , General Relativity, Schrödinger equations, etc)(see Appendix B). In spite of extensive investigations it was not possible to prove, in the general case, the existence of chaos in infinite-dimensional systems.[9,32-34]

Among the **NLPDE** there is a class of equations called soliton equations that are integrable Hamiltonian **PDE** and natural counterparts of finite-dimensional integrable Hamiltonian systems.[9] Many works have also been developed investigating the existence of chaos in perturbed soliton equations.[33,34]

APPENDIX A. Ordinary Differential Equations.

In mathematics, an *ordinary differential equation* is an equation containing a function of *one* independent variable and its derivatives.[17,18] The term "*ordinary*" is used in contrast with the term *partial differential equation* or **PDE** which may be with respect to *more than* one independent variable.

Let x be an independent variable and y = y(x) a linear and continuous function of x. Indicating by $y^{(n)} = d^n y/dx^n$ the derivative of order n of the function y(x) an implicit ordinary differential equation of order n can be generally written as

$$F(x,y,y',\ldots,y^{(n)}) = 0 \qquad (3.1),$$

where F is a continuous linear function of x and of the continuous y(x) and of their derivatives $y^{(n)}(x)$. I this case the equation is defined as *linear differential equation* or simply *ordinary differential equation* (**ODE**).

When nonlinear terms are present, F is an *ordinary nonlinear* differential equation *or* **NLODE**.

Existence and Uniqueness of Solutions of ODE.

It can be shown[17,18,35,36] that there is **one and only one solution** of (3.1) in an interval (x_o - Δ, x_o + Δ), with $\Delta > 0$, given by a continuous function (or *trajectory*)

$$y = y(x,c_o,c_1,c_2,,\ldots, c_n) \qquad (3.2),$$

where $c_o= y(x_o)$ and $c_n= y^{(n)}(x_o)$ (n = 1,…, n) are arbitrary constants (*initial conditions)*. Note that general solutions of **ODE**s involve the knowledge of arbitrary constants. The solution (2.2) can be obtained analytically or by graphical and numerical methods. The existence and uniqueness of the ODE solutions are established by several theorems.[17,18,35,36]

Now let us assume that at x_o there are two different initial conditions: one given $y(x_o,c_o,c_1,c_2,,\ldots, c_n)$ and another $y(x_o,C_o,C_1,C_2,,\ldots, C_n)$ when $C_n = c_n + \delta_n$, with $\delta_n << c_n$. At a point $x \neq x_o$ we have the difference Δy given by $\Delta y = y(x,c_o,c_1,c_2,,\ldots, c_n)$ - $y(x,C_o,C_1,C_2,\ldots, C_n)$. Since y is as a continuous function of the variables x, c_n and C_n, Δy can be expanded in a series in a first order approximation of the increments δ_n. In this way, for arbitrarily small increments δ_n the difference Δy becomes also arbitrarily small.

Conclusion: *"for arbitrarily small variations δ_n of the initial conditions the trajectories are practically the same".* Consequently, chaotic systems cannot be governed by **ODE**.

In absence of analytic solutions, graphical and numerical methods, applied by hand or by computer, may give approximate solutions of **ODE** and perhaps yield useful information..

Existence and Uniqueness of Solutions of NLODE.

There are a few methods of solving **NLODE** analytically**;** those that are known typically depend on equation having particular symmetries. There are no general techniques that work for all such equations, and usually each individual equation has to be studied as a separate problem. In absence of analytic solutions, graphical and numerical methods applied by hand or by computer, may give approximate solutions of **ODE**. One extremely popular is the Runge-Kutta [35]method. **NLODE** can exhibit very complicated behavior over extended time intervals, characteristic of chaos. The questions of existence and uniqueness of solutions of **NLODE** and **PDE** are hard problems and their resolution are of fundamental importance to the mathematical theory.[35] However, if the differential equation is a correctly formulated representation of a meaningful physical process, then one expects it to have a **unique solution.**[37]

Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions. For example, the harmonic oscillator equation is an approximation to the nonlinear pendulum equation that is valid for small amplitude oscillations.

APPENDIX B. Partial Differential Equations and Chaos.

The formulation of a physical problems in mathematical terms often results in a partial differential equation (**PDE**) that contains unknown multivariable functions $u(x_1,x_2,\ldots,x_n)$ and their partial derivatives [38,39] $\partial u/\partial x_1,\ldots, \partial u/\partial x_n$, $\partial^2 u/\partial x_1\partial x_1,\ldots,$ $\partial^2 u/\partial x_1\partial x_n$ and so on…

A **PDE** for the function $u(x_1,x_2,\ldots,x_n)$ can be written in an implicit form:

$$F(x_1,x_2,\ldots,x_n, u, \partial u/\partial x_1,\ldots, \partial u/\partial x_n , \partial^2 u/\partial x_1\partial x_1,\ldots, \partial^2 u/\partial x_1\partial x_n\ldots) = 0 \qquad (A.1),$$

which must generally satisfy additional conditions, which are dependent on the nature of the problem. This is the so-called *boundary value problem*. F can be a *linear* (**LPDE**) or *nonlinear* (**NLPDE**) function of u and its derivatives. [38,39] Common examples of **PDE** include sound and heat equations, fluid flow or Navier-Stokes equation, electrostatics, wave equation, electrodynamics, Laplace´s equation, quantum mechanics, Klein-Gordon and Poisson´s equations and gravitation. **PDE** as **ODE** often model multidimensional systems.

Existence and Uniqueness of Solutions.

Although the issue of *existence and uniqueness* of solutions of **ODE** which has a very satisfactory answer, as seen in Section 1, that is not the case for **PDE**. General solutions of **ODE** involve *arbitrary constants*. Solutions of **PDE** are much more complicate because they involve *arbitrary functions*. A solution of a **PDE** is *generally not unique*: it depends on additional conditions that must be specified on the boundary of region where the solution is defined. The Cauchy-Kowalevski theorem states that the Cauchy problem for any **LPDE** whose coefficients are analytic in the unknown function and its derivatives, has a locally unique analytic solution. Although this result might appear to settle the existence and uniqueness of solutions, there are examples of **LPDE** which have no solutions at all.

The **NLPDE** are more difficult to integrate analytically :[38-40] there are almost no general techniques that work for all such equations, and usually each individual equation has to be studied as a separate problem. A fundamental problem for any **PDE** is the existence and uniqueness of a solution for given boundary conditions. For **LPDE** these questions are in general very hard. It is often possible to obtain analytic solutions as occurs, for instance, with solitons in hydrodynamics, electromagnetic waves and non-linear quantum mechanics. *Numerical solution* on a computer is almost the only method that can be used for getting information about arbitrary **PDE**. A list of **NLPDE** is given in reference. [39] As said in Appendix A, if the differential equation is a correctly formulated representation of a meaningful physical process and if a solution can be found consistently with all the given boundary conditions, it is accepted without proof that this solution is **unique**.[37].

Simplest Chaotic Partial Differential Equation

As commented before in spite of extensive investigations it was not possible to prove, in the general case, the existence of chaos in infinite-dimensional systems.[9,32-34] However,it was shown that very simple **NLPDE** permit chaos.[41] These equations have the form

$$\partial u(x,t)/\partial t = F(u(x,t)),$$

where F(u(x,t)) can consist of derivatives in space but not in time, can contain a constant term, and must contain exactly one quadratic nonlinearity (e.g., u^2 or $u.\partial^n u(x,t)/\partial x^n$, etc…). For instance,

$$\partial u/\partial t = - u.(\partial u/\partial x) - A (\partial^2 u/\partial x^2) - (\partial^4 u/\partial x^4).$$

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