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Water Threads - Instabilities of Cylindrical Fluid Jets

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Abstract. We present to graduate and postgraduate students of Physics and engineering the basic ideas necessary to understand instabilities in fluid jets. To do this we study the particular case of instabilities in water jets accelerated under the influence of gravity.
Key words: *fluid jets*; *hydrodynamic equations; surface tension; flow instabilities.*

(I)Introduction.

Our intention is to present to graduate and postgraduate students of Physics the fundamental ideas necessary to understand the instabilities that appear in fluid jets. To do this we study the particular case of instabilities that occur in "water threads" accelerated under the influence of gravity. First, let us begin reanalyzing these water jets seen in basic hydrodynamic courses.[1] So, let us assume that water is ejected in the air with a flux Q by a pipette or a tap with circular orifice of radius a (see **Figure1**). In general case the flow is governed by the Navier-Stokes equation[2]

$$\partial \mathbf{v}/\partial t + (\mathbf{v}.\mathrm{grad})\mathbf{v} = - \mathrm{grad}(p)/\rho - \mathrm{grad}(\varphi)/\rho + \nu\ \mathrm{lapl}(\mathbf{v}) \qquad (I.1),$$

where φ is an external potential, $\nu = \eta/\rho$ is the *cinematic viscosity* and ρ the fluid density. For a gravitational field $\mathrm{grad}(\varphi)/\rho = g\ \mathbf{k}$. Let us consider that water is incompressible ρ = constant or $\mathrm{div}(\mathbf{v}) = 0$, that the jet Reynolds number $Re = Q/(a\eta)$ is sufficiently high that the influence of viscosity is negligible and that nonlinear effects are negligible putting $(\mathbf{v}.\mathrm{grad})\mathbf{v} = 0$. Furthermore, in a first approach the *superficial tension effects* (pressure effects)[3] between water and air will not be taken into account. In this way the pressure changes inside the column of water along the flow would be very small and we can put $\mathrm{grad}(p) = 0$. So, Eq.(I.1) can be written as

$$\partial \mathbf{v}/\partial t = - g\ \mathbf{k} \qquad (I.2).$$

that integrated, show that water moves in a free fall along the z-axis, that is, **v**(t) = gt **k (**neglecting the initial speed**)** or $v(z) = (2gz)^{1/2}$ (see Bernouilli theorem).[1,2,3] If water is ejected by a circular orifice of radius a the flow will have a cylindrical symmetry. As water is incompressible the flux of mass $\Phi = v(z)A(z)$ = constant along the flow, where A(z) is the cross section area of the flow tube. Putting $A(z) = \pi r(z)^2$ where r(z) is the radius of the circular cross section. If Q is the initial flow rate we verify that r(z) is given by[1]

$$r(z) = \{Q/\pi\sqrt{2gz}\}^{1/2} \qquad \text{(I.3)},$$

that is, the radius of the flux tube decreases as z increases. In **Figure1** is shown the *ideal* water thread flow under the influence of gravity with r(z) described by Eq.(I.3). [4]

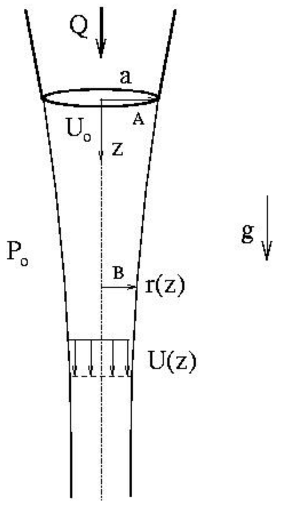


Figure 1. Jet flow of an ideal fluid extruded from an orifice of radius a accelerated under the influence of gravity. Its shape is influenced both by the gravitational acceleration and by the surface tension γ.

In **Figure 2** is shown a typical *real* water flow under the influence of gravity showing instabilities and formation of droplets.[4] To explain the observed jets of real fluids is necessary to estimate the water flow solving the Navier-Stokes differential equation with *boundary conditions* where are taken into account forces acting on the air-fluid interface due to surface tension, atmospheric pressure and viscosity[5,6] (see Appendix A). It will be seen that the surface tension and the curvature ratios of the interface air-fluid play the crucial role in drops formation. As the *curvature ratio* r(z) decreases the surface tension increases breaking the stream into drops.

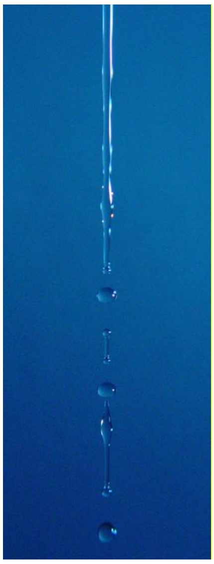


Figure 2. Observed instabilities of the falling water thread with formation of droplets.

(1)Boundary Conditions.

Let us write the basic equations involving superficial tension, viscous forces and difference of pressure in the interface of two fluids which are in relative motion; the interface between the fluids will be indicated by the tridimensional surface $\Omega = \Omega(\mathbf{x}) = \Omega(x,y,z)$. According to the Theory of Superficial Phenomenon[5] as seen in Appendix A, the interface force between two viscous fluids,1 and 2, in relative motion, can be estimated using the following tensor equation for $\mathbf{x} \in \Omega$:

$$n_k \{\sigma_{ik}(2) - \sigma_{ik}(1)\} = \gamma\, (1/R_\alpha + 1/R_\beta)\, n_i \qquad (2.1),$$

where n_α $(\alpha = i,j,k)$ are 3 orthonormal basis ("Darboux frame", see Appendix B) defined on Ω ; only n_i is normal to the interface, the remaining ones are tangent to the interface and σ_{ik} is the *hydrodynamic stress tensor* given by [5]

$$\sigma_{ik} = - p\delta_{ik} + \eta\{(\partial v_i/\partial x_k) + (\partial v_k/\partial x_i)\} = - p\delta_{ik} + \sigma'_{ik} \qquad (2.2),$$

γ the *surface tension*, R_α an R_β the *main curvature rays* [5] of the interface, p the external pressure applied on the interface, η the viscosity coefficient between the fluids and **v** is the relative velocity between the fluids. For two fluids in contact using Eqs.(1) and (2) we have explicitly,

$$(p_1 - p_2)\, n_i = \eta\{\sigma'_{ik}(1) + \sigma'_{ik}(2)\}n_k + \gamma\, (1/R_\alpha + 1/R_\beta)\, n_i \qquad (2.3),$$

where p_1 and p_2 are the *normal* external pressures applied (at the interface) on the fluids 1 and 2, respectively; $\sigma'_{ik}(1)$ and $\sigma'_{ik}(2)$ are the hydrodynamic tensors of the fluids 1 and 2, respectively. Note that $p = p(\mathbf{x},t)$, $n_i = n_i(\mathbf{x},t)$, $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$ and $R = R(\mathbf{x},t)$, where $\mathbf{x} \in \Omega$. In addition, we can also have, for instance, $\eta = \eta(\mathbf{x},t,T)$ and $\gamma = \gamma(\mathbf{x},t,T)$[5] where T is the temperature. The interface velocities considered in (2.3) are obtained calculating the bulk flow of fluids 1 and 2 using the *Navier-Stokes hydrodynamic* equation

$$\partial\mathbf{v}/\partial t + (\mathbf{v}.\mathrm{grad})\mathbf{v} = - \mathrm{grad}(p)/\rho - \mathrm{grad}(\varphi)/\rho + \nu\ \mathrm{lapl}(\mathbf{v}) \qquad (2.4),$$

for *incompressible fluids*, where φ is an external potential, $\nu = \eta/\rho$ is the *cinematic viscosity* and ρ (= constant) the density of the fluids.

Eq.(2.4) is solved using (2.3) assuming the following *boundary condition:* the interface pressure forces are balanced by viscous and capillarity forces and *no-slipping* of the two fluids in motion.

(3)Water Threads in Atmospheric Air.

Let us return to the water jet under the influence of the gravity in a steady atmospheric air seen in the Introduction. Air is stationary and water is in motion with velocity **v.** Neglecting viscous and nonlinear effects that is, $(\mathbf{v}.\mathrm{grad})\mathbf{v} = \eta = 0$, as in the Introduction, water moves as an ideal fluid obeying the Euler equation.[2] So, for two points A and B along the water flow lines we have according to Bernouilli theorem[2]

$$\rho v_A^2/2 + \rho g z_A + p_A = \rho v_B^2/2 + \rho g z_B + p_B \qquad (3.1).$$

Taking water as fluid (1) and air as fluid (2) the *boundary condition* governed by (2.3) will specified indicating by P = p(2) = atmospheric or ambient pressure and by p(1) = p($\mathbf{x}$,t) the internal water pressure. In these conditions using Eq.(2.3) one can shown that, for $\mathbf{x} \in \Omega$,

$$(p - P)\, n_i = \eta\{(\partial v_i/\partial x_k) + (\partial v_k/\partial x_i)\}\, n_k + (\gamma/R)\, n_i = (\gamma/R)\, n_i \qquad (3.2),$$

since $\eta = 0$ and putting $(1/R_\alpha + 1/R_\beta) \approx 1/R$, where R($\mathbf{x}$,t) is the *average curvature radius* of the interface Ω at point $\mathbf{x}$ and time t.

Due to fluid incompressibility, to the cylindrical symmetry of the water flow and to g**k** the fluid speed v(z) **=** U(z) depends only of the distance z measured from the top O of the liquid column(see **Figure1**).

Thus, according to (3.2), since $R = r(z)$ we get $p(z) \approx P + \gamma/r(z)$.[4]

Putting $U_o = U(0)$ and $p_o = p(0)$ into (3.1) we have

$$\rho U_o^2/2 + p_o = \rho U(z)^2/2 - \rho g z + p(z) \qquad (3.3).$$

From (3.2), $p_o = p(0) \approx P + \gamma/a$ and $p(z) \approx P + \gamma/r(z)$. Thus, (3.3) becomes

$$U(z)/U_o = \{1 + (2/F_r)(z/a) + (2/W_e)(1 - a/r(z))\}^{1/2} \qquad (3.4),$$

where F_r = **Froude Number** = U_o^2/ga = inertia/gravity and W_e = **Weber Number** = $\rho a U_o^2/\gamma$ = inertia/curvature.

Now flux conservation requires that

$$Q = 2\pi \int U(z)\, r(z)\, dr = \pi a^2 U_o = \pi r(z)^2 U(z) \qquad (3.5),$$

from which one obtains using (3.4),

$$r(z)/a = (U_o/U(z))^{1/2} = \{1 + (2/F_r)(z/a) + (2/W_e)(1 - a/r(z))\}^{-1/4} \qquad (3.6).$$

This equation may be solved algebraically to yield the thread shape $r(z)/a$ and into (3.4) to obtain the velocity profile $U(z)/U_o$. In the limit of $W_e \to \infty$ one obtains

$$r(z)/a = (1+2gz/U_o^2)^{-1/4} \quad \text{and} \quad U(z)/U_o = (1+2gz/U_o^2)^{1/2} \qquad (3.7),$$

showing that as z increases the fluid speed increases and the radius $r(z)$ decreases. Thus, we would have $r(z) \to 0$ and $v(z) \to \infty$ as $z \to \infty$.

However, actually we observe [4] that a stream of water emerging from a tap will break up into droplets, no matter how smoothly the stream is emitted from the tap (see **Figure 2**).This is due to a phenomenon called the *Rayleigh- Plateau instability*,[4,7] which is entirely a consequence of surface tension effects. The explanation of this instability begins with the existence of tiny perturbations in the stream. These are always present, no matter how smooth the stream is. If the perturbations are resolved into sinusoidal components (see **Figure 3**), we find that some components grow with time while others decay with time. Among those that grow with time, some grow at faster rates than others. Whether a component decays or grows, and how fast it grows is entirely a function of its wave number *k* (a measure of how many peaks and troughs per centimeter) and the radii of the original cylindrical stream.

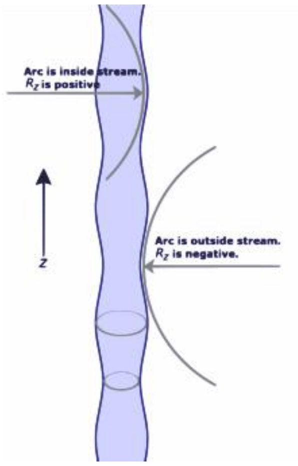


Figure 3. The breakup of streams into drops. Intermediate stage of a jet breaking into drops. Radii of curvature in the axial direction are shown. Equation for the radius of the stream is $R(z) = R_o + A_k \cos(kz)$,where R_o is the radius of the unperturbed stream, A_k is the amplitude of the perturbation and *k* is the wave number.[6]

In next section we explain the reason why the circular jet geometry is rendered unstable by capillarity, as well as the characteristic time and length scales associated with this transition from cylinder to drops. This is achieved using the powerful tool of linear stability analysis, pioneered by Rayleigh. [4,7] As we will show in the next subsection, any perturbation of sufficiently long wavelength will result in a gain in surface energy, so the perturbation grows. Rayleigh [7] was the first to point out the crucial significance of the *most unstable wavelength*, which is only found by studying the dynamics, as we will do throughout this section.

(4) The Rayleigh-Plateau Instabilities.

We briefly show how estimate the Rayleigh-Plateau instabilities of the falling cylindrical water jets.[4,7] First let us assume that steady state consists of an infinite long quiescent cylindrical inviscid fluid column of radius R_o, density ρ and surface tension γ as seen in **Figure 5(**I**).** The influence of gravity is neglected. The pressure p is constant inside the column and may be calculated balancing the normal stresses with the surface tension at the boundary. Assuming zero external pressure (P = 0) yield

$$p_o = \gamma/R_o \qquad (4.1).$$

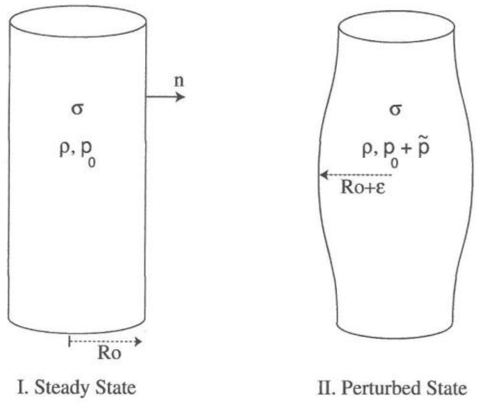


Figure 5. (I)A cylindrical column of initial radius Ro of an inviscid fluid with density ρ and surface tension γ; **n = r** is normal to surface. (II)Perturbed state of the fluid when $R_o \rightarrow R_o + \epsilon$.

Now we assume (see **Figure 5(**II**))** that the initial interface radius R_o is deformed by infinitesimal perturbations ϵ which create varicose on the cylindrical surface, enabling us to *linearize* the governing equations. That is, we put

$$R^* = R_o + \epsilon \exp[\omega t + ikz] \qquad (4.2),$$

where $R_o >> \epsilon$, ω is the growth rate of the instability and k the wave number of the disturbance in the z-direction. The corresponding wavelength λ of the varicose perturbations is necessarily $\lambda = 2\pi/k$.

Putting $p(r,z) = p_o + p^*(r,z,t)$ and $\mathbf{v}(r,z,t) = \mathbf{v}_o + \mathbf{v}^*(r,z,t)$, where $\mathbf{v}^*(r,z,t) = v_r(r,z,t)\mathbf{r} + v_z(r,z,t)\mathbf{k}$ we get, using equation (2.4), $\partial\mathbf{v}/\partial t = -\,\mathrm{grad}(p)/\rho$, that is:

$$\partial v_r^*/\partial t = -(1/\rho)(\partial p^*/\partial r) \quad \text{and} \quad \partial v_z^*/\partial t = -(1/\rho)(\partial p^*/\partial z) \qquad (4.3).$$

Taking into account that $\mathrm{div}(\mathbf{j}) = \rho\,\mathrm{div}(\mathbf{v}_o + \mathbf{v}^*) = 0$ we obtain

$$\partial v_r^*/\partial r + v_r^*/r + \partial v_z^*/\partial z = 0 \qquad (4.4).$$

Assuming that disturbances in velocity and pressure have the form as the surface disturbance (4.2) we write

$$v_r^*(r,z) = \alpha(r) \exp[\omega t + ikz] ,$$

$$v_z^*(r,z) = \beta(r) \exp[\omega t + ikz] \qquad (4.5)$$

$$p^*(r,z) = \pi(r) \exp[\omega t + ikz]$$

Substituting (4.5) into (4.3) and (4.4) yields the linearized equations governing the perturbation fields

$$\omega\alpha = -(1/\rho)d\pi/dr \qquad (4.6)$$

$$\omega\beta = -(ik/\rho)\pi \qquad (4.7)$$

$$d\alpha/dr + \alpha/r + ik\beta = 0 \qquad (4.8).$$

Eliminating β(r) and π(r) results the following differential equation for α(r):

$$r^2(d^2\alpha/dr^2) + r\,(d\alpha/dr) - [1+(kr)^2]\,\alpha = 0 \qquad (4.9).$$

This corresponds to the modified Bessel Equation of order 1[8] whose solutions may be written in terms of the modified Bessel functions $I_1(kr)$ and $K_1(kr)$. The last function will not be considered because $K_1(kr) \to \infty$ as $r \to 0$. Therefore,

$$\alpha(r) = C\, I_1(kr) \qquad (4.10),$$

where the constant C will be determined by boundary conditions.

From (4.6) and (4.10), considering that $I_o'(x) = I_1(x)$[8] we obtain π(r):

$$\pi(r) = -(\omega\rho C/k)\, I_o(r) \qquad (4.11).$$

Now let us consider the boundary conditions. The first one is the kinematic condition on the free surface at r = Ro:

$$\partial R^*/\partial t = \epsilon\omega \exp[\omega t + ikz] = \mathbf{v}^*(r,z)\,.\mathbf{n} = v_r^* = \alpha(r)\exp[\omega t + ikz] \qquad (4.12).$$

From (4.10) and (4.12) we get $\alpha(R_o) = C\, I_1(kR_o) = \epsilon\,\omega$, that is,

$$C = \epsilon\omega/I_1(kR_o) \qquad (4.13).$$

The second condition is the normal stress balance on the free surface,

$$p_o + p^* = \gamma(1/R_1 + 1/R_2) \qquad (4.14),$$

where R_1 is the *radial curvature*, that is, perpendicular to z-axis

$$1/R_1 = 1/R_o\{1 + \epsilon/R_o \exp[\omega t + ikz]\}^{-1} \approx 1/R_o - (\epsilon/R_o^2)\exp[\omega t + ikz] \qquad (4.15).$$

As R_2 is the *axial curvature* (along the z-axis) we can show that[4]

$$1/R_2 \approx \epsilon k^2 \exp[\omega t + ikz] \qquad (4.16).$$

Substituting (4.15) and (4.16) into (4.14) yields

$$p_o + p^* = \gamma/R_o - (\gamma\epsilon/R_o^2)(1 - k^2R_o^2)\exp[\omega t + ikz] \qquad (4.17).$$

With (4.1) the above equation gives

$$p^* = -(\gamma\epsilon/R_o^2)(1 - k^2R_o^2)\exp[\omega t + ikz] \qquad (4.18).$$

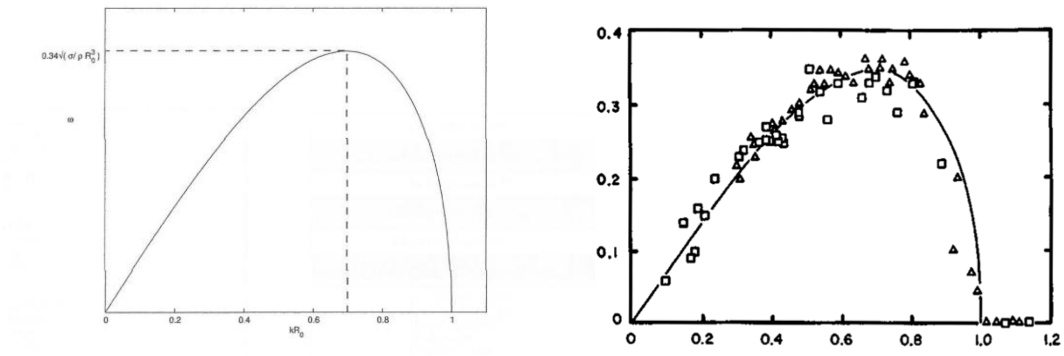
Combining (4.18) with (4.11) and (4.13) yields the dispersion relation that gives the dependence of the growth rate ω on the wave number *k*:

$$\omega^2 = (\gamma/\rho R_o^2)\{I_1(kR_o)/I_o(kR_o)\}(1 - k^2R_o^2) \qquad (4.19)$$

showing that **unstable modes** occur, that is, $\omega^2 > 0$ only for

$$kR_o = 2\pi R_o/\lambda < 1 \qquad (4.20),$$

that is, only to disturbances whose wavelengths exceed the circumference of the cylinder. The plot $\omega(kR_o)$ x kR_o is shown in **Figure 6(a).**[4] **Figure 6(b**) shows (4.19) successfully compared with experimental results[7] measuring the wavelengths λ (high accurate measurements of linear jets stability are surprisingly difficult, even by modern standards).



Figures 6(a) and (b). Plot of the frequency ω as a function of kR_o.

According to **Figure 6(a)** the maximum value for the disturbance frequency $\omega_{max} = 0.34\,(\gamma/\rho R_o^3)^{1/2}$ occurs for $kR_o = 0.697$, i.e. when the disturbance wavelength of the is $\lambda_{max} = 9.02\,R_o$. By inverting the maximum growth rate ω_{max} we get the characteristic break up time

$$t_{breakup} = 2\pi/\omega_{max} \approx 2.91\,(\rho R_o^3/\gamma)^{1/2} \qquad (4.21).$$

With (4.21) one can estimate how much time is necessary to occur the filament break up. According to (4.21) a water jet of $R_o = 1$ cm would have $t_{breakup} \approx 1/8$ s , which is consistent with casual observations of jet break up in a kitchen sink.

APPENDIX A - Surface Physics.

As seen in basic physics courses,[1,3,6] a paperclip, an insect or a needle can float on water. Drops of mercury or water do not spread on a surface. When the clean glass with a small diameter are immersed in water, the water rises in the tube, however, when the liquid is mercury, mercury falls into the tube. This phenomenon is known as capillarity. Bubbles of water with soap can created and float freely in the air. Thin liquid films of water with soap can be created in wire frames. To stretch these films is necessary to apply a force F. In this stretching if the film area is modified by dA is realized a work dW defined by $dW = \gamma\, dA$, where γ is named *surface tension coefficient* or, simply, *surface tension* of the film. It has the dimension of *energy per unit of area* [J/m^2] or *force per unit of length* [N/m]. The thin film behaves as an elastic membrane.

In materials science, *surface tension* is used for either *surface stress* or *surface free energy* and, usually, instead of "force" F we take into account "*tension*" T which is "*force per unit of length*".

These phenomena and many others are observed in *interfaces* of fluids in contact. A clear understanding of the *interface physics* can be only obtained taking into account the molecular structure of the fluids and their mutual interactions.[6]

Usually, in undergraduate physics courses[1,3] are studied only liquid-air interfaces since in this case is easier to understand basic effects derived from surface tension. At liquid-air interfaces, surface tension results from the greater attraction of liquid molecules to each other (due to cohesion) than to the molecules in the air. Thus, the surface becomes under tension from the imbalanced forces. The net effect is an inward force at its surface that causes the liquid to behave as if its surface were covered with a stretched elastic membrane.[1,3] Surface tension is the elastic tendency of a fluid surface which makes it acquire the least surface area possible. An illustrative example is the water droplet free in the air: the liquid assumes a spherical shape which is the least surface area possible, that is, an area with a minimum of elastic energy.

(A.1)Surface curvature and pressure.

If no force acts normal to a tensioned surface (interface between two fluids) $\Omega = \Omega(x,y,z)$ the surface remains flat. But if the pressure on one side of the surface differs from pressure on the other side, the pressure difference times surface area results in a normal force. In order for the surface tension forces to cancel the force due to pressure, the surface must be curved. Let us suppose that every point on the surface undergoes an infinitesimal displacement $\delta\xi$ normal to the surface. So, the elements of volume dV between the fluids is $dV = \delta\xi\, dA$, where dA is the element of area. If p_1 and p_2 are the pressures into the first and second fluid the work δW_p done due to volume variation dV over all surface Ω is given by

$$\delta W_p = \int_\Omega (-p_1 + p_2)\, \delta\xi\, dA \qquad (A.1).$$

Taking into account the surface tension the total work δW performed during the surface deformation is given by

$$\delta W = -\,\delta W_p + \gamma\, \Delta A = -\int_\Omega (-p_1 + p_2)\, \delta\xi\, dA + \gamma\, \Delta A \qquad (A.2),$$

where ΔA is total variation of the area due to the interface deformation. The equilibrium condition is obtained when $\delta W = 0$.

Assume R_α and R_β as the principal ratios of curvature of the surface Ω at a given point $P \in \Omega$ [see Appendix B]. They will be assumed positive if directed in the fluid 1. Let us define dl_α and dl_β the arc elements along the circumferences of ratios R_α and R_β, respectively. When the arc elements move by $\delta\xi$ along the normal **n** they are slightly increased becoming $dl_\alpha = dl_\alpha (1+ \delta\xi/R_\alpha)$ and $dl_\beta = dl_\beta (1+ \delta\xi/R_\beta)$, respectively. In this way, after the displacement $\delta\xi$ the element of area $dA = dl_\alpha\, dl_\beta$ becomes

$$dA' = dl_\alpha (1+ \delta\xi/R_\alpha)\, dl_\beta (1+ \delta\xi/R_\beta) \approx dA\,(1+ \delta\xi/R_\alpha + \delta\xi/R_\beta) \qquad (A.3).$$

Thus, the total area variation ΔA is given by

$$\Delta A = \int_\Omega (1/R_\alpha + 1/R_\beta)\, \delta\xi\, dA \qquad (A.4).$$

Consequently, from (A.2) and (A.4) the equilibrium condition is written as

$$\int_\Omega \{(p_1 - p_2) - \gamma(1/R_\alpha + 1/R_\beta)\}\, \delta\xi\, dA = 0 \qquad (A.5).$$

As this condition must be obeyed for any infinitesimal displacement $\delta\xi$ of the surface Ω the following identity is valid

$$(p_1 - p_2) = \gamma(1/R_\alpha + 1/R_\beta) \qquad (A.6),$$

known as ***Laplace Formula***. For a cylindrical interface $R_\alpha = R$ where R is the *radial curvature* and the *axial curvature* $R_\beta = \infty$, so $(p_1 - p_2) = \gamma/R$; for a spherical interface $R_\alpha = R_\beta = R$, where R is the radius of the sphere, so $(p_1 - p_2) = 2\gamma/R$.

(A.2)Interface Viscosity Effects.

Ignoring the surface tension, when two viscous fluids 1 and 2 are in motion with speeds $v_1(x,t)$ and $v_2(x,t)$ at the interface Ω there are only viscous and pressure forces between them that are described by the tensor σ_{ik} given by[2,5]

$$\sigma_{ik} = - p\delta_{ik} + \eta\{(\partial v_i/\partial x_k) + (\partial v_k/\partial x_i)\} = - p\delta_{ik} + \sigma'_{ik} \qquad (A.7),$$

where p is external pressure and η is the viscosity coefficient. At the boundary Ω the following equilibrium condition must be obeyed

$$n_k \{\sigma_{ik}(2) - \sigma_{ik}(1)\} = 0 \qquad (A.8).$$

depicting the equality of forces at the fluids interface. On the other hand, if surface tension effects are present instead of (A.8) we must now have

$$(p_1 - p_2)\, n_i = \eta\{\sigma'_{ik}(1) + \sigma'_{ik}(2)\}n_k + \gamma\,(1/R_\alpha + 1/R_\beta)\, n_i \qquad (A.9)$$

where p_1 and p_2 are the *normal* external pressures applied (at the interface) on the fluids 1 and 2, respectively; $\sigma'_{ik}(1)$ and $\sigma'_{ik}(2)$ are the hydrodynamic tensors of the fluids 1 and 2, respectively.

APPENDIX B. Surface Curvatures.

Exact calculations of general *surface curvatures* using the differential geometry is a very difficult task.[5,10] This calculation gets easier when surfaces Ω = Ω(x,y,z) are approximately planes. In these conditions let us write the interface equation as z = ξ(x,y), where ξ represent a small normal displacement ("deformation") of the interface at z = 0. Let us now consider an area A of the interface that is given by [5]

$$A = \int_\Omega \{1+(\partial\xi/\partial x)^2 + (\partial\xi/\partial y)^2\}^{1/2}\, dxdy,$$

that, for small ξ can be approximately written as

$$A \approx \int_\Omega [1+(1/2)(\partial\xi/\partial x)^2 + (1/2)(\partial\xi/\partial y)^2]\, dxdy \qquad (B.1).$$

With a small variation δξ we have

$$\Delta A \approx \int_\Omega \{(\partial\xi/\partial x)(\partial\delta\xi/\partial x) + (\partial\xi/\partial y)(\partial\delta\xi/\partial y)\}dxdy \qquad (B.2),$$

that integrating by parts gives, since $\delta\xi \to 0$ outside the interface,

$$\Delta A \approx - \int_{\Omega}\{(\partial^2\xi/\partial x^2) + (\partial^2\xi/\partial y^2)\}\, \delta\xi\, dxdy \qquad \text{(B.3).}$$

Comparing (A.4) with (B.3) we see, as dA = dxdy, that

$$(1/R_{\alpha} + 1/R_{\beta}) = -\{(\partial^2\xi/\partial x^2) + (\partial^2\xi/\partial y^2)\} \qquad \text{(B.4),}$$

which is the general curvature formula for the sum of inverse curvature ratios of a weakly curved surface. **Figure 7** shows the principal curvature planes for a saddle surface and the Darboux frame. [10]

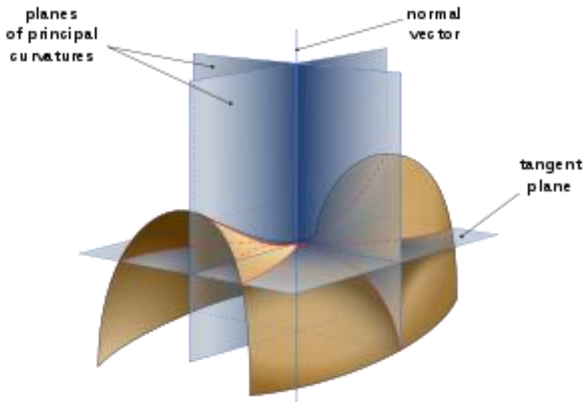


Figure 7. Saddle surface with normal planes in directions of principal curvatures.

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