Instituto de Física
Universidade de São Paulo

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José Maria Filardo Bassalo
Antonio Boulhosa Nassar
Mauro Sérgio Dorsa Cattani

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UNIVERSIDADE DE SÃO PAULO
Instituto de Física
Cidade Universitária
Caixa Postal 66.318
05315-970 - São Paulo - Brasil

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José Maria Filardo Bassalo

Academia Paraense de Ciências (www.apaci.net.org)

jmfbassalo@gmail.com

Antonio Boulhosa Nassar

Extension Program-Department of Sciences, University of California, Los Angeles, California 90024

nassar@ucla.edu

Mauro Sérgio Dorsa Cattani

Instituto de Fsica da USP, 05389-970, São Paulo, SP

mcattani@if.usp.br

Abstract: In this paper we study the Bohmnian Trajectories for the Bateman-Caldirola-Kanai Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm.
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1. Introduction: The Bohmnian Trajectories

In this article, we calculated the *Bohmnian Trajectories* for the Bateman-Caldirola-Kanai Equation. To obtain these trajectories we adopted the quantum mechanical formalism of the de Broglie-Bohm. This was done because this formalism permits to perform essential linear approximations along the classical trajectories that are the basic ingredients of the of the Feynman's principle of minimum action of quantum mechanics. [1]

2. The Bohmnian Trajectories for the Bateman-Caldirola-Kanai Equation

Now, let us calculated the Bohmnian trajectories for the Bateman-Caldirola-Kanai Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm. [2]

2.1. The Bateman-Caldirola-Kanai Equation

In 1931/1941/1948, H. Bateman [3], P. Caldirola [4] and E. Kanai [5] proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\,\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2\,m}\,exp\,(-\lambda\,t)\times\frac{\partial^2\,\Psi(x,t)}{\partial x^2} +$$

$$+\,exp\,(\lambda\,t)\times V(x,t)\times\Psi(x,t)\quad , \qquad (2.1.1)$$

where $\Psi(x,t)$ and $V(x,t)$ are, respectively, the wave function and the time dependent potential of the physical system in study, and λ is a constant.

2.1.1. The Wave Function of the Bateman-Caldirola-Kanai Equation

Writting the wave function $\Psi(x,t)$ in the polar form defined by the Madelung-Bohm transformation [6,7] we obtain:

$$\Psi(x,t) = \phi(x,t)\times exp\,[i\,S(x,t)]\quad , \qquad (2.1.1.1)$$

where $\phi(x,t)$ will be defined in what follows.

Calculating the derivatives, temporal and spatial, of (2.1.1.1), we get [remembering that $exp\,[i\,S]$ is common factor]: [2]

$$\frac{\partial\Psi}{\partial t} = exp\,(i\,S)\,(\,\frac{\partial\phi}{\partial t} + i\,\phi\,\frac{\partial S}{\partial t}\,)\quad , \qquad (2.1.1.2a)$$

$$\frac{\partial^2\Psi}{\partial x^2} = exp\,(i\,S)[\,\frac{\partial^2\phi}{\partial x^2} + 2i\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x} + i\,\phi\,\frac{\partial^2 S}{\partial x^2} - \phi\,(\,\frac{\partial S}{\partial x}\,)^2\,]\quad , \qquad (2.1.1.2b)$$

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.1.1), we have:[2]

$$i\,\hbar\,(\,\frac{\partial\phi}{\partial t} + i\,\phi\,\frac{\partial S}{\partial t}\,) = -\frac{\hbar^2}{2\,m}\,exp\,(-\lambda\,t)\times$$

$$\times[\,\frac{\partial^2\phi}{\partial x^2} + 2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x} + i\,\phi\,\frac{\partial^2 S}{\partial x^2} - \phi\,(\,\frac{\partial S}{\partial x}\,)^2\,] + exp(\,\lambda\,t)\times V(x,t)\times\phi\quad , \qquad (2.1.1.3)$$

Separating the real and imaginary parts of the relation (2.1.1.3), results:

a) <u>imaginary part</u>

$$\frac{\hbar}{\phi}\,\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\,exp\,(-\lambda\,t)\times(\,2\,\frac{1}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\,)\quad , \qquad (2.1.1.4)$$

b) real part

$$-\hbar\frac{\partial S}{\partial t}=-\frac{\hbar^2}{2\,m}\,exp\,(-\lambda\,t)\times[\,\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}-(\frac{\partial S}{\partial x})^2\,]+exp(\lambda\,t)\times V(x,t)\quad . \qquad (2.1.1.5)$$

2.1.2. Dynamics of the Bateman-Caldirola-Kanai Equation

Now, let us see the correlation between the expressions (2.1.1.4-5) and the traditional equations of the Ideal Fluid Dynamics: [8] a) *Continuity Equation*, b) *Euler's equation*. To do this let us perform the following correspondences:

Quantum density probability: $|\,\Psi(x,t)\,|^2=\Psi^*(x,t)\,\Psi(x,t)\quad\leftrightarrow$

Quantum mass density: $\rho(x,t)=\phi^2(x,t)\;\leftrightarrow\;\sqrt{\rho}=\phi$, (2.1.2.1a,b)

Gradient of the wave function: $\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x}\quad\leftrightarrow$

Quantum velocity: $v_{qu}(x,t)\equiv v_{qu}$, (2.1.2.1c,d)

Bohm quantum potential:

$$V_{qu}(x,t)\equiv V_{qu}=-(\frac{\hbar^2}{2m})(\frac{1}{\phi})\frac{\partial^2\phi}{\partial x^2}=-\frac{\hbar^2}{2\,m}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\quad , \qquad (2.1.2.1e,f)$$

Putting the relations (2.1.2.1a-d) into the equation (2.1.1.4) and considering that $\partial(\ell n\,x)/\partial y=(1/x)(\partial x/\partial y)$ and $\ell n(x^m)=m\,\ell nx$, we get:[2]

$$\frac{\partial}{\partial t}[\ell n\,(\phi^2)]=-\frac{\hbar}{m}\times exp\,(-\lambda\,t)\times\{\frac{\partial^2 S}{\partial x^2}+\frac{\partial S}{\partial x}\frac{\partial}{\partial x}[\ell n\,(\phi^2)]\}\;\rightarrow$$

$$\frac{\partial}{\partial t}(\ell n\,\rho)=-\frac{\hbar}{m}\times exp\,(-\lambda\,t)\times[\,\frac{\partial^2 S}{\partial x^2}+\frac{\partial S}{\partial x}\frac{\partial}{\partial x}(\ell n\,\rho)\,]=$$

$$=-\frac{\hbar}{m}\times exp\,(-\lambda\,t)\times[\,\frac{\partial^2 S}{\partial x^2}+\frac{\partial S}{\partial x}\frac{1}{\rho}\frac{\partial\rho}{\partial x}\,]=$$

$$=exp\,(-\lambda\,t)\times[\,-\frac{\partial}{\partial x}(\frac{\hbar}{m}\frac{\partial S}{\partial x})-(\frac{\hbar}{m}\frac{\partial S}{\partial x})\frac{1}{\rho}\frac{\partial\rho}{\partial x}\,]\;\rightarrow$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t}+exp\,(-\lambda\,t)\times\frac{\partial v_{qu}}{\partial x}+\frac{v_{qu}}{\rho}\frac{\partial\rho}{\partial x}=0\;\rightarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, v_{BCK})}{\partial x} = 0, \qquad (2.1.2.2a)$$

where:

$$v_{BCK} = exp(-\lambda\, t) \times v_{qu}, \qquad (2.1.2.2b)$$

is the *Bateman-Caldirola-Kanai velocity*.

We observe that the eq. (2.1.2.2.a) represents the *Continuity Equation* or *Mass Conservation Law* of the Fluid Dynamics. [8] We must note that this expression also shows a coherent effect in the physical system represented by the Bateman-Caldirola-Kanai Equation ($BCK-E$) [eq. (2.1.1)].

Now, let us obtained another dynamic equation of the $BCK-E$. So, differentiating the eq. relation (2.1.1.5) with respect x and using the eqs. (2.1.2.1a-e), we obtain:

$$-\hbar \frac{\partial^2 S}{\partial x\, \partial t} = -\frac{\hbar^2}{2m} \times exp(-\lambda\, t) \times \frac{\partial}{\partial x}[\frac{1}{\phi}\frac{\partial^2 \phi}{\partial x^2} - (\frac{\partial S}{\partial x})^2] + exp(-\lambda\, t) \times \frac{\partial V(x,t)}{\partial x} \quad \rightarrow$$

$$\frac{\partial v_{qu}}{\partial t} = -\frac{exp(-\lambda\, t)}{m} \times \frac{\partial V_{qu}}{\partial x} - exp(-\lambda t) \times v_{qu} \times \frac{\partial v_{qu}}{\partial x} - \frac{exp(\lambda t)}{m} \times \frac{\partial V(x,t)}{\partial x}. \qquad (2.1.2.3)$$

Multiplying the eq. (2.1.2.3) by $exp(-\lambda$ t), using the eq. (2.1.2.2b) and its temporal derivate, we obtain:

$$exp(-\lambda t) \times \frac{\partial v_{qu}(x,t)}{\partial t} =$$

$$= -\frac{exp(-2\,\lambda\, t)}{m} \times \frac{\partial V_{qu}}{\partial x} - exp(-\lambda\, t) \times v_{qu} \times \frac{\partial[exp(-\lambda t) v_{qu}]}{\partial x} - \frac{1}{m} \times \frac{\partial V(x,t)}{\partial x} \quad \rightarrow$$

$$\frac{\partial v_{BCK}}{\partial t} + v_{BCK} \times \frac{\partial v_{BCK}}{\partial x} + \frac{1}{m} \times \frac{\partial}{\partial x}[V(x,t) + V_{BCK}(x,t)] = -\lambda \times v_{BCQ}, \qquad (2.1.2.4a)$$

where:

$$V_{BCK}(x,t) = exp(-2\,\lambda\, t) \times V_{qu}(x,t), \qquad (2.1.2.4b)$$

is the *Bateman-Caldirola-Kanai Potential*.

We observe that, although the eq. (2.1.2.4a) had the aspect of the *Navier-Stokes Equation*,[8] the same represent a conservative system, since when λ $\rightarrow$ ∞, then $v_{BCK}(x,t)$ and $V_{BCQ}(x,t)$ $\rightarrow$ 0, by the eqs. (2.1.2.2b;2.1.2.4b).

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*:[8]

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_{qu}\frac{\partial}{\partial x}, \qquad (2.1.2.5a)$$

and that:

$$v_{qu}(x,t)|_{x=x(t)} = \frac{dx}{dt} \quad , \qquad (2.1.2.5b)$$

the eq. (2.1.2.3) could be written as:

$$m\,\frac{d^2x}{dt^2} + m\,\lambda\, v_{BCK}(x,t) = -\frac{\partial}{\partial x}[\,V(x,t) + V_{BCK}(x,t)\,], \qquad (2.1.2.6)$$

We note that the eq. (2.1.2.6) has a form of the *Second Newton Law*, being the first term of the second member is the *classical newtonian force*, the second is the *quantum bohmnian force*, and the three, is the *Bateman-Caldirola-Kanai force*.

2.1.3 The Quantum Wave Packet of the Linearized Bateman-Caldirola-Kanai Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Bateman-Caldirola-Kanai Equation ($BCK-E$) along a classical trajetory, let us the considerer the *ansatz*: [9]

$$\rho(x,t) = [2\,\pi\, a^2(t)]^{-1/2} \times exp\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\} \qquad (2.1.3.1a)$$

or [use eq. (2.1.2.1b)]:

$$\phi(x,t) = [2\,\pi\, a^2(t)]^{-1/4} \times exp\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \qquad (2.1.3.1b)$$

where $a(t)$ and $q(t) = <x>$ are auxiliary functions of time, to will be determined in what follows, representing the *width* and the *center of mass of wave packet*, respectively.

Differentiating the expression (2.1.3.1a) in the variable t, and remembering that x and t are independent variables, results:

$$\frac{\partial\rho}{\partial t} = -\frac{1}{2}[2\,\pi\, a^2(t)]^{-3/2} \times [4\,\pi\, a(t)\,\dot a(t)] \times exp\,\{-\frac{[x-q(t)]^2}{2a^2(t)}\} +$$

$$+[2\,\pi\, a^2(t)]^{-1/2} \times exp\,\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\} \times \frac{\partial}{\partial t}\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\} =$$

$$= -\,\rho\,\{[2\,\pi\,\dot a(t)] \times [2\pi\, a^2(t)]^{-1} + \frac{4\,a^2(t)\times[x-q(t)]\times[-\dot q(t)] - 4a(t)\dot a(t)\times[x-q(t)]^2}{4a^4(t)}\} \rightarrow$$

$$\frac{\partial\rho}{\partial t} = \rho\,\{-\frac{\dot a(t)}{a(t)} + \frac{\dot q(t)}{a^2(t)}\times[x-q(t)] + \frac{\dot a(t)}{a^3(t)}\times[x-q(t)]^2\,\} \quad . \qquad (2.1.3.2)$$

Substituting the eq. (2.1.3.2) into eq. (2.1.2.2a) and integrating the result, we have (we consider null the integration constant):

$$\rho\,\{-\frac{\dot{a}(t)}{a(t)}+\frac{\dot{q}(t)}{a^2(t)}\times[x-q(t)]+\frac{\dot{a}(t)}{a^3(t)}\times[x-q(t)]^2\}+\frac{\partial(\rho\, v_{BCK})}{\partial x}=0\ \rightarrow$$

$$\int\frac{\partial(\rho\, v_{BCK})}{\partial x}\partial x=\int\rho\,\{\frac{\dot{a}(t)}{a(t)}-\frac{\dot{q}(t)}{a^2(t)}\times[x-q(t)]-\frac{\dot{a}(t)}{a^3(t)}\times[x-q(t)]^2\}\,\partial x\ \rightarrow$$

$$\rho\, v_{BCK}=\int\rho\,\{\frac{\dot{a}(t)}{a(t)}-\frac{[x-q(t)]}{a^2(t)}\times(\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t))\}\,\partial x\ \rightarrow$$

$$v_{BCK}=\frac{1}{\rho}\int\rho\,\{\frac{\dot{a}(t)}{a(t)}-\frac{[x-q(t)]}{a^2(t)}\times(\frac{\dot{a}(t)}{a(t)}[x-q(t)]+\dot{q}(t))\}\,\partial x\,. \qquad (2.1.3.3)$$

Now, using the eq. (2.1.3.1a), we can right that:

$$\frac{\partial}{\partial x}\{\rho\,(\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t))\}=$$

$$=\rho\,\frac{\partial}{\partial x}\{\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\}+\{\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\}\,\frac{\partial\rho}{\partial x}=$$

$$=\rho\,\frac{\dot{a}(t)}{a(t)}+\{\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\}\times\frac{\partial}{\partial x}([2\,\pi\,a^2(t)]^{-1/2}\times exp\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\})=$$

$$=\rho\,\frac{\dot{a}(t)}{a(t)}+\{\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\}\times[2\,\pi\,a^2(t)]^{1/2}\times exp\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\}\times$$

$$\times\frac{\partial}{\partial x}\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\}=$$

$$=\rho\,\frac{\dot{a}(t)}{a(t)}+\{\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\}\times\rho\,\{-\frac{[x-q(t)]}{a^2(t)}\}\ \rightarrow$$

$$\frac{\partial}{\partial x}\{\rho\,(\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t))\}=$$

$$=\rho\,\{\frac{\dot{a}(t)}{a(t)}-\frac{[x-q(t)]}{a^2(t)}\times(\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t))\}\,. \qquad (2.1.3.4)$$

Substituting the eq. (2.1.3.4) into the eq. (2.1.3.3) and using the eqs. (2.1.2.2b;

2.1.2.5b), results:

$$v_{BCK} = \frac{1}{\rho}\int \frac{\partial}{\partial x}\{\rho\,(\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t))\}\,\partial x \quad \to$$

$$v_{BCK}(x,t) = (\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\,), \qquad (2.1.3.5a)$$

and:

$$v_{BCK}(x,t) = exp(-\lambda\, t)\times v_{qu}(x,t) \quad \to$$

$$v_{qu}(x,t) = \frac{dx(t)}{dt} = exp(\lambda\, t)\times(\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\,). \qquad (2.1.3.5b)$$

We observe that the integration of the eq. (2.1.3.5) give us the *bohmnian quantum trajectory* of the physical system considered represented by the eq. (2.1.1). Before, we must calculated the equation for $a(t)$. For that, we expand the functions S(x, t), V(x t), V_{qu}(x, t) and V_{BCK}(x, t) around of $q(t) = < x >$ up to *second Taylor order*. [9] In this way, we have:

$$S(x,t) = S[q(t),t] + S'[q(t),t]\times[x-q(t)] + \frac{S''[q(t)\,t]}{2}\times[x-q(t)]^2, \qquad (2.1.3.6)$$

$$V(x,t) = V[q(t),t] + V'[q(t),t]\times[x-q(t)] + \frac{V''[q(t),t]}{2}\times[x-q(t)]^2, \qquad (2.1.3.7)$$

$$V_{qu}(x,t) = V_{qu}[q(t),t] + V'_{qu}[q(t),t]\times[x-q(t)] +$$

$$+\frac{1}{2}V''_{qu}[q(t),t]\times[x-q(t)]^2, \qquad (2.1.3.8)$$

$$V_{BCK}(x,t) = V_{BCK}[q(t),t] + V'_{BCK}[q(t),t]\times[x-q(t)] +$$

$$+\frac{1}{2}V''_{BCK}[q(t),t]\times[x-q(t)]^2, \qquad (2.1.3.9)$$

where (’) and (") means, respectively: $\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial q^2}$.

Differentiating the eq. (2.1.3.6) in the variable x multiplying the result by $\hbar/m$, using the eqs. (2.1.2.1c,d) and (2.1.3.5b), results:

$$\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x} = \frac{\hbar}{m}\{\,S'[q(t),t] + S''[q(t),t]\times[x-q(t)]\,\} =$$

$$= v_{qu}(x,t) = exp(\lambda\, t) \times (\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t)\,) \quad \rightarrow$$

$$S'[q(t),t] = exp(\lambda\, t) \times \frac{m\,\dot{q}(t)}{\hbar}, \quad S''[q(t),t] = exp(\lambda\, t) \times \frac{m}{\hbar}\,\frac{\dot{a}(t)}{a(t)}. \qquad \text{(2.1.3.10a,b)}$$

Substituting the eqs. (2.1.3.10a,b) into the eq. (2.1.3.6), we have:

$$S(x,t) = S_0(t) + exp(\lambda\, t) \times \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] +$$

$$+ \; exp(\lambda\, t) \times \frac{m}{2\,\hbar}\,\frac{\dot{a}(t)}{a(t)} \times [x - q(t)]^2 \rightarrow$$

$$S(x,t) = S_0(t) +$$

$$+ exp(\lambda\, t) \times (\,\frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\hbar}\,\frac{\dot{a}(t)}{a(t)} \times [x - q(t)]^2\,). \qquad \text{(2.1.3.11a)}$$

where:

$$S_0(t) \equiv S[q(t),t], \qquad \text{(2.1.3.11b)}$$

are the *quantum action*.

Differentiating the eq. (2.1.3.11a) in relation to the time t, we obtain (remembering that $\frac{\partial x}{\partial t} = 0$):

$$\frac{\partial S}{\partial t} = \dot{S}_0(t) +$$

$$+ \frac{\partial}{\partial t}\{\, exp(\lambda t) \times \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)]\,\} + \frac{\partial}{\partial t}\{\, exp(\lambda\, t) \times \frac{m}{2\,\hbar} \times \frac{\dot{a}(t)}{a(t)} \times [x - q(t)]^2\,\} \quad \rightarrow$$

$$\frac{\partial S}{\partial t} = \dot{S}_0(t) + exp(\lambda\, t)\,(\,\frac{m}{2\,\hbar}\,[\,\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \lambda \times \frac{\dot{a}(t)}{a(t)}] \times [x - q(t)]^2 +$$

$$+ [\,\frac{m\,\ddot{q}(t)}{\hbar} + (\lambda - \frac{\dot{a}(t)}{a(t)}) \times \frac{m\,\dot{q}(t)}{\hbar}\,] \times [x - q(t)] - \frac{m\,\dot{q}^2(t)}{\hbar}\,). \qquad \text{(2.1.3.12)}$$

Considering the eqs. (2.1.2.1a,b) and (2.1.3.1a,b), let us write V_{qu} given by eq. (2.1.2.1e,f) in terms of potencies of [x - q(t)]. Before, we calculate the following derivations [remembering that $\frac{d}{dz}\, exp\,(z) = exp\,(z)$]:

$$\frac{\partial\phi(x,t)}{\partial x} = \frac{\partial}{\partial x}\,(\,[2\,\pi a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\,\}\,) =$$

$$= [2\,\pi\,a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\,\} \times \frac{\partial}{\partial x}\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\,\} \quad \rightarrow$$

$$\frac{\partial\phi(x,t)}{\partial x} = -\,[2\,\pi a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\,\} \times \{\frac{[x-q(t)]}{2\,a^2(t)}\} = -\,\phi \times \{\frac{[x-q(t)]}{2\,a^2(t)}\},$$

$$\frac{\partial^2\phi(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\,(\,-\,\phi \times \{\frac{[x-q(t)]}{2\,a^2(t)}\}\,) =$$

$$= -\,\phi\,\frac{\partial}{\partial x}\{-\frac{[x-q(t)]}{2\,a^2(t)}\} - \frac{[x-q(t)]}{2\,a^2(t)}\,\frac{\partial\phi}{\partial x} =$$

$$= -\,\phi \times \frac{1}{2\,a^2(t)} + \frac{[x-q(t)]^2}{4\,a^2(t)} \times \phi \quad \rightarrow$$

$$\frac{1}{\phi(x,t)}\,\frac{\partial^2\,\phi(x,t)}{\partial x^2} = \frac{[x-q(t)]^2}{4\,a^4(t)} - \frac{1}{2\,a^2(t)}. \qquad (2.1.3.13)$$

Substituting the eq. (2.1.3.13) in the eq. (2.1.2.1e), taking into account the eq. (2.1.3.8) and considering the identity of polynomials, results:

$$V_{qu}(x,t) = -\,\frac{\hbar^2}{2\,m}\,\{\frac{[x-q(t)]^2}{4\,a^4(t)} - \frac{1}{2a^2(t)}\,\} =$$

$$= V_{qu}[q(t),t] + V'_{qu}[q(t),t] \times [x-q(t)] +$$

$$+\,\frac{1}{2}\,V''_{qu}[q(t),t] \times [x-q(t)]^2 \quad \rightarrow$$

$$V_{qu}(x,t) = \frac{\hbar^2}{4\,m\,a^2(t)};\;\; V'_{qu}(x,t) = 0;\;\; V''_{qu}(x,t) = -\,\frac{\hbar^2}{8\,m\,a^4(t)} \quad \rightarrow$$

$$V_{qu}(x,t) = \frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8\,m\,a^4(t)} \times [x-q(t)]^2. \qquad (2.1.3.14)$$

Now, let us write V_{BCQ} given by the eq. (2.1.2.4b) in terms of the potencies of [x - q(t)], by using the eq. (2.1.3.14). So, we have:

$$V_{BCQ} = exp(-2\,\lambda\,t) \times (\frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8m\,a^4(t)} \times [x - q(t)]^2\,). \qquad (2.1.3.15)$$

Taking the eq. (2.1.1.5) and using the eqs. (2.1.2.1c,d,e), (2.1.2.2b) and (2.1.2.4b), we obtain:

$$-\hbar\,\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2\,m}\,exp\,(-\lambda\,t) \times [\,\frac{1}{\phi}\,\frac{\partial^2\phi}{\partial x^2} - (\frac{\partial S}{\partial x})^2] + exp(\lambda\,t) \times V(x\,,t) \quad \to$$

$$\hbar\,\frac{\partial S}{\partial t} + exp\,(-\lambda\,t) \times [\,\frac{m}{2}\,v_{qu}^2 + V_{qu}(x,t)\,] + exp\,(\lambda\,t) \times V(x,t) = 0 \quad \to$$

$$\hbar\,\frac{\partial S}{\partial t} + exp\,(\lambda t) \times \{\,\frac{m}{2}\,[exp\,(-\lambda\,t) v_{qu}]^2 + exp\,(-2\,\lambda\,t) \times V_{qu}(x,t) + V(x,t)\,\} = 0 \quad \to$$

$$\hbar\,\frac{\partial S}{\partial t} + exp\,(\lambda t) \times [\,\frac{m}{2}\,v_{BCK}^2 + V(x,t) + V_{BCK}(x,t)\,] = 0 \quad . \qquad (2.1.3.16)$$

Inserting the eqs. (2.1.3.5a), (2.1.3.7), (2.1.3.12) and (2.1.3.15) into eq. (2.1.3.16), we obtain:

$$\hbar\,\dot{S}_0(t) + exp(\lambda\,t)\,(\,-m\,\dot{q}^2(t) + [m\,\ddot{q}(t) + m\,\dot{q}(t)\,(\lambda - \frac{\dot{a}(t)}{a(t)})\,] \times [x - q(t)] +$$

$$+\frac{m}{2}\,[\,\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \lambda\,\frac{\dot{a}(t)}{a(t)}] \times [x - q(t)]^2) +$$

$$+ exp\,(\lambda\,t) \times (\,\frac{m}{2}\{\,\frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\,\}^2\,) +$$

$$+ exp\,(\lambda\,t) \times \{\,V[q(t),t] + V'[q(t),t] \times [x - q(t)] + \frac{1}{2}\,V''[q(t),t] \times [x - q(t)]^2\,\} +$$

$$+ exp\,(-2\,\lambda\,t) \times \{\,\frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8\,ma^4(t)} \times [x - q(t)]^2\,\} = 0 \to$$

$$\hbar\,\dot{S}_0(t) + exp(\lambda\,t)\,(-m\,\dot{q}^2(t) + [m\,\ddot{q}(t) + m\dot{q}(t)\,(\lambda - \frac{\dot{a}(t)}{a(t)}\,)\,] \times [x - q(t)] +$$

$$+\frac{m}{2}\,[\,\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \lambda\,\frac{\dot{a}(t)}{a(t)}] \times [x - q(t)]^2 +$$

$$+ exp\ (\lambda\ t) \times (\frac{m}{2} \{ \frac{\dot{a}^2(t)}{a(t)} \times [x - q(t)]^2 + \dot{q}^2(t) + 2\ \frac{\dot{a}(t)\dot{q}(t)}{a(t)} \} \times [x - q(t)]) +$$

$$+ exp\ (\lambda\ t) \times \{ V[q(t), t] + V'[q(t), t] \times [x - q(t)] + \frac{1}{2} V''[q(t), t] \times [x - q(t)]^2 \} +$$

$$+ exp\ (-\lambda\ t) \times \{ \frac{\hbar^2}{4\ m\ a^2(t)} - \frac{\hbar^2}{8\ ma^4(t)} \times [x - q(t)]^2 \} = 0 . \qquad (2.1.3.17)$$

Since $(x - q)^o = 1$, we can gather together the above expression in potencies of $(x - q)$, obtaining:

$$(\hbar\ \dot{S}_o(t) + exp\ (\lambda t) \times \{ - \frac{m}{2} \dot{q}^2(t) + V[q(t), t] \} + \frac{exp\ (-\lambda\ t)\hbar^2}{4\ m\ a^2}) \times [x - q(t)]^0 +$$

$$+ (exp\ (\lambda\ t) \{ m\ \ddot{q}(t) + V'[q(t), t] \}) \times [x - q(t)] +$$

$$+ (exp\ (\lambda\ t) \times \{ \frac{m}{2} [\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \lambda\ \frac{\dot{a}(t)}{a(t)} +$$

$$+ V''[q(t), t] \} - \frac{exp\ (-\lambda\ t)\hbar^2}{8\ m\ a^4(t)}) \times [x - (q)t)]^2 = 0 \quad . \qquad (2.1.3.18)$$

As the above relation is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar} (m\ \dot{q}^2 + exp\ (\lambda\ t) \times \{ - \frac{m}{2}\ \dot{q}(t) + V[q(t), t] \} -$$

$$- \frac{exp\ (-\lambda\ t)\ \hbar^2}{4\ ma^2(t)}) \quad , \qquad (2.1.3.19)$$

$$\ddot{q}(t) + \frac{V'[q(t), t]}{m} = 0 \quad . \qquad (2.1.3.20)$$

$$\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \lambda\ \frac{\dot{a}(t)}{a(t)} + \frac{V''[q(t), t]}{m} = \frac{exp\ (-2\ \lambda\ t)\ \hbar^2}{4\ m^2\ a^4(t)} \quad . \qquad (2.1.3.21)$$

Now, let us consider that V[q(t), t] is given by:

$$V[q(t), t] = \frac{1}{2}\ m\ \omega^2(t)\ q^2(t) , \qquad (2.1.3.22)$$

which is the Time Dependent Harmonic Oscillator Potencial).
In this case, we have:

$$V'[q(t),t] = m\,\omega^2(t)\,q(t), \quad V''[q(t),t] = m\,\omega^2(t) \quad . \qquad (2.1.3.23a,b)$$

Putting the eqs. (2.1.3.23a,b) into eqs. (2.1.3.20,21), results:

$$\ddot{q}(t) + \omega^2(t)\,q(t) = 0 \quad , \qquad (2.1.3.24)$$

$$\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \lambda\,\frac{\dot{a}(t)}{a(t)} + \omega^2(t) = \frac{exp\,(-2\lambda\,t)\,\hbar^2}{4\,m^2\,a^4(t)} \quad . \qquad (2.1.3.25)$$

2.1.4. The Bohmnian Trajectories for the Bateman-Caldirola-Kanai Equation

The associated Bohmnian Trajectories, [10]-[14] for the Bateman-Caldirola-Kanai Equation ($BCK-E$) of an evolving *ith* particle of the ensemble with an initial position x_{0i} can be calculated by considering that:

$$\dot{x}_i(t) = v_{qu}[x_i(t),t]. \qquad (2.1.4.1)$$

Then substituting the eq. (2.1.4.1) into eq. (2.1.3.5b), results:

$$\dot{x}_i(t) = \frac{dx_i(t)}{dt} = exp(\lambda t)\times(\,\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t)\,) \quad \rightarrow$$

$$x_i(t) = \int_{t_0}^{t} [\,exp(\lambda\,t)\times(\,\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}\,)]\,dt. \qquad (2.1.4.2)$$

The eqs. (2.1.3.24,25) show that a continuous measurement of a quantum wave packet gives specific features to its evolution: the appearance of distinct classical and quantum elements, respectively. This measurement consist of monitoring the position of quantum systems and the result is the measured classical path $q(t)$ for t within a quantum uncertainty $a(t)$.

From the eq. (2.1.3.25), we note that for $\lambda \neq 0$ a stationary regime can be reached and that the width [$a(t)$] of the wave packet can be related to the resolution of measurement as follows. Then considering that $a(t) = cte$ [$\dot{a}(t) = 0$; $a_{t_0} = a_0$] in the eq. (2.1.3.25) and considering the t_0 the $initial\ time$, we have:

$$exp\,(\lambda\,t_0) = \ell n\,[\omega_0 \tau_B], \qquad (2.1.4.3a)$$

where [10]:

$$\tau_B = (\frac{2\,m\,a_0^2}{\hbar}) = 6{,}8\times 10^{-26}s, \qquad (2.1.4.1.3b)$$

is the *Bohmtime constant* which determines the time resolution of the quantum measurement.

We observe that the eq. (2.1.4.3a) to indicated that, as $\lambda \neq 0$, then: $t_0 \neq 0$.

The eqs. (2.1.4.3a,b) means that if an initially free wave packet is kept under a certain continuous measurement, its (a_0) may not spread in time. Then, the associated *Bohmnian Trajectories* of an evolving *ith* particle of the ensemble with an initial position x_{0i} is giving by eq. (2.1.4.2).

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