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Abstract: In this paper we study the Bohmnian Trajectories for the Bialynicki-Birula-Mycielski Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of de Broglie-Bohm.

Keywords: De Broglie-Bohm Quantum Mechanics; Bohmnian Trajectories of the Bialynicki-Birula-Mycielski Equation

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1. Introduction: The Bohmnian Trajectories

In this article, we calculate the *Bohmnian Trajectories* for the Bialynicki-Birula-Mycielski Equation. To obtain these trajectories we adopted the quantum mechanical formalism of de Broglie-Bohm. This was done because this formalism permits to perform essential linear approximations along the classical trajectories that are the basic ingredients of the of the Feynman's principle of minimum action of quantum mechanics. [1]

2. Bohmnian Trajectories for the Bialynicki-Birula-Mycielski Equation

Now, let us calculate the Bohmnian trajectories for the Bialynicki-Birula-Mycielski Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of de Broglie-Bohm. [2]

2.1. The Bialynicki-Birula-Mycielski Equation

In 1976 and 1979, [3] I. Bialynicki-Birula and J. Mycielski proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$\text{i}\ \hbar\,\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2\,m}\,\frac{\partial^2\,\Psi(x,t)}{\partial x^2} +$$

$$+\{V(x,t) - \frac{\hbar\,\lambda}{2}\,\ell n[\Psi(x,t)\,\Psi^*(x,t)]\}\times\Psi(x,t)\ , \quad (2.1.1)$$

where $\Psi(x,t)$ and $V(x,t)$ are, respectively, the wave function and the time dependent potential of the physical system in study, λ *caracterizes the resolution of the measurement*, and (*) means complex conjugate.

2.1.1. The Wave Function of the Bialynicki-Birula-Mycielski Equation

Writting the wave function $\Psi(x,t)$ in the polar form defined by the Madelung-Bohm transformation [4,5] we obtain:

$$\Psi(x,t) = \phi(x,t)\times exp\,[i\,S(x,t)]\ , \quad (2.1.1.1)$$

where $\phi(x,t)$ will be defined in what follows.

Calculating the derivatives, temporal and spatial, of (2.1.1.1), we get [remembering that $exp\,[i\,S]$ is common factor]: [2]

$$\frac{\partial\Psi}{\partial t} = exp\ (i\,S)\,(\,\frac{\partial\phi}{\partial t} + i\,\phi\,\frac{\partial S}{\partial t}\,)\ , \quad (2.1.1.2a)$$

$$\frac{\partial^2\Psi}{\partial x^2} = exp\ (i\,S)\,[\,\frac{\partial^2\phi}{\partial x^2} + 2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x} + i\,\phi\,\frac{\partial^2 S}{\partial x^2} - \phi(\,\frac{\partial S}{\partial x}\,)^2\ ]\ , \quad (2.1.1.2b)$$

$$i\,\hbar\,(\,\frac{\partial\phi}{\partial t} + i\,\phi\,\frac{\partial S}{\partial t}\,) = -\frac{\hbar^2}{2\,m}\,[\,\frac{\partial^2\phi}{\partial x^2} +$$

$$+\,2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x} + i\,\phi\,\frac{\partial^2 S}{\partial x^2} - \phi\,(\,\frac{\partial S}{\partial x}\,)^2\ ] + [V(x,t) - \frac{\hbar\,\lambda}{2}\,\ell n\,(\phi)^2\,]\,\phi\ . \quad (2.1.1.3)$$

Separating the real and imaginary parts of the relation (2.1.1.3), results:

a) <u>imaginary part</u>

$$\frac{\hbar}{\phi}\,\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2\,m}\,(\,2\,\frac{1}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\,)\ , \quad (2.1.1.4)$$

b) <u>real part</u>

$$-\,\hbar\,\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2\,m}\,[\,\frac{1}{\phi}\,\frac{\partial^2\phi}{\partial x^2} - (\,\frac{\partial S}{\partial x}\,)^2\ ] + V(x,t) - \frac{\hbar\,\lambda}{2}\,\ell n\,(\phi)^2\ . \quad (2.1.1.5)$$

2.1.2. Dynamics of the Bialynicki-Birula-Mycielski Equation

Now, let us see the correlation between the expressions (2.1.1.4-5) and the traditional equations of the Ideal Fluid Dynamics: [6] a) *Continuity Equation*, b) *Euler's equation*. To do this let us perform the following correspondences:

<u>Quantum density probability</u>: $\quad |\,\Psi(x,t)\,|^2 = \Psi^*(x,t)\,\Psi(x,t) \quad \leftrightarrow$

Quantum mass density: $\rho(x,t) = \phi^2(x,t) \quad \leftrightarrow \quad \sqrt{\rho} = \phi$, (2.1.2.1a,b)

Gradient of the wave function: $\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x} \quad \leftrightarrow$

Quantum velocity: $v_{qu}(x,t) \equiv v_{qu}$, (2.1.2.1c,d)

Bohm quantum potential:

$$V_{qu}(x,t) \equiv V_{qu} = -(\frac{\hbar^2}{2\,m})(\frac{1}{\phi})\frac{\partial^2 \phi}{\partial x^2} = -\frac{\hbar^2}{2\,m}\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2} \,, \quad (2.1.2.1e,f)$$

Putting the relations (2.1.2.1a-d) into the equation (2.1.1.4) and considering that $\partial(\ell n\, x)/\partial y = (1/x)\,(\partial x/\partial y)$ and $\ell n(x^m) = m \ell n\, x$, we get: [2]

$$\frac{\partial}{\partial t}[\ell n\,(\phi^2)] = -\frac{\hbar}{m}\{\frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x}\frac{\partial}{\partial x}[\ell n\,(\phi^2)]\} \quad \rightarrow$$

$$\frac{\partial}{\partial t}(\ell n\,\rho) = -\frac{\hbar}{m}[\frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x}\frac{\partial}{\partial x}(\ell n\,\rho)] = -\frac{\hbar}{m}[\frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x}\frac{1}{\rho}\frac{\partial \rho}{\partial x}] =$$

$$= -\frac{\partial}{\partial x}(\frac{\hbar}{m}\frac{\partial S}{\partial x}) - (\frac{\hbar}{m}\frac{\partial S}{\partial x})\frac{1}{\rho}\frac{\partial \rho}{\partial x} \quad \rightarrow \quad \frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\partial v_{qu}}{\partial x} + \frac{v_{qu}}{\rho}\frac{\partial \rho}{\partial x} = 0 \quad \rightarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, v_{qu})}{\partial x} = 0\,, \quad (2.1.2.2)$$

which represents the *Continuity Equation* or *Mass Conservation Law* of the Fluid Dynamics. We must note that this expression also shows a coherent effect in the physical system represented by the Bialynicki-Birula-Mycielski Equation ($BBM - E$) [eq. (2.1.1)].

Now, let us obtain another dynamic equation of the $BBM - E$. So, differentiating the eq. relation (2.1.1.5) with respect x and using the eqs. (2.1.2.1a-e), we obtain:

$$-\hbar\frac{\partial^2 S}{\partial x\,\partial t} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}[\frac{1}{\phi}\frac{\partial^2 \phi}{\partial x^2} - (\frac{\partial S}{\partial x})^2] + \frac{\partial}{\partial x}[V(x,t) - \frac{\hbar \lambda}{2}\ell n\,(\phi^2)] \quad \rightarrow$$

$$\frac{\partial}{\partial t}(\frac{\hbar}{m}\frac{\partial S}{\partial x}) =$$

$$= \frac{\partial}{\partial x}[\frac{\hbar^2}{2\,m^2}\frac{1}{\phi(x,t)}\frac{\partial^2 \phi(x,t)}{\partial x^2}] - \frac{1}{2}\frac{\partial}{\partial x}[\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x}]^2 - \frac{1}{m}\frac{\partial V(x,t)}{\partial x} + \frac{\hbar\,\lambda}{2\,m}\frac{\partial\{\ell n[\phi^2(x,t)]\}}{\partial x} =$$

$$= \frac{\partial}{\partial x} (\frac{\hbar^2}{2\,m^2} \frac{1}{\phi(x,t)} \frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{V(x,t)}{m} + \frac{\hbar\,\lambda}{2\,m} \ell n[\phi^2(x,t)]) - \frac{1}{2} \frac{\partial}{\partial x} (\frac{\hbar}{m} \frac{\partial S(x,t)}{\partial x})^2 \rightarrow$$

$$\frac{\partial v_{qu}(x,t)}{\partial t} + v_{qu}(x,t) \frac{\partial v_{qu}(x,t)}{\partial x} +$$

$$+ \frac{1}{m} \frac{\partial}{\partial x} [V(x,t) + V_{qu}(x,t) - V_{BBM}(x,t)] = 0 \;, \quad (2.1.2.3)$$

where:

$$V_{BBM}(x,t) = \frac{\hbar\,\lambda}{2} \ell n[\phi^2(x,t)] = \frac{\hbar\,\lambda}{2} \ell n \rho(x,t) \;, \quad (2.1.2.4a,b)$$

is the *Bialynicki-Birula-Mycielski Potential*. We must observe that the eq. (2.1.2.3) is an equation similar to the *Euler Equation* which governs the motion of an ideal fluid.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*: [6]

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_{qu} \frac{\partial}{\partial x} \;, \quad (2.1.2.5a)$$

and that:

$$v_{qu}(x,t)\,|_{x=x(t)} = \frac{dx}{dt} \;, \quad (2.1.2.5b)$$

the eq. (2.1.2.3) could be written as:

$$m \frac{d^2 x}{dt^2} = - \frac{\partial}{\partial x} [V(x,t) + V_{qu}(x,t) - V_{BBM}(x,t)] \rightarrow$$

$$m \frac{d^2 x}{dt^2} = F_c(x,t)\,|_{x=x(t)} + F_{qu}(x,t)\,|_{x=x(t)} - F_{BBM}(x,t)\,|_{x=x(t)} \;. \quad (2.1.2.6)$$

We note that the eq. (2.1.2.6) has a form of the *Second Newton Law*, being the first term of the second member is the *classical newtonian force*, the second is the *quantum bohmnian force*, and the three, is the *Bialynicki-Birula-Mycielski force*.

2.1.3. The Quantum Wave Packet of the Linearized Bialynicki-Birula-Mycielski Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Bialynicki-Birula-Mycielski Equation ($BBM - E$) along a classical trajetory, let us the considerer the *ansatz*: [7]

$$\rho(x,t) = [2\,\pi\,a^2(t)]^{-1/2} \times exp\{- \frac{[x - q(t)]^2}{2\,a^2(t)}\} \qquad (2.1.3.1a)$$

or [use eq. (2.1.2.1a,b)]:

$$\phi(x,t) = [2\,\pi\,a^2(t)]^{-1/4} \times exp\{- \frac{[x - q(t)]^2}{4\,a^2(t)}\} \qquad (2.1.3.1b)$$

where $a(t)$ and $q(t) = < x >$ are auxiliary functions of time, to will be determined in what follows, representing the *width* and the *center of mass of wave packet*, respectively.

Differentiating the expression (2.1.3.1a) in the variable t, and remembering that x and t are independent variables, results:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2}[2\pi\, a^2(t)]^{-3/2} \times [4\,\pi\, a(t)\,\dot{a}(t)] \times exp\,\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\} +$$

$$+ [2\,\pi\, a^2(t)]^{-1/2} \times exp\,\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\} \times \frac{\partial}{\partial t}\{-\frac{[x-q(t)]^2}{2a^2(t)}\} =$$

$$= -\,\rho\,\{[2\,\pi\,\dot{a}(t)] \times [2\pi\, a^2(t)]^{-1} + \frac{4\,a^2(t) \times [x-q(t)] \times [-\dot{q}(t)] - 4a(t)\dot{a}(t) \times [x-q(t)]^2}{4a^4(t)}\} \rightarrow$$

$$\frac{\partial \rho}{\partial t} = \rho\,\{-\frac{\dot{a}(t)}{a(t)} + \frac{\dot{q}(t)}{a^2(t)} \times [x-q(t)] + \frac{\dot{a}(t)}{a^3(t)} \times [x-q(t)]^2\,\}\;. \qquad (2.1.3.2)$$

Substituting the eq. (2.1.3.2) into eq. (2.1.2.2) and integrating the result, we have (we consider null the integration constant):

$$\rho\,\{-\frac{\dot{a}(t)}{a(t)} + \frac{\dot{q}(t)}{a^2(t)} \times [x-q(t)] + \frac{\dot{a}(t)}{a^3(t)} \times [x-q(t)]^2\} + \frac{\partial(\rho\, v_{qu})}{\partial x} = 0 \quad \rightarrow$$

$$\int \frac{\partial(\rho\, v_{qu})}{\partial x}\partial x = \int \rho\,\{\frac{\dot{a}(t)}{a(t)} - \frac{\dot{q}(t)}{a^2(t)} \times [x-q(t)] - \frac{\dot{a}(t)}{a^3(t)} \times [x-q(t)]^2\}\,\partial x \quad \rightarrow$$

$$\rho\, v_{qu} = \int \rho\,\{\frac{\dot{a}(t)}{a(t)} - \frac{[x-q(t)]}{a^2(t)} \times (\,\frac{\dot{a}(t)}{a(t)} \times [x-q(t)] + \dot{q}(t)\,)\}\,\partial x \quad \rightarrow$$

$$v_{qu} = \frac{1}{\rho}\int \rho\,\{\frac{\dot{a}(t)}{a(t)} - \frac{[x-q(t)]}{a^2(t)} \times (\,\frac{\dot{a}(t)}{a(t)}[x-q(t)] + \dot{q}(t)\,)\,\}\,\partial x\,. \qquad (2.1.3.3)$$

Now, using the eq. (2.1.3.1a), we can right that:

$$\frac{\partial}{\partial x}\{\rho\,(\frac{\dot{a}(t)}{a(t)} \times [x-q(t)] + \dot{q}(t)\,)\} =$$

$$= \rho\,\frac{\partial}{\partial x}\{\,\frac{\dot{a}(t)}{a(t)} \times [x-q(t)] + \dot{q}(t)\} + \{\,\frac{\dot{a}(t)}{a(t)} \times [x-q(t)] + \dot{q}(t)\,\}\,\frac{\partial \rho}{\partial x} =$$

$$= \rho\,\frac{\dot{a}(t)}{a(t)} + \{\frac{\dot{a}(t)}{a(t)} \times [x-q(t)] + \dot{q}(t)\} \times \frac{\partial}{\partial x}\,(\,[2\pi\, a^2(t)]^{-1/2} \times exp\{-\frac{[x-q(t)]^2}{2\,a^2(t)}\}\,) =$$

$$= \rho \frac{\dot{a}(t)}{a(t)} + \{\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t)\} \times [2\,\pi\, a^2(t)]^{1/2} \times exp\{- \frac{[x - q(t)]^2}{2\, a^2(t)}\} \times$$

$$\times \frac{\partial}{\partial x} \{ - \frac{[x - q(t)]^2}{2\, a^2(t)} \} =$$

$$= \rho \frac{\dot{a}(t)}{a(t)} + \{\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t)\} \times \rho\, \{- \frac{[x - q(t)]}{a^2(t)}\} \quad \rightarrow$$

$$\frac{\partial}{\partial x} \{ \rho\, (\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t)) \} =$$

$$= \rho\, \{ \frac{\dot{a}(t)}{a(t)} - \frac{[x - q(t)]}{a^2(t)} \times (\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t))\}\,. \qquad (2.1.3.4)$$

Substituting the eq. (2.1.3.4) into the eq. (2.1.3.3), results:

$$v_{qu} = \frac{1}{\rho} \int \frac{\partial}{\partial x} \{\rho\, (\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t)\,)\} \,\partial x \quad \rightarrow$$

$$v_{qu}(x, t) \equiv \frac{dx(t)}{dt} = \frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t). \qquad (2.1.3.5)$$

We observe that the integration of the eq. (2.1.3.5) give us the *bohmnian quantum trajectory* of the physical system considered. Before, we must calculate the equation for $a(t)$. For that, we expand the functions $S(x, t)$, $V(x, t)$, $V_{qu}(x,t)$ and $V_{BBM}(x, t)$ around of $q(t)$ up the *second Taylor order* [7]. In this way, we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{S''[q(t)\, t]}{2} \times [x - q(t)]^2, \qquad (2.1.3.6)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] \times [x - q(t)] + \frac{V''[q(t), t]}{2} \times [x - q(t)]^2, \qquad (2.1.3.7)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''_{qu}[q(t), t] \times [x - q(t)]^2, \qquad (2.1.3.8)$$

$$V_{BBM}(x, t) = V_{BBM}[q(t), t] + V'_{BBM}[q(t), t] \times [x - q(t)] +$$

$$+\frac{1}{2}V''_{BBM}\,[q(t),t]\times[x-q(t)]^2, \qquad (2.1.3.9)$$

where (’) and (") means, respectively: $\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial q^2}$.

Differentiating the eq. (2.1.3.6) in the variable x multiplying the result by $\hbar/m$, using the eqs. (2.1.2.1c,d) and (2.1.3.5), results:

$$\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x}=\frac{\hbar}{m}\{\,S'[q(t),t]+S''[q(t),t]\times[x-q(t)]\,\}=$$

$$=v_{qu}(x,t)=\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]+\dot{q}(t) \quad\rightarrow$$

$$S'[q(t),t]=\frac{m\,\dot{q}(t)}{\hbar}, \quad S''[q(t),t]=\frac{m}{\hbar}\frac{\dot{a}(t)}{a(t)}. \qquad (2.1.3.10a,b)$$

Substituting the eqs. (2.1.3.10a,b) into the eq. (2.1.3.6), we have:

$$S(x,t)=S_0(t)+\frac{m\,\dot{q}(t)}{\hbar}\times[x-q(t)]+\frac{m}{2\,\hbar}\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]^2, \qquad (2.1.3.11a)$$

where:

$$S_0(t)\equiv S[q(t),t], \qquad (2.1.3.11b)$$

is the *quantum action*.

Differentiating the eq. (2.1.3.11a) in relation to the time t, we obtain (remembering that $\frac{\partial x}{\partial t}=0$):

$$\frac{\partial S}{\partial t}=\dot{S}_0(t)+\frac{\partial}{\partial t}\{\frac{m\,\dot{q}(t)}{\hbar}\times[x-q(t)]\}+\frac{\partial}{\partial t}\{\frac{m}{2\,\hbar}\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]^2\} \rightarrow$$

$$\frac{\partial S}{\partial t}=\dot{S}_0(t)+\frac{m\,\ddot{q}(t)}{\hbar}\times[x-q(t)]-\frac{m\dot{q}^2(t)}{\hbar}+$$

$$+\frac{m}{2\,\hbar}[\frac{\ddot{a}(t)}{a(t)}-\frac{\dot{a}^2(t)}{a^2(t)}]\times[x-q(t)]^2-\frac{m\,\dot{q}(t)}{\hbar}\frac{\dot{a}(t)}{a(t)}\times[x-q(t)]. \qquad (2.1.3.12)$$

Considering the eqs. (2.1.2.1a,b) and (2.1.3.1a,b), let us write V_{qu} given by eq. (2.1.2.1e,f) in terms of potencies of [x - q(t)]. Before, we calculate the following derivations [remembering that $\frac{d}{dz}exp\,(z)=exp\,(z)$]:

$$\frac{\partial\phi(x,t)}{\partial x}=\frac{\partial}{\partial x}(\,[2\,\pi\,a^2(t)]^{-1/4}\times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\}\,)=$$

$$= [2\,\pi\,a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \frac{\partial}{\partial x}\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \;\;\rightarrow$$

$$\frac{\partial\phi(x,t)}{\partial x} = -[2\,\pi\,a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \{\frac{[x-q(t)]}{2\,a^2(t)}\} = -\,\phi \times \{\frac{[x-q(t)]}{2\,a^2(t)}\},$$

$$\frac{\partial^2\phi(x,t)}{\partial x^2} = \frac{\partial}{\partial x}(-\,\phi \times \{\frac{[x-q(t)]}{2\,a^2(t)}\}) =$$

$$= -\,\phi\,\frac{\partial}{\partial x}\{-\frac{[x-q(t)]}{2\,a^2(t)}\} - \frac{[x-q(t)]}{2\,a^2(t)}\,\frac{\partial\phi}{\partial x} =$$

$$= -\,\phi \times \frac{1}{2\,a^2(t)} + \frac{[x-q(t)]^2}{4\,a^2(t)} \times \phi \;\;\rightarrow$$

$$\frac{1}{\phi(x,t)}\,\frac{\partial^2\,\phi(x,t)}{\partial x^2} = \frac{[x-q(t)]^2}{4\,a^4(t)} - \frac{1}{2\,a^2(t)}. \qquad (2.1.3.13)$$

Substituting the eq. (2.1.3.13) in the eq. (2.1.2.1e), taking into account the eq. (2.1.3.8) and considering the identity of polynomials, results:

$$V_{qu}(x,t) = -\frac{\hbar^2}{2\,m}\{\frac{[x-q(t)]^2}{4\,a^4(t)} - \frac{1}{2\,a^2(t)}\} =$$

$$= V_{qu}[q(t),t] + V'_{qu}[q(t),t] \times [x-q(t)] +$$

$$+ \frac{1}{2}V''_{qu}[q(t),t] \times [x-q(t)]^2 \;\;\rightarrow$$

$$V_{qu}(x,t) = \frac{\hbar^2}{4\,m\,a^2(t)};\;\; V'_{qu}(x,t) = 0;\;\; V''_{qu}(x,t) = -\frac{\hbar^2}{8\,m\,a^4(t)} \;\;\rightarrow$$

$$V_{qu}(x,t) = \frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8\,m\,a^4(t)} \times [x-q(t)]^2. \qquad (2.1.3.14)$$

Now, let us write V_{BBM} given by the eq. (2.1.2.4b) in terms of potencies of [x - q(t)]. Initially using the eqs. (2.1.3.1a) and (2.1.3.9) and considering the identity of polynomials [remembering that $\ell n\,(ab) = \ell n\,a + \ell n\,b$, $\ell n\,[exp\,(x)] = x$ and $\ell n\,(a^r) = r\,(\ell n\,a)$], results:

$$V_{BBM}\,(x,t) = \frac{\hbar\,\lambda}{2}\,\ell n \rho =$$

$$= \frac{\hbar\,\lambda}{2} \times \ell n(\,[2\,\pi\,a^2(t)]^{-1/2} \times exp\,\{- \frac{[x-q(t)]^2}{2\,a^2(t)}\,\}\,) =$$

$$= -\,(\frac{\hbar\,\lambda}{2}) \times \{\,\frac{1}{2}\,\ell n\,[2\,\pi\,a^2(t) + \frac{[x-q(t)]^2}{2\,a^2(t)}\,\} =$$

$$= V_{BBM}\,[q(t),t] + V'_{BBM}\,[q(t),t] \times [x-q(t)] +$$

$$+ \frac{1}{2}\,V''_{BBM}\,[q(t),t] \times [x-q(t)]^2 \;\rightarrow$$

$$V_{BBM}\,(x,t) = -\,(\frac{\hbar\,\lambda}{4}) \times \{\,\ell n\,[2\,\pi\,a^2(t)]\,\}, \quad V'_{BBM}\,(x,t) = 0$$

$$V''_{BBM}\,(x,t) = -\,\frac{\hbar\,\lambda}{4a^2(t)} \quad \rightarrow$$

$$V_{BBM}\;(x,t) = -\,(\frac{\hbar\,\lambda}{4}) \times \{\,\ell n\,[2\,\pi\,a^2(t)]\,\} - \frac{\hbar\,\lambda}{4\,a^2(t)} \times [x-q(t)]^2\,. \qquad (2.1.3.15)$$

Using the eqs. (2.1.1.5), (2.1.2.1c-e) and (2.1.2.4a), results:

$$-\,\hbar\,\frac{\partial S(x,t)}{\partial t} = [-\,\frac{\hbar^2}{2\,m}\,\frac{1}{\phi(x,t)}\,\frac{\partial^2\phi(x,t)}{\partial x^2} + \frac{m}{2}(\,\frac{\hbar}{m}\,\frac{\partial S(x,t)}{\partial x}\,)^2\,] + V(x,t) - \frac{\hbar\,\lambda}{2}\,\ell n\,[\phi^2(x,t)] \;\rightarrow$$

$$\hbar\,\frac{\partial S(x,t)}{\partial t} + \frac{m}{2}\,v_{qu}^2(x,t) + V(x,t) + V_{qu}(x,t) - V_{BBM}\,(x,t) = 0\,. \qquad (2.1.3.16)$$

Inserting the eqs. (2.1.3.5,7,9,12,14,15) into eq. (2.1.3.16) we obtain:

$$\hbar\,(\,\dot{S}_0(t) + \frac{m\,\ddot{q}(t)}{\hbar} \times [x-q(t)] - \frac{m\,\ddot{q}(t)}{\hbar} + \frac{m}{2\,\hbar}\,[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}] \times [x-q(t)]^2 -$$

$$-\,\frac{m\,\dot{a}(t)\,\dot{q}(t)}{\hbar\,a(t)} \times [x-q(t)]\,) + \frac{m}{2}\,\{\,\frac{\dot{a}(t)}{a(t)}\,[x-q(t)] + \dot{q}(t)\}^2 +$$

$$+\,V[q(t),t] + V'[q(t),t] + \frac{1}{2}\,V''[q(t),t] \times [x-q(t)]^2 +$$

$$+\frac{\hbar^2}{4\,m\,a^2(t)}-\frac{\hbar^2}{8m\,a^4(t)}\times[x-q(t)]^2+\frac{\hbar\lambda}{4}\times\ell n\,[2\,\pi\,a^2(t)]+\frac{\hbar\lambda}{2\,a^2(t)}\times[x-q(t)]^2=0\ .$$

Since $[x-q(t)]^0=1$, we can gather together the above expression in potencies of [x - q(t)], obtaining:

$$\{\,\hbar\,\dot{S}_0(t)-\frac{1}{2}\,m\,\dot{q}^2(t)+V[q(t),t]+\frac{\hbar^2}{4\,m\,a^2(t)}+\frac{\hbar\lambda}{4}\times\ell n\,[2\,\pi\,a^2(t)]\,\}\times[x-q(t)]^0+$$

$$+\{\,m\,\ddot{q}(t)+V'[q(t),t]\,\}\times[x-q(t)]+$$

$$+\{\,\frac{m}{2}\,\frac{\ddot{a}(t)}{a(t)}+\frac{1}{2}V''[q(t),t]-\frac{\hbar^2}{8\,ma^4(t)}+\frac{\hbar\,\lambda}{2\,a^2(t)}\,\}\times[x-q(t)]^2=0\ . \quad (2.1.3.17)$$

As above relation is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_0(t)=\frac{1}{\hbar}\{\,\frac{1}{2}\,m\,\dot{q}^2(t)-V[q(t),t]-\frac{\hbar^2}{4\,m\,a^2(t)}-\frac{\hbar\,\lambda}{4}\times\ell n\,[2\,\pi\,a^2(t)]\,\}, \quad (2.1.3.18)$$

$$\ddot{q}(t)+\frac{1}{m}V'[q(t),t]=0\ . \quad (2.1.3.19)$$

$$\ddot{a}(t)+\frac{a(t)}{m}V''[q(t),t]-\frac{\hbar^2}{4\,m\,a^3(t)}+\frac{\hbar\lambda}{a(t)}=0\ . \quad (2.1.3.20)$$

Now, let us consider that V[q(t), t] is given by:

$$V[q(t),t]=\frac{1}{2}\,m\,\omega^2(t)\,q^2(t), \quad (2.1.3.21)$$

which is the Time Dependent Harmonic Oscillator Potencial).
In this case, we have:

$$V'[q(t),t]=m\,\omega^2(t)\,q(t), \quad V''[q(t),t]=m\omega^2(t)\ . \quad (2.1.3.22a,b)$$

Putting the eqs. (2.1.3.22a,b) into eqs. (2.1.3.19,20), results:

$$\ddot{q}(t)+\omega^2(t)\,q(t)=0\ , \quad (2.1.3.23)$$

$$\ddot{a}(t)+a(t)\,\omega^2(t)=\frac{\hbar^2}{4\,m\,a^3(t)}-\frac{\hbar\lambda}{a(t)}\ . \quad (2.1.3.24)$$

Assuming that the following initial conditions are obeyed:

$$q(0)=x_o\ , \quad \dot{q}(0)=v_o\ , \quad a(0)=a_o\ , \quad \dot{a}(0)=b_o\ , \quad (2.1.3.25a\text{-}d)$$

and that [see eqs.(2.1.2.1c,d) and (2.1.3.11b)]:

$$S_o(0) = \frac{m\, v_o\, x_o}{\hbar} \;, \quad (2.1.3.26)$$

the integration of the expression (2.1.3.18) will be given by:

$$S_o(t) = \frac{1}{\hbar}\int_o^t dt'\{\frac{1}{2}\, m\, \dot{q}^2(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} - \frac{\hbar\lambda}{4}\, \ell n\, [2\, \pi\, a^2(t')]\} +$$

$$- \frac{m\, v_o\, x_o}{\hbar} \;. \quad (2.1.3.27)$$

Taking into account the expressions (2.1.3.11a,b) in the equation (2.1.3.27) results:

$$S(x, t) = \frac{1}{\hbar}\int_o^t dt'\{\frac{1}{2}\, m\, \dot{q}^2(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} - \frac{\hbar\lambda}{4}\, \ell n\, [2\, \pi\, a^2(t')]\} +$$

$$+ \frac{m\, v_o\, x_o}{\hbar} + \frac{m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m\, \dot{a}(t)}{2\, \hbar\, a(t)} \times [x - q(t)]^2 \;. \quad (2.1.3.28)$$

This result obtained above permit us, finally, to obtain the wave packet for the $BB - ME$, linearized along a classical trajetory. Indeed, considering the equations (2.1.3.1b) and (2.1.3.23), we get: [2]

$$\Psi(x, t) = [2\, \pi\, a^2(t)]^{-1/4}\, exp\, [\, \frac{i\, m}{2\, \hbar}\, \frac{\dot{a}(t)}{a(t)} - \frac{1}{4\, a^2(t)}\,] \times [x - q(t)]^2\} \times$$

$$\times\, exp\, \{\, \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\, m v_o\, x_o}{\hbar}\, \} \times$$

$$\times\, exp\, [\, \frac{i}{\hbar}\int_o^t dt'\, \{\frac{1}{2}\, m\, \dot{q}^2(t') - V[q(t'), t'] -$$

$$- \frac{\hbar^2}{4\, m\, a^2(t')} - \frac{\hbar\lambda}{4}\, \ell n\, [2\, \pi\, a^2(t')]\}\,] \;. \quad (2.1.3.29)$$

2.1.4. The Bohmnian Trajectories for the Bialynicki-Birula-Mycielski Equation

The associated Bohmnian Trajectories, [8]-[12] for the Bialynicki-Birula-Mycielski Equation ($BBM - E$) of an evolving *ith* particle of the ensemble with an initial position x_{0i} can be calculated by considering that:

$$\dot{x}_i(t) = v_{qu}[x_i(t), t] \,. \quad (2.1.4.1)$$

Then substituting the eq. (2.1.4.1) into eq. (2.1.3.5), results:

$$\dot{x}_i(t) = \frac{\dot{a}(t)}{a(t)} \times [x_i(t) - q(t)] + \dot{q}(t) \;\rightarrow\; \frac{\dot{x}_i(t) - \dot{q}(t)}{x_i(t) - q(t)} = \frac{\dot{a}(t)}{a(t)} \;\rightarrow$$

$$\int_0^t \frac{d}{dt}\{\ \ell n\,[x_i(t)-q(t)]\,\}\,dt = \int_0^t \frac{d}{dt}\{\ell n\,[a(t)]\,\}\,dt \quad \to$$

$$\ell n\,\{\frac{[x_i(t)-q(t)]}{[x_{0i}-q_0]}\} = \ell n\,\{\frac{a(t)}{a_0}\} \quad \to \quad \frac{[x_i(t)-q(t)]}{[x_{0i}-q_0]} = \frac{a(t)}{a_0} \quad \to$$

$$x_i(t) = q(t) + (x_{0i}-q_0)\times\frac{a(t)}{a_0}. \quad (2.1.4.2)$$

The eqs. (2.1.3.23,24) show that a continuous measurement of a quantum wave packet gives specific features to its evolution: the appearance of distinct classical and quantum elements, respectively. This measurement consist of monitoring the position of quantum systems and the result is the measured classical path $q(t)$ for t within a quantum uncertainty $a(t)$.

2.1.4.1. The Bohmnian Trajectories for the Bialynicki-Birula-Mycielski Equation in a Stationary Regime

From the eqs. (2.1.3.23,24), we note that for $\lambda \neq 0$ a stationary regime can be reached and that the width [$a(t)$] of the wave packet can be related to the resolution of measurement as follows. Then considering that $a(t) = cte$ [$\dot{a}(t) = 0$] in the eqs. (2.1.3.23,24), we have:

$$a_0\ \omega_0^2 = \frac{\hbar^2}{4\ ma_0^3} - \frac{\hbar\,\lambda}{a_0} \quad \to$$

$$\lambda = \frac{1}{2\,\tau_B} - \frac{a_0^2\omega_0^2}{\hbar}, \quad (2.1.4.1.1a)$$

where [8, 13]:

$$\tau_B = \frac{2\,m\,a_0^2}{\hbar}, \quad (2.1.4.1.1b)$$

is the *Bohmtime constant* which determines the time resolution of the quantum measurement, and:

$$\ddot{q}(t) + \omega^2(t)\,q(t) = 0 \quad \to \quad q(t) = q_0\ exp(\pm\, i\ \omega_0\ t)\ . \quad (2.1.4.1.2)$$

3. Conclusion

The eqs. (2.1.4.1a,b) means that if an initially free wave packet is kept under a certain continuous measurement, its (a_0) may not spread in time. Then, the associated *Bohmnian Trajectories* [eq. (2.1.4.1.2)] of an evolving *ith* particle of the ensemble with an initial position x_{0i} is giving by:

$$x_i(t) = q_0 \times exp\ (\pm\,\omega_0\ t) + (x_{0i}-q_0)\times\frac{a(t)}{a_0}\ . \quad (2.1.4.1.3)$$

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13. If the initial wave packet width (a_0) is taken to be equal to $2{,}8\times10^{-15}\,m$ (the approximate size of an electron of mass m) then τ_B to be about 10^{-25} sec, for a continuous measurement. We note that (see 8.), experiments to measure the size of the electron consist on colliding two beams of electrons against each other and counting how many are scattered and altered their trajectories. By counting the collisions, and knowing how many particles we have thrown, we can estimate the average size of each particle in the beam.