Instituto de Física
Universidade de São Paulo

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Mauro Sérgio Dorsa Cattani

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UNIVERSIDADE DE SÃO PAULO
Instituto de Física
Cidade Universitária
Caixa Postal 66.318
05315-970 - São Paulo - Brasil

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M.Cattani
Instituto de Física, Universidade of São Paulo, Brasil (mcattani@if.usp.br)

Abstract.
This paper was written to graduate and postgraduate students of Physics. We study, using General Relativity and Quantum Mechanics, the emission of gravitational waves by binaries composed by micro non-charged black holes. We made an attempt to introduce quantum effects in the "classical" General Relativity. We analyzed stability, inspiral motion of the binaries and emission of gravitational waves. We concluded that, at least for micro black hole binaries, seems possible that effects of weak gravitation interaction can be quantized in the non relativistic limit of Schrödinger´s equation.
Key words. micro black hole binary; gravitational quantum effects; gravitational waves.

(I) Introduction.

This is a didactical paper written to graduate and postgraduate students of physics. Our intention is to investigate only basic aspects about emission of gravitational waves (**GW**) by *microscopic* binaries composed by two non-charged micro black holes (**mBBH**). Are used classical mechanics[1], classical electrodynamics,[2],quantum mechanics (**QM**),[3,4] special relativity **(SR)** and general relativity(**GR**).[5] In Section 1 are given significant parameters associated with micro black holes (**mBH**). In Section 2 using GR are estimated gravitational luminosity L_{GW} and the "spiral time" τ of mBBH. In Section 3 we study the *microscopic* mBBH with Schrödinger equation and show how one could calculate the total gravitational energy per unit of time dE/dt emitted by the mBBH using an "hybrid" GR and QM approach. It is also shown that the dE/dt and the "spiral time" τ of the mBBH calculated with the hybrid approach is in good agreement with the L_{GW} and τ estimated with the "classical" GR theory. Finally, in Section 4 are presented conclusions of our analysis.

(1) Significant Parameters associated with mBH.

In Figure 1 is shown a binary that we assume composed by two non-charged micro black holes (mBH).[5,7]

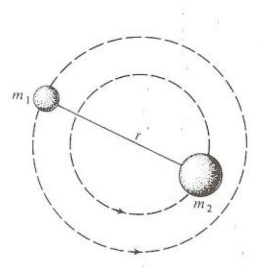

Figure 1. Binary system of non-charged mini black holes (mBBH).

The BH mass M according to the classical GR[5] can be arbitrarily small, however, the smallest M is estimated by Planck mass[8] $M_P = (\hbar c/G)^{1/2}$. The BH Schwarzschild radius[9] r_s and lifetime τ_{ev} [10] ("evaporation time") are estimated, respectively, by $r_s = 2GM/c^2$ and $\tau_{ev} = 5120\pi G^2M^3/(\hbar c^4)$. Here the Planck mass M_P, the radius r_s , the lifetime τ_{ev} , the "gravitational Bohr radius" $(a_o)_g$, the metric tensor component $g_{oo}(r)$ and a relativistic parameter β are written in terms of the constants c,G and ħ, in the MKS system,

$$M_P = (\hbar c/G)^{1/2} \sim 2\ 10^{-8}\ \text{(kg)} \qquad (1.1),$$

$$r_s = 2GM/c^2 \sim 1.5\ 10^{-27}\ M \quad \text{(m)} \qquad (1.2),$$

$$\tau_{ev} = 5120\pi G^2M^3/(\hbar c^4) \sim 4\ 10^{-18}\ M^3\ \text{(s)} \qquad (1.3),$$

$$(a_o)_g = \hbar^2/GM^3 \sim 10^{-58}/M^3\ \text{(m)} \qquad (1.4),$$

$$g_{oo}(r) = -1 - 2GM/rc^2 \qquad (1.5)$$

and

$$\beta = (r_s/r)^{1/2} \qquad (1.6).$$

(2)Gravitational Luminosity of the mBBH according to GR.

Gravitational waves emitted by a binary black hole **(BBH)** formed by BH with M ~ 10 - 30 solar masses have been recently detected by Abbott et al.[11,12] The BBH motion is unstable; this unstable motion can be divided into three stages:[11-13] "inspiral", "merger" (or "plunge") and "ringdown". During this motion the BBH emits GW. The "inspiral" is the first stage of the BBH life which resembles a gradually shrinking orbit and take a longer time; the emitted GW are weak when BH are distant from each other.

During the "inspiral" motion of a BBH binary with $m_1 = m_2 = M$ the gravitational luminosity L_{GW} would be given by [5,13-16] (see **Appendix A**)

$$L_{GW} = dE/dt = -(8G/5c^5)\,M^2r^4\omega^6 \qquad (2.1),$$

where r is distance between the BH and ω is the **orbital rotational frequency**. With Kepler´s law[1,5] $r(t)^3\omega(t)^2 = 2GM$ we get

$$|L_{GW}(\omega)| = (8G/5c^5)\,M^2r^4\omega^6 \sim 10^{-192/3}\,(M\omega)^{10/3} \qquad (2.2)$$

or

$$|L_{GW}(r)| = -(8G/5c^5)\,M^2r^4\omega^6 \sim 10^{-84}\,(M/r)^5 \qquad (2.3).$$

In addition, as $r_s = 2GM/c^2 \sim 1.5\ 10^{-27}$ M we get

$$\omega_{max} \sim 10^{26}M^{1/2} \qquad (2.4).$$

The "**spiral time**" τ [5,16] of the BBH is estimated writing the total mechanical energy E of the BBH as $E = I\omega^2/2 - GM^2/r$ that can be written, using the "virial" theorem,[1] as $E = -GM^2/2r$. Taking into account this equation and (2.1) we verify that[5]

$$dr/dt = - (128/5c^5)\, G^3M^3/r^3 \quad (2.5),$$

that is,

$$r^3\, dr/dt = (1/4)d(r^4)/dt = -(128/5c^5)\, G^3M^3 \quad (2.6).$$

Integrating (2.6) from r_o up to $2r_s$ we get

$$r_o^4 = (2r_s)^4 - (128/5c^5)\, G^3M^3\, \tau \quad (2.7),$$

where τ, that is also called **"time to fall"** from a generic orbit $r = r_o$ to the closest distance $2r_s$ between two BH, is given by :

$$\tau = [5c^5/(128\, G^3M^3)]\, (r_o^4 - 16r_s^4) \quad (2.8).$$

In Section 1 we saw that the minimum mBH mass is $M \sim M_P = 10^{-8}$ k; this would imply that its "lifetime" is $\tau_{ev} = 5120\pi G^2M^3/(\hbar c^4) \sim 4\ 10^{-18}\ M^3 \sim 9\ 10^{-40}$ s, which is an *astronomically* small lifetime. So, if GW are emitted, for instance, in a time interval $\Delta t \sim 10^{-8}$ s, masses $M \sim 10^{-8}$ kg cannot be considered to study the GW emission. Thus, in this paper we will suppose that the lifetime of the mBH is $\tau_{ev} \sim 60$ s. So, to satisfy this condition we see, using (1.3), that the mBH mass must be **$M \sim 10^6$ kg**. For this mass, using (1.1)-(1.4) the Schwarzschild radius $r_s \sim 1.5\ 10^{-27}\ M \sim 10^{-21}$ m. For these mass values the mBBH can be taken as a *microscopic system.*

(2.1) Numerical Estimations of L_{GW} and τ for $M = 10^6$ kg.

Kepler´s law establishes a constraint between $\omega(t)$ and $r(t)$. The maximum values of $\omega(t)$ occurs for the minimum value of $r(t)$ and vice-versa. So, putting $M = 10^6$ kg in (1.2) and (2.2)-(2.4) we get $r_s \sim 10^{-21}$ m, $\omega_{max} \sim 10^{29}$ rad/s and the maximum luminosity

$$|L_{GW}|_{max} = |L_{GW}(\omega_{max})| = |L_{GW}(r_s)| \sim 10^{41}\ J/s = 10^{41} W \quad (2.9).$$

The time τ to fall from $r_o \sim 100\ r_s$ m up to $2r_s \sim 10^{-21}$ m given by (2.8) is

$$\tau = [5c^5/(128\, G^3M^3)]\, (r_o^4 - 16r_s^4) \sim 3\ 10^{53}\ 10^{-76} \sim 10^{-17}\ s \quad (2.10),$$

that is, the gravitational energy would be "instantaneously" emitted , like a "flash".

In recent GW observations[11,12] known as GW150914 and GW151226 the BBH were composed by BH with masses $M \sim 10^{30}$ kg. The measured GW frequencies are in the range 30-500 Hz, the peaked luminosities $L_{GW} \sim 10^{49}$ W and spiral times $\tau \sim 1$ s.

(3)mBBH described by Schrödinger´s Equation.

Let us admit (as will be shown below) that the mBBH is a *small* system in Dirac´s[6] sense that can be described, in the spiral stage, by the Bohr hydrogen one-electron atom (**HLA**)[3,4] theory for very large quantum numbers. So, mBBH would obey Schrödinger´s equation (**SE**) replacing the electrostatic interaction by the gravitational interaction given by the Hamiltonian

$$H = \{(\hbar^2/2\mu)\Delta - GM^2/r\}\Psi(r,\theta,\varphi) = E\Psi(r,\theta,\varphi) \quad (3.1),$$

where r is distance between the mBH, Δ the laplacian operador in spherical coordinates and $\mu = m_1 m_2/(m_1+m_2) = M/2$ is the reduced mass of the system. Note that in semi-classical limit [3,4] would be possible to construct a phase space (p,q) to describe the quantization of the mBBH orbits. Solving (3.1)[3,4] the gravitational energies E^g_n of the mBBH are given by (in the MKS system of units),

$$E^g_n = - \Theta_{grav}/n^2, \qquad (3.2),$$

where n =1,2,3,...and $\Theta_{grav} = (M/2)(GM^2)^2/2\hbar^2 = G^2M^5/4\hbar^2$. Since $G \sim 10^{-10}$ MKS and $\hbar \sim 10^{-34}$ MKS we have

$$\Theta_{grav} = G^2M^5/4\hbar^2 \sim 10^{47}\, M^5 \quad J \qquad (3.3),$$

showing that the mBBH, which is a *microscopic system,* can also be taken as *small* in Dirac´s sense depending on the mass M as will be seen below.

For the HLA with charge Z we have,[3,4]

$$E^{elet}_n = - \Theta_{eletr}/n^2 \qquad (3.4),$$

where $\Theta_{eletr} = Z^2 m_e e^4/2\hbar^2$. That is,[3,4]

$$\Theta_{eletr} = Z^2\, 13.6\ eV \sim Z^2\, 10^{-18} \quad J \qquad (3.5).$$

The mBBH would be described by the normalized energy eigenfunctions $u_{n\ell m}(r,\theta,\varphi)$ given by

$$u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\, |\ell m> \qquad (3.6),$$

where the functions $R_{n\ell}(r)$ and $|\ell m> = Y_{\ell m}(\theta,\varphi)$ are shown in references,[3,4] remembering that $n = 1,2,...,\ \ell = 0,1,2,..,n-1$ and $m = -\ell, -\ell+1,...,\ell-1, \ell$.

For the HLA the "electromagnetic Bohr radius" $(a_o)_{elet}$ is given by[3,4] $(a_o)_{elet} = \hbar^2/me^2 \sim 0.5\ 10^{-10}$ m and the "**gravitational Bohr radius**" for the mBBH is given by $(a_o)_g = \hbar^2/G^2M^3$. From (3.2)-(3.5) we verify that the energies $E^g_n = E^{elet}_n$ if $M \sim 10^{-13}$ kg; in this case the mBBH would be *small* in Dirac´s sense. The orbit radius r_n are given by $(r_n)_{elet} = n^2(a_o)_{Bohr} = n^2(\hbar^2/me^2) \sim n^2\, 0.5\ 10^{-10}$ m and $(r_n)_g = n^2(a_o)_g = n^2(\hbar^2/G^2M^3)$.

According to Kepler´s law $\omega(t)^2 r(t)^3 = Ze^2/\mu$ for the HLA and $\omega(t)^2 r(t)^3 = 2GM$ for mBBH. Since $v = \omega r$ the orbital relativistic parameter $\beta = (v/c)$ for the mBH will be given $\beta = (1/r)^{1/2}(2GM/c^2)^{1/2}$. If we believe that only the fundamental state n = 1 is stable, similar to HLA,[3,4] it seems reasonable believe that gravitational waves (GW) would be emitted in "spontaneous" decay transitions between the quantum states $u_{n\ell m}(r,\theta,\varphi) \rightarrow u_{n'\ell' m'}(r,\theta,\varphi)$. At this point we ask**:"what kind of interaction would be responsible for these transitions?"** It will be analyzed in Section(3.3).

(3.1)mBBH Stability.

The HLA ground state $u_{n\ell m}(r,\theta,\varphi)$ (n = 1) is stable.[3,4] In this state the atomic radius $r \sim 10^{-10}$ m is much larger the nuclear radius $\sim 10^{-15}$ m. That is, the electron can be thought as moving in a orbit very far from nucleus. Supposing that this condition is essential to the stability of the HLA we "take for granted" that mBBH ground state cannot be stable if inside the sphere with radius $(a_o)_g = \hbar^2/G^2M^3$ there is "contact" between the mBH, which one with radius r_s. So, we assume that stability cannot exist if $4r_s > (a_o)_g$. Using (3.1) and (3.4) this condition is written as $8GM/c^2 > \hbar^2/G^2M^3$. Thus,

$M^4 > (\hbar^2/G^3c^2)/8$, that is, $M > 0.5\ (\hbar/c)G^{-3/2} \sim 10^{-27}$ kg. So, we see that the mBBH would be **unstable** if

$$M > 10^{-14}\ \text{kg} \qquad (3.7).$$

Since it we have assumed in Section 1 that $M = 10^6$ kg our mBBH is **unstable**. The BH unstable motion can be divided into three stages:[11-13] "inspiral", "merger" (or "plunge") and "ringdown". During this motion the system emits GW. The "inspiral" is the first stage of the mBBH life which resembles a gradually shrinking orbit and take a longer time; the emitted GW are weak when the mBH are distant from each other ($r >> r_s$). As the mBBH orbit shrinks, the mBH speed increases, and the GW emission increases. When the mBH are close ($r \sim r_s$) the GW cause the orbit to shrink rapidly. In the final fraction of a second the mBH can reach extremely high velocity. This is followed by a plunging orbit and the mBH will "merge" once they are close enough ($r \leq r_s$). At this time the GW amplitude reaches its peak. Once merged, the single hole settles down to a stable form, via a stage called “ringdown", where any distortion in the shape is dissipated as more gravitational waves.

(3.2) Inspiral motion.

For $M = 10^6$ kg, by (1.2) the Schwarzschild radius $r_s \sim 1.5\ 10^{-27} M \sim 10^{-21}$ m, $(a_o)_g = \hbar^2/G^2M^3 \sim 10^{-66}$ m and the binary "quantum radius" would be given by $(r)_g = n^2\ 10^{-66}$ m. The energies E_n, (3.2) and (3.3), are given by $E^g{}_n = -10^{77}/n^2$ J $\sim -10^{96}/n^2$ eV. As $r_s \sim 10^{-21}$ m the mBH would be well distant when $(r)_g > 10^{-21}$ m, that is, only when $n > 10^{22}$. For $r \geq 10^{-20}$ m the binary is still a ***microscopic system***, about 10^7 times smaller than the HLA. For $r \geq 10^{-20}$ m we get, using (1.5), that $g_{oo}(r) \sim -1$ showing that gravitational distortions of the metric are negligible.[5] If $r_s \sim 10^{-21}$ m and the mBBH radius $r = r_n = n^2 10^{-66}$ m see, from (1.6), that $\beta \sim 3\ 10^{27}/n^2$. That is, for $n > 10^{22}$ we have $\beta < 1$ showing that relativistic effects are also negligible. Rigorously we can say that the inspiral motion is restricted to distances $r > r_s$, that is, for $n > 10^{22}$. Higher energy GW would be generated by transitions for distances $(r)_g \sim r_s$. Let us assume that the inspiral motion is restricted for n values in the range $n \sim 10^{21} - 10^{23}$. For these very large n values we see that energies $\hbar\omega$ in the transitions $n \rightarrow n + 1$ are given by

$$\hbar\omega = E^g{}_{n+1} - E^g{}_n = -10^{77}[1/(n+1)^2 - 1/n^2] \approx 10^{96}/n^4\ \text{eV} \approx 10^{77}/n^4\ \text{J}. \qquad (3.8).$$

In the inspiral region the frequencies ω are in the range $10^{29} - 10^{19}$ rad/s. Note that the recently observed GW frequencies[11,12] are $\omega \sim 100 - 200$ rad/s.

(3.3)GW emitted in the inspiral motion according to Schrödinger´s approach.

Let us see how we could estimate the gravitational luminosity of the mBHH supposing that GW with energies $\hbar\omega = E^g{}_{n+1} - E^g{}_n$ given by (3.8) are emitted in transitions $n \rightarrow n + 1$. To do this we admit (*without proving*) that there is some kind of interaction (what kind?) that induces transitions between the quantum states $|\, n >$. It will be done using the perturbation theory derived from Schrödinger´s equation. So, assuming that the this interaction is represented by an time operator $W(t)$ which depends harmonically on the time given by[4]

$$W^{\pm}(t) = w^{\pm} \exp[\pm i\omega t] \qquad (3.9),$$

where $w^{\pm}$ is time independent. It can be shown[4] that the transition probability m → n per unit of time $P^{\pm}_{nm}$ will be given by the equation

$$P^{\pm}_{nm} = (2\pi/\hbar) |< n | w^{\pm} | m >|^2 \delta(E_n - E_m \pm \hbar\omega) \qquad (3.10),$$

where the + and - correspond to the signs in the exponential in (3.9). Thus, under the action of the perturbation, periodic in time, transitions take place to states with energies satisfying the condition $E_m = E_n \pm \hbar\omega$. Thus, if the perturbation is of the form $W^+(t) = w^+ \exp(i\omega t)$ the system loses an energy ħω (energy is emitted), since $E_n = E_m - \hbar\omega$ in the transition while if it is of the form $W^-(t) = w^- \exp(-i\omega t)$ it gains an energy ħω , since $E_n = E_m + \hbar\omega$. Our main problem is to determine the function $W^{\pm}(t)$. The gravitational "luminosity" $(L_{GW})_{nm}$ in the *inspiral* stage would estimated by $(L_{GW})_{nm} = \hbar\omega P^+_{nm}$ for very large quantum numbers.

Before to propose a model to obtain $W^+(t)$ let us remember that according to Bohr correspondence principle (CP)[3] for very large quantum numbers, classical and quantum physics are expected to give the same answer, at least in average. The probabilistic interpretation of the phenomenon obtained with the Schrödinger´s equation will give, in average the same results obtained by classical laws. Ehrenfest, for instance, showed that Newton's laws hold on average: the quantum statistical expectation value of the position and momentum obey Newton's laws. Thus, we expect that in the inspiral stage mBBH properties estimations given by the "classical" GR and QM laws agree *in average*. In addition, we show that (see **Appendix B** and **C)** that in Classical Electrodynamics the luminosities L_ω emitted dipolar and quadrupole radiation are given, respectively, by $L_\omega = dE/dt = (ck^4/3) |\mathbf{D}|^2 = (\omega^4/3c^3) |\mathbf{D}|^2$ and $L_\omega = dE/dt = (\omega^6/360c^5)\sum_{\alpha\beta} |Q_{\alpha\beta}|^2$. In Quantum Electrodynamics these are given, respectively, by $L_\omega = (4\omega^4/3c^3) |\boldsymbol{D}_{nm}|^2$ and $L_\omega \approx (\omega^6/2\pi c^5)|Q_{nm}|^2$, where $\omega = \omega_{nm}$, $\boldsymbol{D}_{nm} = < n |\mathbf{D} |m >$ and $Q_{nm} = < n | Q | m >$. Finally, according to the "classical" GR estimations the luminosity, in the inspiral stage, L_{GW} is given by the quadrupolar radiation (2.1):

$$L_{GW} = (32\mu^2G/5c^5)r^4\omega^6 = (8G\omega^6/5c^5) M^2r^4 = (8G\omega^6/5c^5)Q^2 \qquad (3.11),$$

where $Q = Mr^2$ is the mBBH mass quadrupole. Thus, by analogy with the predicted electromagnetic radiation and based in the CP we propose that the QM gravitational luminosity $(L_{GW})_{nm}$ can be estimated by

$$(L_{GW})_{nm} = \hbar\omega P^+_{nm} \approx (8G\omega^6/5c^5) | < n | Q | m >|^2 \qquad (3.12).$$

A somewhat different approach proposed by Weinberg[15] to calculate $(L_{GW})_{nm}$ is shown in **Appendix D.**

Now**,** let us give a reasonable justification for (3.12). Thus, let us suppose that $W^+(t)$ is proportional to the small perturbations $h_{\mu\upsilon}$ of the tensor metric $g_{\mu\upsilon}$ created by the quadrupole temporal oscillations $Q_{\alpha\beta}(t)$[14,19] of the mBBH that are given by

$$Q_{xx}(t) = 3\mu r^2[1 + \cos(2\omega t)]/2 \quad \text{and} \quad Q_{yy}(t) = 3\mu r^2[1- \cos(2\omega t)]/2 \qquad (3.13).$$

where $\mu = m_1m_2/(m_1 + m_2)$ and ω is the orbital angular frequency (see **Appendix A).** That is, $g_{\mu\nu}$ is slightly modified, $g_{\mu\upsilon} \approx g_{\mu\upsilon}^{(o)} + h_{\mu\upsilon}$, where $h_{\mu\upsilon}$ is due to quadrupolar effects pointed above. Taking into account that [14,19] $h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2r)(\partial^2Q_{\alpha\beta}/\partial t^2)$ and that the "classical" gravitational luminosity L_{GW} we saw that (**Appendix A**)

$$L_{GW} = (G/45c^5) < \dddot{Q}_{\alpha\beta}{}^2 > = (G/45c^5) [< \dddot{Q}_{xx}{}^2 > + < \dddot{Q}_{yy}{}^2 >] =$$

$$= (32\mu^2 G/5c^5) r^4 \omega^6 = (8G\omega^6/5c^5) Q^2 \qquad (3.14),$$

where $Q = Mr^2$ is the mBBH mass quadrupole. So, admitting that $(L_{GW})_{nm} = \hbar\omega P^+_{nm}$, $w^+(t) \sim h_{\alpha\beta}(t)$ and using (3.10) we will admit that the QM gravitational luminosity $(L_{GW})_{nm}$ could be estimated by

$$(L_{GW})_{nm} = \hbar\omega P^+_{nm} \approx (8G\omega^6/5c^5) \,|< n \,|\, Q \,|\, m >|^2 \qquad (3.15),$$

in agreement with (3.12). At this point it is important to analyze this proposed mechanism to explain the decay transitions in mBBH. Indeed, as seen in **Appendix A**, the amplitude of the emitted GW are given by $\Psi_{\alpha\beta}(t,\mathbf{x}) = h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2R)(\partial^2 Q_{\alpha\beta}/\partial t^2)$. That is, GW are essentially emitted due to the metric modification $h_{\alpha\beta}(t)$. To obtain (3.15) a similar hypothesis is assumed: the metric modification is responsible for a potential interaction W^+ that induces transitions $n \rightarrow m$ between quantum states. The gravitational luminosity would now be given by $(L_{GW})_{nm} = \hbar\omega P^+_{nm}$. That is, quantum transitions are created by interactions of the mBBH with the "metric" not with the usual "vacuum" of the quantum field theory.

(3.4)Numerical Estimation of $(L_{GW})_{nm}$.

Let us compare the L_{GW} emitted in the **inspiral stage** given by (2.1), using the "classical" GR, with our hybrid GR&QM approach given by (3.12). Thus, putting in (3.12) $M = 10^6$ kg and taking $| n > \rightarrow | m > = | n+1 >$, $\omega = \omega_{nm} = (E_n - E_m)/\hbar$ and

$$|< n \,|\, Q \,|\, m >|^2 \sim [2M < n |r^2| m >]^2 = 4 M^2 |< n |r^2| n+1 >|^2 = 4 M^2 \,|(r^2)_{n,n+1}|^2$$

we have

$$(L_{GW})_{nm} \sim 10^{-41} \omega_{n,n+1}{}^6 \,|(r^2)_{n,n+1}|^2 \qquad (3.16).$$

As in the inspiral stage, according to (3.8), $\hbar\omega = \hbar\omega_{n,n+1} = E^g{}_{n+1} - E^g{}_n \approx 10^{77}/n^4$ J, the most significant contributions to the luminosity occurs when n is the range $n \sim 10^{21}$-10^{23} with frequencies in the range $\omega \sim 10^{29}$ - 10^{19} rad/s. Taking, e.g., $\omega \sim 5\ 10^{27}$ rad/s and $| r_{n,n+1}| \sim 10^{-20}$ m we get $\mathbf{(L_{GW})_{nm} \sim 10^{41}\ W}$, in good agreement with $\mathbf{|L_{GW}|_{max} \sim 10^{41}\ W}$ given by (2.9) evaluated with the "classical" GR. We think that this agreement attained for very large quantum numbers $n > 10^{21}$ would be expected according to the CP.

To evaluate the QM "spiral time" τ we must remember that in this stage, according to Section 3 the energy levels $E^g{}_n = -\Theta_{grav}/n^2$ are very close since quantum numbers are very large, $n > 10^{21}$. As we have a "continuum of levels" it is expected, according to the CP, that the mBBH description given by quantum mechanics approaches asymptotically a state of motion obtained with the "classical" GR. Indeed, for the inspiral stage (3.11) can be written as

$$(L_{GW})_{ab} = (dE/dt)_{ab} \approx (8G\omega^6/5c^5) M^2 r^4 = (8 M^2 G\omega^6/5c^5)\, r^4 \qquad (3.17).$$

which is similar to (2.1) given by the "classical" GR. Integrating (3.14) as was done in Section 2 we get for the spiral time τ the same result predicted by (2.8).

Finally, if $| a >$ and $| b >$ of the mBBH are represented by $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\, |\ell m >$ the quadrupole matrix elements would be written as (**Appendix C.3**).

$$Q_{ab} = \int dr\, r^4\, R_a(r)\, R_b(r) < \ell_b m_b | Y_{2m}^*(\theta,\varphi) | \ell_a m_a > \qquad (3.18),$$

showing that, according to the Wigner-Eckart Theorem,[4] quadrupole **t**ransitions $a \rightarrow b$ are allowed only if $\ell_b = \ell_a \pm 2$ and $m_b = m_a + 2$. If GW are composed by "gravitons", as electromagnetic waves are composed by photons with spin 1, selection rules (3.18) would suggest that "gravitons" have spin 2 (see **Appendix D**).

(4)Conclusions.

From the exposed above it seems reasonable to believe (at least for the fantastic mBBH) that effects of weak gravitation interaction can be quantized in non relativistic limit of Schrödinger´s equation.

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Appendix A. Gravitational Waves Emitted by BBH.

In GR [5,14-16], assuming that the gravitation field is *weak* and that the bodies have small velocities compared with the light velocity, the space-time metric tensor $g_{\mu\upsilon}$ we can put $g_{\mu\upsilon} \approx g_{\mu\upsilon}^{(o)} + h_{\mu\upsilon}$, where $h_{\mu\upsilon}$ is as mall perturbation of $g_{\mu\upsilon}^{(o)}$.[5,14-16] In the Newtonian limit we have $g_{oo} = - 1 - 2\varphi/c^2$, where $\varphi = GM/r$.[5] In these conditions the Ricci tensor R_{ik} can be written as

$$R_{ik} = - (1/2)\Box h_{\mu\upsilon} \qquad (A.1).$$

Defining the gravitational field as $\Psi_{\mu\upsilon} = h_{\mu\upsilon} - (1/2)\delta_{\mu\upsilon} h$, where $h = h_\alpha{}^\alpha$, in *weak field* limit the field $\Psi_{\mu\upsilon}$ obeys the equations[5, 14-16]

$$\Box\Psi_{\mu\upsilon} = - (16\pi G/c^4)\tau_{\mu\upsilon} \quad \text{and} \quad \partial_\mu \Psi^{\mu\upsilon} = 0 \quad (\textit{gauge condition}) \quad (A.2),$$

where $\tau_{\mu\upsilon}$ is a pseudo-tensor mass-energy momentum.

The solution of (A.2) for retarded times is given by[5,18]

$$\Psi_{\mu\upsilon}(\mathbf{x},t) = - (4G/c^4) \int \tau_{\mu\upsilon}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x})\, d^3\mathbf{x}' / |\mathbf{x} - \mathbf{x}'| \qquad (A.3),$$

where the integration is over the volume V of the system.

Supposing that gravitational effects are observed very far from the origin O ("wave zone") where they are produced, that is, $|\mathbf{x}| = R >> |\mathbf{x}'|$ we get from (A.3), remembering that we have a retarded time function $\tau_{\mu\upsilon}$:

$$\Psi_{\mu\upsilon}(x,t) \approx - (4G/c^4 R) \int \tau_{\mu\upsilon}\, d^3\mathbf{x}' \qquad (A.4).$$

Integrating (A.4) over the volume V we obtain the gravitational field[5,13]

$$\Psi_{\alpha\beta}(x,t) = (2G/c^2 R)\, (\partial^2 Q_{\alpha\beta}/\partial t^2) \qquad (A.5),$$

where $Q_{\alpha\beta}$ is the *mass quadrupole moment* of the emitting system defined by

$$Q_{\alpha\beta} = \int \rho_o(\mathbf{x'})(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\, d^3\mathbf{x'}$$

where ρ_o is the mass density. At this point it opportune to remember that gravitational multipoles are defined by the potential expansion [14]

$$\varphi(\mathbf{x}) = -G \int \rho_o(\mathbf{x'})d^3\mathbf{x'}/|\mathbf{x} - \mathbf{x'}| \approx - Gm/r - (G/r^3)\, \mathbf{x.D} - (G/2r^5)\Sigma_{\alpha\beta}\, Q^{\alpha\beta}\, x^\alpha x^\beta + ... \quad \text{(A.6)},$$

where $m = \int \rho_o(\mathbf{x'})\, d^3\mathbf{x'}$, $\mathbf{D} = \int \rho_o(\mathbf{x'})\, \mathbf{x'} d^3\mathbf{x'}$ and $Q_{\alpha\beta} = \int \rho_o(\mathbf{x'})(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\, d^3\mathbf{x'}$.

The *mass dipole moment* is null ($\mathbf{D} = 0$) since the origin of coordinates O is chosen to coincide with the center of mass.

In vacuum we have the *traditional wave equations*

$$\Box\Psi_{\mu\upsilon} = \Box h_{\mu\upsilon} = 0 \qquad \text{with the "gauge "} \quad \partial(h^\mu{}_\nu)/\partial x^\mu = 0 \qquad \text{(A.7)}$$

showing that the gravitational field propagates with the light velocity. Note that the tensor field $h_{\mu\upsilon}$ is obtained integrating (A.4) as will be seen later.

At this point we find a fruitful analogy with the electromagnetism. The Maxwell equations in *Lorentz gauge* in empty space are $\Box A_\mu = 0$ and $\partial A^\mu/\partial x^\mu = 0$.

Let us consider a plane GW, that is, a field that changes only in one direction z of the space. Choosing $z > 0$ as the direction of propagation of the wave we can write $h_{ik} = h_{ik}(t - z/c)$. So, the wave equation (A.7) becomes

$$[\partial^2/\partial z^2 - (1/c^2)\,(\partial^2/\partial t^2)]\, h_{ik} = 0 \qquad \text{(A.8)}$$

that has the familiar solution with the gauge condition,

$$h_{ik}(z,t) = A_{ik} \cos(k_\mu x_\mu) \qquad \text{(A.9)},$$

where $k_\mu = (0,0,k,\omega)$, $k = k_z = |\mathbf{k}| = \omega/c$ is the wave vector and ω is the frequency of the wave. As $h_{ik}(z)$ obey (A.8) the following conditions are obeyed: $A_{\beta\alpha}k^\alpha = 0$ and $k_\alpha k^\alpha = 0$. Under these conditions the **amplitude tensor A_{ik}** has only 4 non-null components $A_{11} = - A_{22}$, $A_{12} = A_{21}$ with the condition $Tr(A_{ik}) = A_i{}^i = 0$ and only the following **transversal components** to the z-direction of propagation: $A_{xx} = -A_{yy}$ and $A_{xy} = A_{yx}$.

$$A_{ik} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The transversal fields h_{xx}, h_{yy} and h_{xy} are represented using (2x2) matrices called polarization matrices $(\varepsilon_+)_{ik}$ and $(\varepsilon_x)_{ik}$:

$$(\varepsilon_+)_{ik} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad (\varepsilon_x)_{ik} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(A.10)}$$

The general solution of (B.8) can be written as a linear combination of the fields h_{ik}, with polarizations (+) and (x), respectively:

$$h_{ik}^{(+)} = h_+ (\varepsilon_+)_{ik} \cos(\omega t - kz) \quad \text{and} \quad h_{ik}^{(x)} = h_x (\varepsilon_+)_{ik} \cos(\omega t - kz + \alpha) \qquad (A.11),$$

where $h_+ = A_{11}$, $h_x = A_{12}$ and α is an arbitrary phase. The tensorial polarization of the GW creates an effect much more complicate than the linear polarization of the electromagnetic waves. These fields deform the space-time creating tidal (shear) on the matter . The line forces due to the polarizations (X) and (+) are shown in Figure 2.

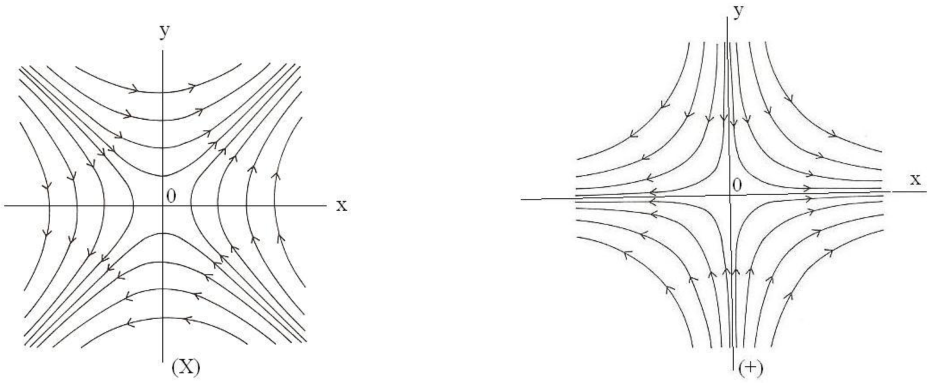

Figure 2. Line forces due to the polarizations (X) and (+).

The total energy emitted per unit of time dE/dt or "gravitational luminosity" L_{GW} is given by[5,14]

$$L_{GW} = dE/dt = -(G/45c^5) < (\partial^3 Q_{\alpha\beta}/\partial t^3)^2 > \qquad (A.12),$$

where the brackets indicates a time average and are taken into account the effect of all components of the quadrupole tensor. Note that the GW is a tensor function not a scalar function like an electromagnetic wave.

(A.1)GW emitted by BBH.

For a binary system (see **Fig.1**) composed by stars with masses m_1 and m_2 separated by a distance r one can show[14,19] that

$$Q_{xx} = 3\mu r^2[1 + \cos(2\omega t)]/2 \quad \text{and} \quad Q_{yy} = 3\mu r^2[1 - \cos(2\omega t)]/2 \qquad (A.13),$$

where $\mu = m_1 m_2/(m_1 + m_2)$ and ω is the **orbital angular frequency.** In these conditions one see that $h_{\alpha\beta}(t,\mathbf{x})$, using (C.11) and (C.13), would be given by

$$\Psi_{\alpha\beta}(t,\mathbf{x}) = h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2R)(\partial^2 Q_{\alpha\beta}/\partial t^2) \sim h \cos(2\omega t) \qquad (A.14),$$

where $h = 6\mu G r^2/Rc^2$. Showing that the GW frequency is $\boldsymbol{\omega_g = 2\omega}$.

Using (A.12) and (A.13) we obtain

$$L_{GW} = (G/45c^5) < \dddot{Q}_{\alpha\beta}^{\,2} > = (G/45c^5)\,[< \dddot{Q}_{xx}^{\,2} > + < \dddot{Q}_{yy}^{\,2} >] = (32\mu^2 G/5c^5) r^4 \omega^6 \quad (A.15).$$

As the energy of the GW in the radiation zone is transported by a **plane wave** with amplitude h and rotation frequency ω one can show that[13,14]

$$h^2 = (8\pi G/\omega^2 c^3)\,(L_{GW}/4\pi R^2) \qquad \text{(A.16).}$$

Kepler´s law for a binary [1,5] says that $\omega^2 r^3 = G(m_1 + m_2)$; as $M = m_1 \sim m_2$ results $r = (2GM/\omega^2)^{1/3}$. Substituting this r value in (C.16) we get h as a function of the **orbital angular frequency ω (rad/s):**

$$h(\omega) = (4GM/Rc^4\sqrt{36})(2GM/\omega^2)^{2/3}\omega^2 = (4^{2/3}/\sqrt{36})\,[(GM)^{5/3}/Rc^4]\,\omega^{2/3} \qquad \text{(A.17),}$$

that shows a good agreement with the Abbott et al[11,12] experimental results.

Appendix B. Classical Electromagnetic Radiation.

According to classical Electrodynamics[2]

$$\Box \mathbf{A}(\mathbf{x},t) = -\mu_o \mathbf{J}(\mathbf{x},t) \qquad \text{(B.1),}$$

where □ is the d´Alembertian operator $\Box = \partial_\mu \partial^\mu$. The solution of (A.1)is given by[2]

$$\mathbf{A}(\mathbf{x},t) = \mu_o \int d^3\mathbf{x}' \int dt' [\mathbf{J}(\mathbf{x}',t')/|\mathbf{x} - \mathbf{x}'|]\, \delta(t' + |\mathbf{x} - \mathbf{x}'|/c - t) \qquad \text{(B.2).}$$

With the sinusoidal time dependence $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x}) \exp(-i\omega t)$ (A.1) becomes given by

$$\mathbf{A}(\mathbf{x},t) = \mu_o \int \mathbf{J}(\mathbf{x}') \exp(ik|\mathbf{x} - \mathbf{x}'|)/|\mathbf{x} - \mathbf{x}'|\, d^3\mathbf{x}' \qquad \text{(B.3),}$$

that can be expanded in series taking into account that the fields are very far from the source, that is, $r >> d$ and that $d << \lambda$, where d is the dimension of the source and λ the wavelength of the emitted radiation. The rate of the emitted electromagnetic radiation dE/dt can be calculated expanding $\mathbf{A}(\mathbf{x},t)$ using *electric and magnetic multipoles.* [2]

In vacuum (A.1) obeys the equation

$$\Box \mathbf{A}(\mathbf{x},t) = 0 \qquad \text{(B.4).}$$

The general solutions of the above equations for **A** is formed by superposing transverse waves[2] of the field $\mathbf{A}(x_\mu)$. In *second quantization* context [4,21] planes waves **A** are written as (omitting details of normalization constant, wave polarization,...) where $k_\mu = (\mathbf{k}, i\omega/c)$,

$$\mathbf{A}(x_\mu) = \sum_{k\omega} [\mathbf{a}_{k\omega} \exp(ik_\mu x_\mu) + \mathbf{a}^*_{k\omega} \exp(-ik_\mu x_\mu)] \qquad \text{(B.5),}$$

(B.1) Emitted electromagnetic energy per unitof time dE/dt.

If the emitted radiation is mainly due to the electric dipole $\mathbf{D} = \int \mathbf{x}' \rho_e(\mathbf{x}')\, d^3\mathbf{x}'$ we have

$$dE/dt = (ck^4/3)\,|\mathbf{D}|^2 = (\omega^4/3c^3)\,|\mathbf{D}|^2 \qquad \text{(B.6),}$$

where $\rho_e(\mathbf{x}')$ is the electric charge density and $k = 2\pi/\lambda = \omega/c$.

If the energy is mainly emitted by electric quadrupole $Q_{\alpha\beta}$ and by magnetic dipole **m** we can show that

$$dE/dt = (ck^6/360)\Sigma_{\alpha\beta} |Q_{\alpha\beta}|^2 \qquad (B.7),$$

where $Q_{\alpha\beta} = \int\rho_e(\mathbf{x}')(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\, d^3\mathbf{x}'$ and $\mathbf{m} = \int \mathbf{x}' \mathrm{x}\, \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}'$.

(B.2)Larmor Acceleration Formula.

According to the classical electrodynamics accelerated charges emit radiation and the dominant energy loss is from electric dipole which obeys the Larmor formula (in Gaussian units),[2,17]

$$dE/dt = (2/3c^3)|d^2\mathbf{D}/dt^2| \qquad (B.8).$$

This formula can be used to estimate the classical lifetime of the **Bohr atom**.[17] For very large quantum numbers n, Bohr's *correspondence principle* **(CP)** demands that classical physics and quantum physics give the same answer, at least in average. In these conditions as the energy levels are very close the radiate energy is estimated using the classical electrodynamics.[17] So, putting $\mathbf{D} = e\mathbf{r}$ it is assumed that the electron moves in circular orbits around the nucleus emits continuously radiating energy according to,

$$dE/dt = (2/3c^3)e^2\mathbf{a}(t)^2 \qquad (B.9),$$

where **a** the electron acceleration, which is essentially the radial one $a_r = r\omega^2$. In this *adiabatic approximation* the electronic orbit remains nearly circular at all times whith $\omega \approx$ constant. According to reference [17] the electron will fall to the origin, following a spiral motion, after a time $t_{fall} \sim 10^{-11}$ s. The observed lifetime of the $2p^{1/2}$ state of the hydrogen is $\sim 10^{-9}$ s (see Appendix C). In quantum mechanics the ground state, however, "appears" to have infinite lifetime. The accelerated electron along a radius r(t) with a tangential speed $v_\Theta(t)$ and angular speed $\omega = d\Theta/dt = v_\Theta(t)/r$ emits a wave with frequency ω called *synchrotron radiation*.

Taking into account that $|a| \sim a_r = r\omega^2$ (B.9) becomes written as

$$dE/dt \approx (2e^2\omega^4/3c^3)\, \mathbf{r}(t)^2 \qquad (B.10).$$

Appendix C. Quantum Electromagnetic Radiation.

In Special Relativity (**SR**) [2,4] the generalized vector potential is defined by $A_\mu = (\mathbf{A}, iA_o) = (\mathbf{A}, i\varphi)$. A free particle with a mass m has a 4-momentum $p_\mu = (\mathbf{p}, iE)$ where E is the total energy $E = (m^2c^2 + p^2c^2)^{1/2}$. The 4-momentum a charged particle submitted to an electromagnetic field becomes given by $p_\mu \rightarrow p_\mu - (e/c) A_\mu$. That is, $E \rightarrow E - e\varphi$ and $\mathbf{p} \rightarrow \mathbf{p} - (e/c)\mathbf{A}$.

The relativistic wave equation [4] for a charged spin zero particle submitted to an external electromagnetic field is obtained through the transformation

$$p_\mu - (e/c) A_\mu \rightarrow - i\hbar\, \partial/\partial_{x\mu} - (e/c) A_\mu \qquad (C.1),$$

that is

$$\{ \sum_\mu (- i\hbar\, \partial/\partial_{x\mu} - (e/c)\, A_\mu)^2 + m^2c^2 \}\Psi = 0 \qquad (C.2),$$

or

$$(1/c^2)[i\hbar\, \partial/\partial t - e\varphi]^2\, \Psi = [(i\hbar\, \text{grad} - (e/c)\mathbf{A})^2 + m^2c^2\,]\Psi \qquad (C.3).$$

According to quantum mechanics[4] the interaction of a charged spin less particle with the electromagnetic radiation is given by the operator, putting $\boldsymbol{p}$ = -iħ grad,

$$W(t) = -(e/mc)(\mathbf{A}.\boldsymbol{p}) + (e^2/2mc^2)\mathbf{A}^2 \qquad (C.4),$$

where the vector potential **A** is written in the form of a plane wave with wave vector **k** and frequency ω, $\mathbf{A}(\mathbf{r},t) = A_o\, \mathbf{u} \cos[\mathbf{k}.\mathbf{r} - \omega t]$, with **u** the unit vector determining the polarization of the radiation (direction of the electric field vector). With the perturbation theory to evaluate the transitions probabilities, in a first order approximation, we neglect the term $(e^2/2mc^2)\mathbf{A}^2$ since it is gives a small contribution, of the order of $\alpha = e^2/hc$ ~1/137.[4] In this way we retain only the first term of (C.4),

$$W(t) = - (e/mc)(\mathbf{A}.\boldsymbol{p}) \qquad (C.5).$$

The amplitude a_o will be determined in such a way that there are an average N photons of energy ħω and polarization **u** in a volume V . So, from

$$\mathbf{E} = - (1/c)\partial\mathbf{A}/\partial t = A_o\, \mathbf{u}\, (\omega/c) \sin[\mathbf{k}.\mathbf{r} - \omega t] \qquad \text{and}$$

from the condition

$$N\hbar\omega/V = < \mathbf{E}^2(t)>/4\pi = (A_o^2\omega^2/4\pi c^2) < \sin^2[\mathbf{k}.\mathbf{r} - \omega t] > = A_o^2\omega^2/8\pi c^2$$

we see that $A_o = 2c(2\pi\hbar N/\omega V)^{1/2}$.

Writing $W(t) = w \exp(i\omega t) + w^*\exp(- i\omega t)$ where $w = A_o \exp(-i\mathbf{k}.\mathbf{r})(\mathbf{u}.\boldsymbol{p})$ the transition probability per unit of time for a transition from a (initial) state |b > to a (final)state |a > with the *emission* of a quantum ħω will be determined by the expression

$$P_{ab} = (2\pi/\hbar)\, |< a\, |w|\, b >|^2\, \rho(E_{fin}) \qquad (C.6),$$

where the initial energy E_{init} = final energy E_{fin} or $E_a = E_b + \hbar\omega$ and $\rho(E_{fin}) = \rho(\hbar\omega)$ [4] is the density of final photon states $dN/d\varepsilon = \rho(\hbar\omega) = [V\omega^2/(2\pi c)^3\hbar]d\Omega$, remembering that for photons ε = ħω and p = ε/c. The matrix element < a |w| b > is given by

$$< a\, |w|\, b > = - A_o < a \,|\, e^{-i\,\mathbf{k}.\mathbf{r}}\, (\mathbf{u}.\boldsymbol{p}) \,|\, b > \qquad (C.7),$$

remembering that $\boldsymbol{p}$ = -iħ grad. Since the integration of matrix element is will be essentially over the region (**r**) of the size (a) of emitting system it is convenient to expand the exponential factor in a power series,

$$e^{-i\,\mathbf{k}.\mathbf{r}} = 1 - i\, (\mathbf{k}.\mathbf{r}) + [-i(\mathbf{k}.\mathbf{r})]^2/2! + ... = \qquad (C.8).$$

(B.1) Dipole radiation.

When $ka = 2\pi/\lambda << 1$, where λ is the wavelength of the emitted photon, it is enough to consider only of the first term of (C.8) obtaining:[4]

$$< a\, |w|\, b > = - i\, \omega_{ab}\, A_o(\mathbf{u}.\boldsymbol{D})_{ab} \qquad (C.9),$$

where $\boldsymbol{D} = \Sigma_i q_i \boldsymbol{r}_i$ is the ***electric dipole moment operator*** of the emitting system with discrete charges q_i. One can show that

$$< a |w| b > = - i \omega_{ab} A_o \mathbf{u}.(\boldsymbol{D}_{ab}) \qquad (C.10),$$

where the vector $\boldsymbol{D}_{ab}$ = < a |**D** |b > is called the *electrical dipole moment of the b → a tarnsition.* In this way, using (C.6)-(C.10) we obtain the probability per unit of time dP_{ab}^+ that a photon with polarization **u** and frequency $\omega = |\omega_{ab}| = (E_a - E_b)/\hbar$ is emitted within a solid angle dΩ ,

$$(dP_{ab}^+)_{dip} = N (\omega^3/2\pi\hbar c^3) |\mathbf{u}.(\boldsymbol{D}_{ab})|^2 d\Omega \qquad (C.11).$$

The polarization **u** is perpendicular to the direction of propagation **k.** If we denote by θ the angle between **k** and the dipole moment of the transition $\boldsymbol{D}_{ab}$ we have $|\mathbf{u}.(\boldsymbol{D}_{ab})|^2 = |\boldsymbol{D}_{ab}|^2 \sin^2\theta$. Thus,

$$(dP_{ab}^+)_{dip} = N (\omega^3/2\pi\hbar c^3) |\boldsymbol{D}_{ab}|^2 \sin^2\theta \, d\Omega \qquad (C.12).$$

Integrating (C.12) with N =1[4] over all directions of the radiation we get the *total transition probability per unit of time* P_{ab} involving the *emission of one photon*:

$$(P_{ab}^+)_{dip} = (4\omega^3/3\hbar c^3) |\boldsymbol{D}_{ab}|^2 \qquad (C.13).$$

To estimate the order of magnitude of (C.13) for atomic systems with linear dimension a we put $\boldsymbol{D} = e\boldsymbol{r}$ taking $|\boldsymbol{r}_{ab}| = a \approx e^2/\hbar\omega$. Thus, $(P_{ab}^+)_{dip}$ can be written as

$$(P_{ab}^+)_{dip} \approx (e^2\omega/\hbar c)(\omega a/c)^2 \approx \omega/(137)^3,$$

that for optical radiation ($\omega \sim 10^{15}$/s) gives $(P_{ab})_{dip} \sim 10^9$/s. The observed lifetime $\tau \sim 1/(P_{ab})_{dip}$ of the $2p^{1/2}$ state of the hydrogen is $\tau \sim 10^{-9}$ s.[4]

Consequently, energy emitted per unit of time dE/dt will be given by $(dE_{ab})_{dip} = \hbar\omega(P_{ab}^+)_{dip}$, that is,

$$(dE/dt)_{dip} = (4\omega^4/3c^3) |\boldsymbol{D}_{ab}|^2 \qquad (C.14).$$

In case of the Bohr atom with **D** = e**r** (C.14) becomes written as

$$(dE/dt)_{dip} = (4e^2\omega^4/3c^3) |\boldsymbol{r}_{ab}|^2 \qquad (C.15).$$

It becomes equal to (B.8) if the average energy (averaged over the time) emitted per unit of time is due to a dipole $\mathbf{D}(t) = e\mathbf{r}(t) = 2 (|\boldsymbol{D}_{ab}|^2)^{1/2} \cos(\omega t) = 2e |\boldsymbol{r}_{ab}| \cos(\omega t)$.

(B.2)Quadrupole radiation.

If it is necessary to take into account the second term of the expansion (B.8) the matrix element < a |*w*| b > given by (C.7) will be

$$< a |w| b > = -i A_o < b |(\mathbf{k}.\mathbf{r}')(\mathbf{u}.\mathbf{p}')| a > = A_o (\hbar k/2)\mu\omega < b| \mathbf{r}'(\mathbf{n}.\mathbf{r}')| \mathbf{a} > \qquad (C.16),$$

where $\omega_{ab} = \omega$, μ the electron mass and $\mathbf{n} = \mathbf{r}'/r'$. (C.16) would be responsible for *electric quadrupole* transitions involving matrix elements of the products xy, xz and yz and *dipole magnetic* transitions of matrix elements of the angular momentum operators

L_x, L_y and L_z. In quantum systems with spherically symmetric potential magnetic dipole transitions give no contributions to photons emission.[4] So, following the same procedure used for dipole radiation we can calculate the total emission probability per unit of time within the solid angle dΩ. The general angular distribution of the quadrupole radiation is very complicated. [2,20] As we only intend to obtain an order of magnitude of the quadrupole radiation we put

$$(P_{ab}{}^{+})_Q \approx (\omega^5/2\pi\hbar c^5)|Q_{ab}|^2 \qquad (C.17),$$

where, the quadrupole matrix element is represented by Q_{ab}. So, the total energy per unit of time $(dE/dt)_Q$ emitted by the quadrupole is given by

$$(dE/dt)_Q \approx (\omega^6/2\pi c^5)|Q_{ab}|^2 \qquad (C.18).$$

In classical electrodynamics we have[2]

$$(dE/dt)_{class} \approx (ck^6/240)Q_o{}^2 = (\omega^6/240c^5)\, Q_o{}^2 \qquad (C.19).$$

Let us estimate $(P_{ab}{}^{+})_Q$, given by (C.17), for systems emitting optical frequencies $\omega \sim 10^{15}$/s and with atomic dimensions a ~10^{-7} cm. Taking $Q_{ab} \sim ea^2$ we verify that

$$(P_{ab}{}^{+})_Q \approx (\omega^5/2\pi\hbar c^5)|Q_{ab}|^2 \sim 10^5/s \qquad (C.20),$$

that is, $(P_{ab}{}^{+})_Q \sim 10^{-4}\, (P_{ab}{}^{+})_{dip}$.

(C.3)Multipole tensor operators $T_{\ell m}(\theta,\varphi)$.

Since calculations of quadrupole and magnetic dipole transitions and of higher order terms of the expansion (B.8) are very intricate it is convenient to use a different approach to estimate these matrix elements. In this way are used the *tensor multipole operators* $T_{\ell m}(\theta,\varphi)$ defined by [2,4,20]

$$T_{\ell m}(r,\theta,\varphi) = [4\pi/(2\ell+1)]^{1/2}\, r^\ell\, Y_{\ell m}(\theta,\varphi) = [4\pi/(2\ell+1)]^{1/2}\, r^\ell\, |\ell m\rangle \qquad (C.21),$$

where ℓ =1, 2,...correspond to dipole, quadrupole ,... and the angle θ is between **k** and **r.**

If the state functions are given by $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\,|\ell m\rangle$ the transition probabilities per unit of time P_{ab} will directly proportional to $|a_E(\ell,m)|^2$ where the amplitudes $a_E(\ell,m)$ are given, for ka << 1, by[4]

$$a_E(\ell,m) = -\,[4\pi/(2\ell+1)!!](\ell+1/\ell)^{1/2}\, k^{\ell+2}\, Q_{\ell m} \qquad (C.22),$$

where

$$Q_{\ell m} = \int dr\, r^{\ell+2}\, R_a(r)\, R_b(r) \langle \ell_b m_b|Y_{\ell m}{}^*(\theta,\varphi)|\, \ell_a m_a\rangle.$$

The matrix element $\langle n'j'm'|T_k{}^q\,|\, n\, j\, m\rangle$ according to the Wigner-Eckart Theorem (WET)[22] is given by $\langle n'j'm'|T_k{}^q\,|\, n\, j\, m\rangle = (jkmq|j'm')\,(n'j'||Tk||nj)$, where $(jkmq|j'm') \neq 0$ only when $m + q = m'$ and $|j - k| \leq j' \leq j + k$.

For **dipole (ℓ=1)** using (C.18) the transition probabilities per unit of time P_{ab} between states $|\, a\rangle$ and $|\, b\rangle$ are proportional to $|\mathbf{D}_{ab}|^2$ where,

$$|\mathbf{D}_{ab}| = (4\pi/3)^{1/2} \int dr\, r^3 R_a(r) R_b(r) < \ell_b m_b|Y_{10}(\theta,\varphi)|\, \ell_a m_a > \qquad (C.23).$$

Thus, following the WET the a → b transition is allowed only if we have:

$$\ell_b = \ell_a \pm 1 \quad \text{and} \quad m_b = m_a .$$

This kind radiation is called *electrical dipole radiation* and is denoted by E1.

For electric **quadrupole (ℓ=2)** P_{ab} is proportional to $|Q_{ab}|^2$ where

$$Q_{ab} = \int dr\, r^4 R_a(r) R_b(r) < \ell_b m_b|Y_{2m}^*(\theta,\varphi)|\, \ell_a m_a > \qquad (C.24),$$

showing that quadrupole **t**ransitions a → b are allowed only if

$$\ell_b = \ell_a \pm 2 \quad \text{and} \quad m_b = m_a + 2 \qquad (C.25).$$

This kind of radiation is called *electric quadrupole radiation* and is denoted by E2.

(B.4)Second quantization approach.

Basic ideas on the quantization of radiation can be seen in many books. In *vacuum*, with the Lorentz gauge the electromagnetic field $\mathbf{A}(x^\mu)$ is given by[4,21]

div(**A**) = 0, $\partial_\mu \partial^\mu = \Box\mathbf{A} = 0$, $\mu = 1,2,3,4$, $x_\mu = (\mathbf{x}, ict)$ and $A_\mu = (\mathbf{A}, i\varphi)$.

The general solutions of the above equations for **A** is formed by superposing transverse waves[2,4] of the field $\mathbf{A}(x_\mu)$. In the *second quantization* context planes waves **A** are written as (omitting details of normalization constant, wave polarization,...)

$$\boldsymbol{A}(x_\mu) = \Sigma_{k\omega} [\mathbf{a}_{k\omega} \exp(ik_\mu x_\mu + \mathbf{a}^*_{k\omega} \exp(-ik_\mu x_\mu)]/\surd\omega \qquad (C.26),$$

where $k_\mu = (\mathbf{k}, i\omega/c)$, $\mathbf{a}_{k\omega}$ and $\mathbf{a}^*_{k\omega}$ are the **creation and annihilation photon operators**, respectively.

In this approach transition probabilities P_{ab} are now estimated using in (C.6) the field operator **A** defined by (C.22). Taking into account transitions involving *vacuum states* and wavefunctions $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)|\ell m>$ we get the same results obtained before without the second quantization approach. The main difference now is that the electromagnetic radiation is composed by *photons*. Selection rules obeyed in *electrical dipole radiation* (E1) show that **photons** must have spin 1.

Appendix D. Quantum Theory of Gravitation.

Classical electrodynamics, quantum theory and their connections are very well established. To introduce basis of a quantum field theory in GR Weinberg[15] analyzed, for instance, the possibility to quantize the gravitational wave field $h_{\mu\nu}$ that in free obeys the equations (see **Appendix A**) $\Box h_{\rho\nu} = 0$ and $\partial h_\rho^{\ \nu}/\partial x^\nu = 0$. The general solutions of these equations are given by the superposition of transverse plane tensor waves $h_{\rho\nu}(x)$ which propagates with the light velocity c and helicities $\mu = \pm 2$. This would be done in order to construct, similarly to the Electromagnetic field, a *Lorentz invariant Hamiltonian* in terms of creation and annihilation operators of gravitons. That is, the Hamiltonian would be built up of quantum fields $h_{\rho\nu}(x)$ [*transverse plane waves*] that in a **second quantization** framework would be given by[15]

$$h_{\rho\nu}(x) = \sum_\mu \int d^3\mathbf{k}\{a(\mathbf{k},\mu)\, e_{\rho\nu}(\mathbf{k},\mu)\, \exp(ik_\lambda x^\lambda) + a^+(\mathbf{k},\mu)\, e^*_{\rho\nu}(\mathbf{k},\mu)\, \exp(-ik_\lambda x^\lambda)\} \quad (4.1),$$

where $e_{\rho\nu}(\mathbf{k},\mu)$ is the polarization tensor for a graviton of momentum h**k** and **helicity** $\mu = \pm 2$, and $a(\mathbf{k},\mu)$ and $a^+(\mathbf{k},\mu)$ are the corresponding annihilation and creation operators, characterized by the commutation relations

$$[a(\mathbf{k},\mu)\, ,\, a^+(\mathbf{k}',\mu')] = \delta^3(\mathbf{k} - \mathbf{k}')\, \delta_{\mu'\mu}$$

$$[a(\mathbf{k},\mu)\, ,\, a(\mathbf{k}',\mu')] = [a^+(\mathbf{k},\mu)\, ,\, a^+(\mathbf{k}',\mu')] = 0 \quad (4.2)$$

The difficult in this approach comes from the fact that the operator (4.1) is not a "Lorentz tensor"(*which is invariant by Lorentz group*). Remembering that $\tau_{\mu\nu}$ is a Lorentz tensor if it transforms as $\tau'_{\mu\nu} = \Lambda_\mu^{\ \rho}\, \Lambda_\nu^{\ \sigma}\, \tau_{\rho\sigma}$, where Λ is the Lorentz matrix.[15] As shown by Weinberg[15] in Section 10.2 a "true" plane wave tensor would have helicities 0, ±1 as well ± 2. This is in contradiction with (4.1) where there are only helicities $\mu = \pm 2$. Of course, we can start with a true tensor and then subject $e_{\mu\nu}$ to a gauge transformation that will eliminate the unphysical helicities 0 and ±1, but once we choose a gauge in this way, $h_{\rho\nu}(x)$ is no longer a Lorentz tensor. This gauge condition is not Lorentz invariant. Many other attempts are mentioned by Weinberg.[15] According to him at present does not exist any complete and self-consistent quantum theory of gravitation . In his book he presents to the reader some taste of what a quantum theory of gravitation would be like. Instead of using Lagragian or Hamiltonian formalisms he adopts a different way. In this way he proposed, for instance, that for a general system the emission rate $d\Gamma_{GW}$ of a gravitational wave ("gravitons") with frequency ω in a solid angle $d\Omega$ is given by

$$d\Gamma_{GW} = (G\omega/\hbar\pi)[T^{\lambda\nu *}(k,\omega)\, T_{\lambda\nu}(k,\omega) - (1/2)\, |T^\lambda_{\ \lambda}(k,\omega)|^2]\, d\Omega \quad (4.3),$$

where $T_{\lambda\nu}(k,\omega)$ is the energy-momentum tensor. Using (4.1) one can show[15] that in the quadrupole approximation the total power emitted at a single discrete frequency ω is given by

$$\Gamma_{GW} = (2G\omega^6/5)[D^*_{ij}(\omega)D_{ij}(\omega) - (1/2)|D_{ij}(\omega)|^2] \quad (4.4),$$

where $D_{ij}(\omega) = \int x^i x^j T^{oo}(\mathbf{x},\omega)\, d^3x$ which is the quadrupole matrix operator and $T^{oo}(\mathbf{x},\omega)$ the energy density operator written as ρ. In this way, Γ_{GW} given by (4.4) could interpreted as matrix element of ρ between final and initial states ψ_a and ψ_b. That is, in a quantum transition a → b the total rate $(\Gamma_{GW})_{ab}$ would be given by

$$(\Gamma_{GW})_{ab} = (2G\omega^5/5\hbar)[D^*_{ij}(a \to b)D_{ij}(a \to b) - (1/3)|D_{ij}(a \to b)|^2] \quad (4.5),$$

where $D_{ij}(a \to b) \equiv \int \psi_b^*(\mathbf{x})\, \rho\, x_i\, x_j\, \psi_a(\mathbf{x})\, d^3\mathbf{x}$ which is a quadrupole matrix element. He applied this formula to calculate GW emitted by 3d → 1s transition of hydrogen and concluded that there is no chance to be observe the event. Probably, he ought to have applied his formula to calculate GW emitted by mBBH.

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