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Abstract: In this paper we study the Bohmnian Trajectories for the Diósi-Halliwell-Nassar Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm.

Keywords: De Broglie-Bohm Quantum Mechanics; Bohmnian Trajectories of the Diósi-Halliwell-Nassar Equation

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1. Introduction: The Bohmnian Trajectories

In this article, we calculated the *Bohmnian Trajectories* for the Diósi-Halliwell-Nassar Equation. To obtain these trajectories we adopted the quantum mechanical formalism of de Broglie-Bohm. This was done because this formalism permits to perform essential linear approximations along the classical trajectories that are the basic ingredients of the of the Feynman's principle of minimum action of quantum mechanics. [1]

2. The Bohmnian Trajectories for the Diósi-Halliwell-Nassar Equation

Now, let us calculate the Bohmnian trajectories for the Diósi-Halliwell-Nassar Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm. [2]

2.1. The Diósi-Halliwell-Nassar Equation

In 1998, L. Diósi and J. J. Halliwell [3] proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\,\frac{\partial\Psi(x,\ t)}{\partial t} \;=\; -\,\frac{\hbar^2}{2\ m}\,\frac{\partial^2\ \Psi(x,\ t)}{\partial x^2}\;+$$

$$+\;\Big[\;V(x,\ t)\;+\;\lambda\;x\;X(t)\;-\;i\,\hbar\,\Big(\;\frac{[x\;-\;q(t)]^2}{a^2}\;-\;\frac{\eta(t)\ [x\;-\;q(t)]}{a}\;\Big)\;\Big]\;\times\;\Psi(x,\ t)\ , \qquad (2.1.1)$$

where $\Psi(x,\ t)$ is a wave function which describes a given system, $X(t)$ is the position of classical particle submitted to a time dependent potential $V(x,\ t)$, $q(t)\ = < x(t) >$, and λ and a are constants.

However, as the eq. (2.1.1) is not normalized, Nassar [4] considered $a(t)$ and $\eta(t)\ = a(t)/[x,\ q(t)]$, and proposed that:

$$i\,\hbar\,\frac{\partial\Psi(x,\ t)}{\partial t} \;=\; -\,\frac{\hbar^2}{2\ m}\,\frac{\partial^2\ \Psi(x,\ t)}{\partial x^2}\;+$$

$$+\;\Big[\;V(x,\ t)\;+\;\lambda\;x\;X(t)\;-\;\frac{i\,\hbar}{4\ \sigma}\,\Big(\;\frac{[x\;-\;q(t)]^2}{a^2}\;-\;1\;\Big)\;\Big]\;\times\;\Psi(x,\ t)\ , \qquad (2.1.2)$$

where σ is a constant. The eq. (2.1.2) represents a *Schrödinger Equation for Continuous Quantum Measurements* or *Diósi-Halliwell-Nassar Equation* $(DHN - E)$.

2.1.1. The Wave Function of the Diósi-Halliwell-Nassar Equation

Initially, let us write the wave function $\Psi(x,\ t)$ in the polar form defined by the Madelung-Bohm transformation [5,6]:

$$\Psi(x,\ t)\;=\;\phi(x,\ t)\;\times\;exp\;[i\;S(x,\ t)]\ , \qquad (2.1.1.1)$$

where $\phi(x,\ t)$ will be defined in what follows.

Calculating the derivatives, temporal and spatial, of (2.1.1.1), we get [remembering that $exp\;[i\;S]$ is common factor]: [2]

$$\frac{\partial\Psi}{\partial t}\;=\;exp\;(i\;S)\;(\;\frac{\partial\phi}{\partial t}\;+\;i\;\phi\;\frac{\partial S}{\partial t}\;)\ , \qquad (2.1.1.2a)$$

$$\frac{\partial^2\Psi}{\partial x^2}\;=\;exp\;(i\;S)\;[\;\frac{\partial^2\phi}{\partial x^2}\;+\;2\;i\;\frac{\partial S}{\partial x}\;\frac{\partial\phi}{\partial x}\;+\;i\;\phi\;\frac{\partial^2 S}{\partial x^2}\;-\;\phi\;(\;\frac{\partial S}{\partial x}\;)^2\;]\ , \qquad (2.1.1.2b)$$

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.1.2), we have:[2]

$$i\,\hbar\,\Big(\;\frac{\partial\phi}{\partial t}\;+\;i\;\phi\;\frac{\partial S}{\partial t}\;\Big)\;=\;-\;\frac{\hbar^2}{2\ m}\;\Big[\;\frac{\partial^2\phi}{\partial x^2}\;+\;2\;i\;\frac{\partial S}{\partial x}\;\frac{\partial\phi}{\partial x}\;+\;i\;\phi\;\frac{\partial^2 S}{\partial x^2}\;-\;\phi\;\Big(\;\frac{\partial S}{\partial x}\;\Big)^2\;\Big]\;+$$

$$+ \left[V(x,\ t) \ + \ \lambda\ x\ X(t) \ - \ \frac{i\ \hbar}{4\ \sigma} \left(\frac{[x\ -\ q(t)]^2}{a^2(t)} \ - \ 1 \right) \right] \times \phi(x,\ t)\ , \qquad (2.1.1.3)$$

Separating the real and imaginary parts of the eq. (2.1.1.3), results:

a) imaginary part

$$\frac{\hbar}{\phi}\ \frac{\partial \phi}{\partial t} \ = \ - \ \frac{\hbar^2}{2\ m} \left(2\ \frac{1}{\phi}\ \frac{\partial S}{\partial x}\ \frac{\partial \phi}{\partial x} \ + \ \frac{\partial^2 S}{\partial x^2} \right) \ - \ \frac{\hbar}{4\ \sigma} \left(\frac{[x\ -\ q(t)]^2}{a^2(t)} \ - \ 1 \right)\ , \qquad (2.1.1.4)$$

b) real part

$$- \ \hbar\ \frac{\partial S}{\partial t} \ = \ - \ \frac{\hbar^2}{2\ m} \left[\frac{1}{\phi}\ \frac{\partial^2 \phi}{\partial x^2} \ - \ \left(\frac{\partial S}{\partial x} \right)^2 \right] \ + \ V(x,\ t) \ + \ \lambda\ x\ X(t)\ . \qquad (2.1.1.5)$$

2.1.2. Dynamics of the Diósi-Halliwell-Nassar Equation

Now, let us see the correlation between the expressions (2.1.1.4-5) and the traditional equations of the Ideal Fluid Dynamics: [8] a) *Continuity Equation*, b) *Euler's equation.* To do this let us perform the following correspondences:

Quantum density probability: $\mid \Psi(x,\ t) \mid^2 \ = \ \Psi^*(x,\ t)\ \Psi(x,\ t) \qquad \longleftrightarrow$

Quantum mass density: $\rho(x,\ t) \ = \ \phi^2(x,\ t) \ \longleftrightarrow \ \sqrt{\rho} \ = \ \phi\ , \qquad (2.1.2.1a,b)$

Gradient of the wave function: $\frac{\hbar}{m}\ \frac{\partial S(x,\ t)}{\partial x} \qquad \longleftrightarrow$

Quantum velocity: $v_{qu}(x,\ t) \equiv v_{qu}\ , \qquad (2.1.2.1c,d)$

Bohm quantum potential:

$$V_{qu}(x,\ t) \equiv V_{qu} \ = \ - \ (\frac{\hbar^2}{2\ m})\ (\frac{1}{\phi})\ \frac{\partial^2 \phi}{\partial x^2} \ = \ - \ \frac{\hbar^2}{2\ m}\ \frac{1}{\sqrt{\rho}}\ \frac{\partial^2 \sqrt{\rho}}{\partial x^2}\ , \qquad (2.1.2.1e,f)$$

Putting the relations (2.1.2.1a-d) into the equation (2.1.1.4) and considering that $\partial(\ell n\ x)/\partial y \ = \ (1/x)\ (\partial x/\partial y)$ and $\ell n(x^m) \ = \ m\ \ell n\ x$, we get:[2]

$$\frac{\partial}{\partial t}\ [\ell n\ (\phi^2)] \ = \ - \ \frac{\hbar}{m} \times \left\{ \frac{\partial^2 S}{\partial x^2} \ + \ \frac{\partial S}{\partial x}\ \frac{\partial}{\partial x}\ [\ell n\ (\phi^2)] \right\} \ - \ \frac{1}{4\ \sigma} \left(\frac{[x\ -\ q(t)]^2}{a^2(t)} \ - \ 1 \right) \ \to$$

$$\frac{\partial}{\partial t}\ (\ell n\ \rho) \ = \ - \ \frac{\hbar}{m} \times \left[\frac{\partial^2 S}{\partial x^2} \ + \ \frac{\partial S}{\partial x}\ \frac{\partial}{\partial x}\ (\ell n\ \rho) \right] \ - \ \frac{1}{4\ \sigma} \left(\frac{[x\ -\ q(t)]^2}{a^2(t)} \ - \ 1 \right) \ =$$

$$= \ - \ \frac{\hbar}{m} \times \left[\frac{\partial^2 S}{\partial x^2} \ + \ \frac{\partial S}{\partial x}\ \frac{1}{\rho}\ \frac{\partial \rho}{\partial x} \right] \ - \ \frac{1}{4\ \sigma} \left(\frac{[x\ -\ q(t)]^2}{a^2(t)} \ - \ 1 \right) \ =$$

$$= - \frac{\partial}{\partial x} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right) - \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right) \frac{1}{\rho} \frac{\partial \rho}{\partial x} \,] - \frac{1}{4\,\sigma} \left(\frac{[x - q(t)]^2}{a^2(t)} - 1 \right) \rightarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_{qu})}{\partial x} = - \frac{\rho}{4\,\sigma} \left(\frac{[x - q(t)]^2}{a^2(t)} - 1 \right), \qquad (2.1.2.2)$$

expression that indicates <u>decoherence</u> of the considered physical system represented by the Diósi-Halliwell-Nassar Equation ($DHN - E$) [eq. (2.1.2)]; then the *Continuity Equation* its not preserved.

Now, let us obtained another dynamic equation of the $DHN - E$. So, differentiating the eq. (2.1.1.5) with respect x and using the eqs. (2.1.2.1a-e), we obtain:

$$- \hbar \frac{\partial^2 S}{\partial x \, \partial t} = - \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left[\frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} - \left(\frac{\partial S}{\partial x} \right)^2 \right] + \frac{\partial V(x,\, t)}{\partial x} + \lambda \, X(t) \rightarrow$$

$$\frac{\partial}{\partial t} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right) =$$

$$= - \frac{\partial}{\partial x} \left(- \frac{\hbar^2}{2\, m^2} \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right)^2 - \frac{1}{m} \frac{\partial V(x,\, t)}{\partial x} - \frac{\lambda}{m} X(t) \rightarrow$$

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \frac{\lambda}{m} X(t) = - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) \, . \qquad (2.1.2.4)$$

We observe that the eq. (2.1.2.4) has the aspect of the *Euler Equation* [8] for a ideal fluid in movement.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*: [8]

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_{qu} \frac{\partial}{\partial x} \, , \qquad (2.1.2.5a)$$

and that:

$$v_{qu}(x,\, t) \, |_{x=x(t)} = \frac{dx}{dt} \, , \qquad (2.1.2.5b)$$

the eq. (2.1.2.4) could be written as:

$$m \, \frac{d^2 x}{dt^2} = - \frac{\partial}{\partial x} \left[\lambda \, x \, X(t) + V(x,\, t) + V_{qu}(x,\, t) \right] \rightarrow$$

$$m \, \frac{d^2 x}{dt^2} = - \lambda \, x + F_C(x,\, t)|_{x=x(t)} + F_Q(x,\, t)|_{x=x(t)} \, , \qquad (2.1.2.5)$$

eq. that has the form of the *Second Newton Law*, being the terms of the second member, respectively, the *classical newtonian force* and the *quantum bohmnian force*.

2.1.3 The Quantum Wave Packet of the Linearized Diósi-Halliwell-Nassar Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Diósi-Halliwell-Nassar Equation($DHN - E$) along a classical trajetory, let us the considerer the *ansatz*: [9]

$$\rho(x,\ t)\ =\ [2\ \pi\ a^2(t)]^{-1/2}\ \times\ exp\ \{-\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}\} \qquad (2.1.3.1a)$$

or [use eq. (2.1.2.1b)]:

$$\phi(x,\ t)\ =\ [2\ \pi\ a^2(t)]^{-1/4}\ \times\ exp\ \{-\ \frac{[x\ -\ q(t)]^2}{4\ a^2(t)}\} \qquad (2.1.3.1b)$$

where $a(t)$ and $q(t)\ = < x >$ are auxiliary functions of time, to will be determined in what follows, representing the *width* and the *center of mass of wave packet*, respectively.

Differentiating the expression (2.1.3.1a) in the variable t, and remembering that x and t are independent variables, results:

$$\frac{\partial \rho}{\partial t}\ =\ -\ \frac{1}{2}\ [2\ \pi\ a^2(t)]^{-3/2}\ \times\ [4\ \pi\ a(t)\ \dot{a}(t)]\ \times\ exp\ \{-\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}\}\ +$$

$$+\ [2\ \pi\ a^2(t)]^{-1/2}\ \times\ exp\ \{-\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}\ \}\ \times\ \frac{\partial}{\partial t}\ \{\ -\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}\ \}\ =$$

$$=\ -\ \rho\ \{[2\ \pi\ \dot{a}(t)]\ \times\ [2\ \pi\ a^2(t)]^{-1}\ +\ \frac{4\ a^2(t)\ \times\ [x\ -\ q(t)]\ \times\ [-\dot{q}(t)]\ -\ 4a(t)\dot{a}(t)\ \times\ [x-q(t)]^2}{4a^4(t)}\}\ \rightarrow$$

$$\frac{\partial \rho}{\partial t}\ =\ \rho\ \{-\ \frac{\dot{a}(t)}{a(t)}\ +\ \frac{\dot{q}(t)}{a^2(t)}\ \times\ [x\ -\ q(t)]\ +\ \frac{\dot{a}(t)}{a^3(t)}\ \times\ [x\ -\ q(t)]^2\ \}\ . \qquad (2.1.3.2)$$

Substituting the eq. (2.1.3.2) into eq. (2.1.2.2) and integrating the result, we have (we consider null the integration constant):

$$\rho\,\{-\ \frac{\dot{a}(t)}{a(t)}\ +\ \frac{\dot{q}(t)}{a^2(t)}\ \times\ [x - q(t)]\ +\ \frac{\dot{a}(t)}{a^3(t)}\ \times\ [x - q(t)]^2\}\ +\ \frac{\partial(\rho\ v_{qu})}{\partial x}\ =\ -\ \frac{\rho}{4\ \sigma}\ \Big(\ \frac{[x\ -\ q(t)]^2}{a^2(t)}\ -\ 1\Big)\ \rightarrow$$

$$\int\ \frac{\partial(\rho\ v_{qu})}{\partial x}\ \partial x\ =$$

$$=\ \int\ \rho\ \{\frac{\dot{a}(t)}{a(t)}\ -\ \frac{\dot{q}(t)}{a^2(t)}\ \times\ [x\ -\ q(t)]\ -\ \frac{\dot{a}(t)}{a^3(t)}\ \times\ [x\ -\ q(t)]^2 +\ \frac{1}{4\ \sigma}\ \Big(\ \frac{[x\ -\ q(t)]^2}{a^2(t)}\ -\ 1\ \Big)\ \}\ \partial x\ \rightarrow$$

$$\rho\ v_{qu}\ =\ \int\ \rho\ \{\frac{\dot{a}(t)}{a(t)}\ -\ \frac{[x-q(t)]}{a^2(t)}\ \times\ \Big(\ \frac{\dot{a}(t)}{a(t)}\ \times\ [x-\ q(t)]\ +\ \dot{q}(t)\ \Big)\ +\ \frac{1}{4\ \sigma}\ \Big(\ \frac{[x\ -\ q(t)]^2}{a^2(t)}\ -\ 1\ \Big)\ \}\ \partial x\ \rightarrow$$

$$v_{qu}\ =\ \frac{1}{\rho}\ \int\ \rho\ \{\frac{\dot{a}(t)}{a(t)}\ -\ \frac{[x-q(t)]}{a^2(t)}\ \times\ \Big(\ \frac{\dot{a}(t)}{a(t)}\ [x-\ q(t)]\ +\ \dot{q}(t)\ \Big)\ +$$

$$+\ \frac{1}{4\ \sigma}\ \Big(\ \frac{[x\ -\ q(t)]^2}{a^2(t)}\ -\ 1\ \Big)\}\ \partial x\ . \qquad (2.1.3.3)$$

Now, using the eq. (2.1.3.1a), we can right that:

$$\frac{\partial}{\partial x}\ \{\rho\ \Big(\ \frac{\dot{a}(t)}{a(t)}\ \times\ [x-\ q(t)]\ +\ \dot{q}(t)\ \Big)\}\ =$$

$$= \rho \frac{\partial}{\partial x} \{ \frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \} + \{ \frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \} \frac{\partial \rho}{\partial x} =$$

$$= \rho \frac{\dot{a}(t)}{a(t)} + \{ \frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \} \times \frac{\partial}{\partial x} \Big([2 \pi a^2(t)]^{-1/2} \times exp\{ - \frac{[x - q(t)]^2}{2 a^2(t)} \} \Big) =$$

$$= \rho \frac{\dot{a}(t)}{a(t)} + \{ \frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \} \times [2 \pi a^2(t)]^{1/2} \times exp \{ - \frac{[x- q(t)]^2}{2 a^2(t)} \} \times$$

$$\times \frac{\partial}{\partial x} \{ - \frac{[x - q(t)]^2}{2 a^2(t)} \} =$$

$$= \rho \frac{\dot{a}(t)}{a(t)} + \{\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t)\} \times \rho \{ - \frac{[x - q(t)]}{a^2(t)} \} \rightarrow$$

$$\frac{\partial}{\partial x} \{ \rho \Big(\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \Big) \} =$$

$$= \rho \{ \frac{\dot{a}(t)}{a(t)} - \frac{[x - q(t)]}{a^2(t)} \times \Big(\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \Big) \} . \qquad (2.1.3.4)$$

Substituting the eq. (2.1.3.4) into the eq. (2.1.3.3) and using the eq. (2.1.2.5b), results:

$$v_{qu} = \frac{1}{\rho} \int \frac{\partial}{\partial x} \{\rho \Big(\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \Big)\} \partial x + \int \frac{\rho}{4 \sigma} \Big(\frac{[x - q(t)]^2}{a^2(t)} - 1 \Big) \partial x \rightarrow$$

$$v_{qu}(x, t) = \frac{dx(t)}{dt} = \Big(\frac{\dot{a}(t)}{a(t)} \times [x - q(t)] + \dot{q}(t) \Big) + F(x, t) , \qquad (2.1.3.5a)$$

where:

$$F(x, t) = \frac{1}{\rho} \int \frac{\rho}{4 \sigma} \Big(\frac{[x - q(t)]^2}{a^2(t)} - 1 \Big) \partial x. \qquad (2.1.3.5b)$$

We observe that the integration of the eq. (2.1.3.5a) give us the *bohmnian quantum trajectory* [x(t)] of the physical system considered represented by the eq. (2.1.2). Before, we must calculated the equation for $a(t)$. For that, we expand the functions S(x, t), V(x t) and V_{qu}(x, t) around of $q(t) = < x >$ up to *second Taylor order.* [9] In this way, we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{S"[q(t)\ t]}{2} \times [x - q(t)]^2, \qquad (2.1.3.6)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] \times [x - q(t)] + \frac{V"[q(t), t]}{2} \times [x - q(t)]^2, \qquad (2.1.3.7)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V"_{qu}[q(t), t] \times [x - q(t)]^2, \qquad (2.1.3.8)$$

where (') and (") means, respectively: $\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial q^2}$.

Differentiating the eq. (2.1.3.6) in the variable x multiplying the result by $\hbar/m$, using the eqs. (2.1.2.1c,d) and (2.1.3.5b), results:

$$\frac{\hbar}{m}\,\frac{\partial S(x,\ t)}{\partial x} \ = \ \frac{\hbar}{m}\ \{\ S'[q(t),\ t]\ +\ S"[q(t),t] \times [x\ -\ q(t)]\ \}\ =$$

$$=\ v_{qu}(x,\ t)\ =\ \Big(\ \frac{\dot{a}(t)}{a(t)} \times [x\ -\ q(t)]\ +\ \dot{q}(t)\ \Big)\ +\ F(x,\ t)\ \ \rightarrow$$

$$S'[q(t),\ t]\ =\ (\tfrac{m}{\hbar}) \times [\dot{q}(t)\ +\ F(x,\ t)]; \qquad S"[q(t),\ t]\ =\ \frac{m}{\hbar}\,\frac{\dot{a}(t)}{a(t)}. \qquad (2.1.3.10a,b)$$

Substituting the eqs. (2.1.3.10a,b) into the eq. (2.1.3.6), we have:

$$S(x,\ t)\ =\ S_0(t)\ +$$

$$+\ (\tfrac{m}{\hbar}) \times [\dot{q}(t)\ +\ F(x,\ t)] \times [x\ -\ q(t)]\ +\ (\tfrac{m}{2\ \hbar})\,\frac{\dot{a}(t)}{a(t)} \times [x\ -\ q(t)]^2\ , \qquad (2.1.3.11a)$$

where:

$$S_0(t) \equiv S[q(t),\ t], \qquad (2.1.3.11b)$$

is the *quantum action*.

Differentiating the eq. (2.1.3.11a) in relation to the time t, we obtain (remembering that $\frac{\partial x}{\partial t}\ =\ 0$):

$$\frac{\partial S}{\partial t}\ =\ \dot{S}_0(t)\ +$$

$$+\ \frac{\partial}{\partial t}\ \{\ \frac{m}{\hbar} \times [\dot{q}(t)\ +\ F(x,\ t)] \times [x\ -\ q(t)]\ \}\ +\ \frac{\partial}{\partial t}\ \{\ \frac{m}{2\ \hbar}\,\frac{\dot{a}(t)}{a(t)} \times [x\ -\ q(t)]^2\ \}\ \ \rightarrow$$

$$\frac{\partial S}{\partial t}\ =\ \dot{S}_0(t)\ +\ \frac{m}{2\ \hbar}\ \Big[\ \frac{\ddot{a}(t)}{a(t)}\ -\ \frac{\dot{a}^2(t)}{a^2(t)}\ \Big] \times [x\ -\ q(t)]^2\ +$$

$$+\ (\tfrac{m}{\hbar}) \times \Big(\ [\ddot{q}(t)\ +\ \{\tfrac{\delta}{\delta t}\}\ [F(x,\ t)]\ -\ \frac{\dot{a}(t)\ \dot{q}(t)}{a(t)}]\ \Big) \times [x\ -\ q(t)]\ -$$

$$-\ (\tfrac{m}{\hbar})\ \dot{q}(t)\ [\dot{q}(t)\ +\ F(x,\ t)]\ . \qquad (2.1.3.12)$$

Considering the eqs. (2.1.2.1a,b) and (2.1.3.1a,b), let us write V_{qu} given by eq. (2.1.2.1e,f) in terms of potencies of [x - q(t)]. Before, we calculate the following derivations [remembering that $\frac{d}{dz}\ exp\ (z)\ =\ exp\ (z)$]:

$$\frac{\partial \phi(x,\ t)}{\partial x} \ = \ \frac{\partial}{\partial x} \Big(\ [2 \ \pi \ a^2(t)]^{-1/4} \ \times \ exp \ \{ \ - \ \frac{[x \ - \ q(t)]^2}{4 \ a^2(t)} \ \} \ \Big) \ =$$

$$= \ [2 \ \pi \ a^2(t)]^{-1/4} \ \times \ exp \ \{ \ - \ \frac{[x \ - \ q(t)]^2}{4 \ a^2(t)} \ \} \ \times \ \frac{\partial}{\partial x} \ \{ \ - \ \frac{[x \ - \ q(t)]^2}{4 \ a^2(t)} \ \} \ \ \rightarrow$$

$$\frac{\partial \phi(x,\ t)}{\partial x} \ = \ - \ [2 \ \pi \ a^2(t)]^{-1/4} \ \times \ exp \ \{ \ - \ \frac{[x \ - \ q(t)]^2}{4 \ a^2(t)} \ \} \ \times \ \{ \ \frac{[x \ - \ q(t)]}{2 \ a^2(t)} \ \} \ = \ - \ \phi \ \times \ \{ \ \frac{[x \ - \ q(t)]}{2 \ a^2(t)} \ \},$$

$$\frac{\partial^2 \phi(x,\ t)}{\partial x^2} \ = \ \frac{\partial}{\partial x} \Big(\ - \ \phi \ \times \ \{ \ \frac{[x \ - \ q(t)]}{2 \ a^2(t)} \ \} \ \Big) \ =$$

$$= \ - \ \phi \ \frac{\partial}{\partial x} \ \{ \ \frac{[x \ - \ q(t)]}{2 \ a^2(t)} \ \} \ - \ \frac{[x \ - \ q(t)]}{2 \ a^2(t)} \ \frac{\partial \phi}{\partial x} \ =$$

$$= \ - \ \phi \ \times \ \frac{1}{2 \ a^2(t)} \ + \ \frac{[x \ - \ q(t)]^2}{4 \ a^2(t)} \ \times \ \phi \ \ \rightarrow$$

$$\frac{1}{\phi(x,\ t)} \ \frac{\partial^2 \ \phi(x,\ t)}{\partial x^2} \ = \ \frac{[x \ - \ q(t)]^2}{4 \ a^4(t)} \ - \ \frac{1}{2 \ a^2(t)}. \qquad (2.1.3.13)$$

Substituting the eq. (2.1.3.13) in the eq. (2.1.2.1e), taking into account the eq. (2.1.3.8) and considering the identity of polynomials, results:

$$V_{qu}(x,\ t) \ = \ - \ \frac{\hbar^2}{2 \ m} \ \{ \ \frac{[x \ - \ q(t)]^2}{4 \ a^4(t)} \ - \ \frac{1}{2 \ a^2(t)} \ \} \ =$$

$$= \ V_{qu}[q(t),\ t] \ + \ V'_{qu}[q(t),\ t] \ \times \ [x \ - \ q(t)] \ +$$

$$+ \ \frac{1}{2} \ V"_{qu}[q(t),\ t] \ \times \ [x \ - \ q(t)]^2 \ \ \rightarrow$$

$$V_{qu}(x,\ t) \ = \ \frac{\hbar^2}{4 \ m \ a^2(t)}; \quad V'_{qu}(x,\ t) \ = \ 0; \quad V"_{qu}(x,\ t) \ = \ - \ \frac{\hbar^2}{8 \ m \ a^4(t)} \ \ \rightarrow$$

$$V_{qu}(x,\ t) \ = \ \frac{\hbar^2}{4 \ m \ a^2(t)} \ - \ \frac{\hbar^2}{8 \ m \ a^4(t)} \ \times \ [x \ - \ q(t)]^2. \qquad (2.1.3.14)$$

Taking the eq. (2.1.1.5) and using the eqs. (2.1.2.1c,d,e), we obtain:

$$- \ \hbar \ \frac{\partial S}{\partial t} \ = \ - \ \frac{\hbar^2}{2 \ m} \ \frac{1}{\phi} \ \frac{\partial^2 \ \phi}{\partial x^2} \ - \ (\ \frac{\partial S}{\partial x})^2 \ + \ V(x \ , \ t) \ + \ \lambda \ x \ X(t) \ \ \rightarrow$$

$$\hbar \ \frac{\partial S}{\partial t} \ + \ \Big[\ \frac{m}{2} \ v_{qu}^2 \ + \ V_{qu}(x,\ t) \ \Big] \ + \ V(x,\ t) \ + \lambda \ x \ X(t) \ = \ 0. \quad (2.1.3.15)$$

Inserting the eqs. (2.1.3.5a), (2.1.3.7), (2.1.3.12) and (2.1.3.14) into eq. (2.1.3.15), we obtain:

$$\hbar\ \dot{S}_0(t)\ +\ exp(\lambda\ t)\ \Big(\ -\ m\ \dot{q}^2(t)\ +\ [m\ \ddot{q}(t)\ +\ m\ \dot{q}(t)\ (\lambda\ -\ \tfrac{\dot{a}(t)}{a(t)}\ )\ ]\ \times\ [x\ -\ q(t)]\ +$$

$$+\ \tfrac{m}{2}\ \Big[\ \tfrac{\ddot{a}(t)}{a(t)}\ -\ \tfrac{\dot{a}^2(t)}{a^2(t)}\ +\ \lambda\ \tfrac{\dot{a}(t)}{a(t)}\Big]\ \times\ [x\ -\ q(t)]^2\ \Big)\ +$$

$$+\ exp\ (\lambda\ t)\ \times\ \Big(\ \tfrac{m}{2}\ \{\ \tfrac{\dot{a}(t)}{a(t)}\ [x\ -\ q(t)]\ +\ \dot{q}(t)\ \}^2\ \Big)\ +$$

$$+\ exp\ (\lambda\ t)\ \times\ \{\ V[q(t),\ t]\ +\ V'[q(t),\ t]\ \times\ [x\ -\ q(t)]\ +\ \tfrac{1}{2}\ V''[q(t),\ t]\ \times\ [x\ -\ q(t)]^2\ \}\ +$$

$$+\ exp\ (-\ 2\ \lambda\ t)\ \times\ \{\ \tfrac{\hbar^2}{4\ m\ a^2(t)}\ -\ \tfrac{\hbar^2}{8\ m\ a^4(t)}\ \times\ [x\ -\ q(t)]^2\ \}\ =\ 0\ \rightarrow$$

$$\hbar\ \dot{S}_0(t)\ +\ exp(\lambda\ t)\ \Bigg(\ -\ m\ \dot{q}^2(t)\ +\ [m\ \ddot{q}(t)\ +\ m\ \dot{q}(t)\ (\lambda\ -\ \tfrac{\dot{a}(t)}{a(t)}\ )\ ]\ \times\ [x\ -\ q(t)]\ +$$

$$+\ \tfrac{m}{2}\ \Big[\ \tfrac{\ddot{a}(t)}{a(t)}\ -\ \tfrac{\dot{a}^2(t)}{a^2(t)}\ +\ \lambda\ \tfrac{\dot{a}(t)}{a(t)}\Big]\ \times\ [x\ -\ q(t)]^2\ +$$

$$+\ exp\ (\lambda\ t)\ \times\ \Big(\ \tfrac{m}{2}\ \{\ \tfrac{\dot{a}^2(t)}{a(t)}\ \times\ [x\ -\ q(t)]^2\ +\ \dot{q}^2(t)\ +\ 2\ \tfrac{\dot{a}(t)\ \dot{q}(t)}{a(t)}\ \}\ \times\ [x\ -\ q(t)]\ \Big)\ +$$

$$+\ exp\ (\lambda\ t)\ \times\ \{\ V[q(t),\ t]\ +\ V'[q(t),\ t]\ \times\ [x\ -\ q(t)]\ +\ \tfrac{1}{2}\ V''[q(t),\ t]\ \times\ [x\ -\ q(t)]^2\ \}\ +$$

$$+\ exp\ (-\ \lambda\ t)\ \times\ \{\ \tfrac{\hbar^2}{4\ m\ a^2(t)}\ -\ \tfrac{\hbar^2}{8\ m\ a^4(t)}\ \times\ [x\ -\ q(t)]^2\ \}\ =\ 0\ . \qquad (2.1.3.17)$$

Since $(x\ -\ q)^o\ =\ 1$, we can gather together the above expression in potencies of $(x\ -\ q)$, obtaining:

$$\Bigg(\ \hbar\ \dot{S}_o(t)\ +\ exp\ (\lambda\ t)\ \times\ \{\ -\ \tfrac{m}{2}\ \dot{q}^2(t)\ +\ V[q(t),\ t]\ \}\ +\ \tfrac{exp\ (-\ \lambda\ t)\ \hbar^2}{4\ m\ a^2}\ \Bigg)\ \times\ [x\ -\ q(t)]^0\ +$$

$$+\ \Bigg(\ exp\ (\lambda\ t)\ \{\ m\ \ddot{q}(t)\ +\ V'[q(t),\ t]\ \}\ \Bigg)\ \times\ [x\ -\ q(t)]\ +$$

$$+\ \Bigg(\ exp\ (\lambda\ t)\ \times\ \{\tfrac{m}{2}\ [\ \tfrac{\ddot{a}(t)}{a(t)}\ -\ \tfrac{\dot{a}^2(t)}{a^2(t)}\ +\ \lambda\ \tfrac{\dot{a}(t)}{a(t)}\ +$$

$$+\ V''[q(t),\ t]\ \}\ -\ \tfrac{exp\ (-\ \lambda\ t)\ \hbar^2}{8\ m\ a^4(t)}\ \Bigg)\ \times\ [x\ -\ (q)t)]^2\ =\ 0\ . \qquad (2.1.3.18)$$

As the above relation is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) \;=\; \tfrac{1}{\hbar} \Big(m\, \dot{q}^2 \;+\; exp\,(\lambda\, t) \times \{ \;-\; \tfrac{m}{2}\, \dot{q}(t) \;+\; V[q(t),\, t] \;\} \;-$$

$$-\; \tfrac{exp\;(-\;\lambda\;t)\;\hbar^2}{4\;m\;a^2(t)} \Big)\;, \qquad (2.1.3.19)$$

$$\ddot{q}(t) \;+\; \tfrac{V'[q(t),\;t]}{m} \;=\; 0\;. \qquad (2.1.3.20)$$

$$\tfrac{\ddot{a}(t)}{a(t)} \;-\; \tfrac{\dot{a}^2(t)}{a^2(t)} \;+\; \lambda\, \tfrac{\dot{a}(t)}{a(t)} \;+\; \tfrac{V"[q(t),\;t]}{m} \;=\; \tfrac{exp\;(-\;2\;\lambda\;t)\;\hbar^2}{4\;m^2\;a^4(t)}\;. \qquad (2.1.3.21)$$

Now, let us consider that V[q(t), t] is given by:

$$V[q(t),\; t] \;=\; \tfrac{1}{2}\, m\, \omega^2(t)\, q^2(t), \qquad (2.1.3.22)$$

which is the Time Dependent Harmonic Oscillator Potencial).

In this case, we have:

$$V'[q(t),\; t] \;=\; m\, \omega^2(t)\, q(t), \quad V"[q(t),\; t] \;=\; m\, \omega^2(t)\;. \qquad (2.1.3.23a,b)$$

Putting the eqs. (2.1.3.23a,b) into eqs. (2.1.3.20,21), results:

$$\ddot{q}(t) \;+\; \omega^2(t)\, q(t) \;=\; 0\;, \qquad (2.1.3.24)$$

$$\tfrac{\ddot{a}(t)}{a(t)} \;-\; \tfrac{\dot{a}^2(t)}{a^2(t)} \;+\; \lambda\, \tfrac{\dot{a}(t)}{a(t)} \;+\; \omega^2(t) \;=\; \tfrac{exp\;(-\;2\;\lambda\;t)\;\hbar^2}{4\;m^2\;a^4(t)}\;. \qquad (2.1.3.25)$$

2.1.4. The Bohmnian Trajectories for the Diósi-Halliwell-Nassar Equation

The associated Bohmnian Trajectories, [10]-[14] for the Diósi-Halliwell-Nassar Equation $(DHN - E)$ of an evolving *ith* particle of the ensemble with an initial position x_{0i} can be calculated by considering that:

$$\dot{x}_i(t) \;=\; v_{qu}[x_i(t),\; t]. \qquad (2.1.4.1)$$

Then substituting the eq. (2.1.4.1) into eq. (2.1.3.5b), results:

$$\dot{x}_i(t) \;=\; \tfrac{dx_i(t)}{dt} \;=\; exp(\lambda\; t) \times \Big(\tfrac{\dot{a}(t)}{a(t)} \times [x \;-\; q(t)] \;+\; \dot{q}(t) \Big) \;\;\rightarrow$$

$$x_i(t) \;=\; \int_{t_0}^{t} \Big[exp(\lambda\; t) \times \Big(\tfrac{\dot{a}(t)}{a(t)} \times [x \;-\; q(t)] \;+\; \dot{q} \Big) \Big]\; dt. \qquad (2.1.4.2)$$

The eqs. (2.1.3.24,25) show that a continuous measurement of a quantum wave packet gives specific features to its evolution: the appearance of distinct classical and quantum elements, respectively. This measurement consist of monitoring the position of quantum systems and the result is the measured classical path $q(t)$ for t within a quantum uncertainty $a(t)$.

From the eq. (2.1.3.25), we note that for $\lambda \neq 0$ a stationary regime can be reached and that the width $[a(t)]$ of the wave packet can be related to the resolution of measurement as follows. Then considering that $a(t) \ = \ cte \ [\dot{a}(t) \ = \ 0; \ a_{t_0} \ = \ a_0]$ in the eq. (2.1.3.25) and considering the t_0 the *initial time*, we have:

$$exp \ (\lambda \ t_0) \ = \ \ell n \ [\omega_0 \ \tau_B], \qquad (2.1.4.3a)$$

where [10]:

$$\tau_B \ = \ \left(\frac{2 \ m \ a_0^2}{\hbar} \right) \ = \ 6,8\times \ 10^{-26} s, \qquad (2.1.4.1.3b)$$

is the *Bohmtime constant* which determines the time resolution of the quantum measurement.

We observe that the eq. (2.1.4.3a) to indicated that, as $\lambda \neq 0$, then: $t_0 \neq 0$.

The eqs. (2.1.4.3a,b) means that if an initially free wave packet is kept under a certain continuous measurement, its (a_0) may not spread in time. Then, the associated *Bohmnian Trajectories* of an evolving *ith* particle of the ensemble with an initial position x_{0i} is giving by eq. (2.1.4.2).

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