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Publicação IF 1706
21/07/2017

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THE FEYNMAN DE BROGLIE BOHM PROPAGATOR OF THE LINEARIZED SCHUCH-CHUNG-HARTMANN EQUATION ALONG A CLASSICAL TRAJETORY

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Abstract: In this paper we study the Feynman-de Broglie-Bohm propagator of the linearized Schuch-Chung-Hartmann equation along a classical trajetory.

PACS 03.65 - Quantum Mechanics

1. **Introduction**

In the present work we investigate the Feynman-de Broglie-Bohm propagator of the linearized Schuch-Chung-Hartmann equation along a classical trajetory by using the Quantum Mechanical of the de Broglie-Bohm.[1]

2. **The Schuch-Chung-Hartmann Equation**

In 1983-1985,[2] D. Schuch, K. M. Chung and H. Hartmann proposed a non-linear Schrödinger to represent time dependent physical systems, given by:

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(x,\ t)\ =\ -\ \frac{\hbar^2}{2\ m}\,\frac{\partial^2\psi(x,\ t)}{\partial x^2}\ +\ \Big(\ V(x,\ t)\ +$$

$$+\ \frac{\hbar\ \nu}{i}\ [\ell n\ \psi(x,\ t)-\ <\ell n\ \psi(x,\ t)>]\ \Big)\ \psi(x,\ t)\ , \qquad (2.1)$$

where $\psi(x,\ t)$ and $V(x,\ t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, and ν is a constant.

Writting the wavefunction $\psi(x,\ t)$ in the polar form, defined by the Madelung-Bohm [3, 4]:

$$\psi(x,\ t)\ =\ \phi(x,\ t)\ exp\ [i\ S(x,\ t)], \qquad (2.2)$$

where $S(x,\ t)$ is the classical action and $\phi(x,\ t)$ will be defined in what follows, and using eq. (2.2) into eq. (2.1), we get (remember that $\ell n\ e^{i\ S}\ =\ i\ S$): [1]

$$i\,\hbar\,(i\,\frac{\partial S}{\partial t}\ +\ \frac{1}{\phi}\,\frac{\partial \phi}{\partial t})\,\psi\ =$$

$$=\ -\ \frac{\hbar^2}{2\,m}\,[i\,\frac{\partial^2 S}{\partial x^2}\ +\ \frac{1}{\phi}\,\frac{\partial^2 \phi}{\partial x^2}\ -\ (\frac{\partial S}{\partial x})^2\ +\ 2\,\frac{i}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial \phi}{\partial x}]\,\psi\ +$$

$$+\ \Big(\,V(x,\ t)\ +\ \frac{\hbar\,\nu}{i}\,[\ell n\ (\phi\ e^{i\ S})\ -\ <\ell n\ (\phi\ e^{i\ S})>]\,\Big)\,\psi\quad\rightarrow$$

$$i\,\hbar\,(i\,\frac{\partial S}{\partial t}\ +\ \frac{1}{\phi}\,\frac{\partial \phi}{\partial t})\,\psi\ =\ -\ \frac{\hbar^2}{2\,m}\,[i\,\frac{\partial^2 S}{\partial x^2}\ +\ \frac{1}{\phi}\,\frac{\partial^2 \phi}{\partial x^2}\ -$$

$$-\ (\frac{\partial S}{\partial x})^2\ +\ 2\,\frac{i}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial \phi}{\partial x}]\,\psi\ +$$

$$+\ \Big(\,V(x,\ t)\ -\ i\,\hbar\,\nu\,[\ell n\ \phi\ +\ i\ S\ -\ <\ell n\ \phi>\ -\ i\ <S>]\,\Big)\,\psi\ . \qquad (2.3)$$

Taking the real and imaginary parts of eq. (2.3), we obtain:

a) imaginary part

$$\frac{1}{\phi}\,\frac{\partial \phi}{\partial t}\ =\ -\ \frac{\hbar}{2\,m}\,(\frac{\partial^2 S}{\partial x^2}\ +\ \frac{2}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial \phi}{\partial x})\ -$$

$$-\ \nu\ (\ell n\ \phi\ -\ <\ell n\ \phi>). \qquad (2.4)$$

b) real part

$$-\ \hbar\,\frac{\partial S}{\partial t}\ =\ -\ \frac{\hbar^2}{2\,m}\,[\frac{1}{\phi}\,\frac{\partial^2 \phi}{\partial x^2}\ -\ (\frac{\partial S}{\partial x})^2]\ +$$

$$+\ V(x,\ t)\ +\ \hbar\,\nu\ (S\ -\ <S>). \qquad (2.5)$$

2.1 **Dynamics of the Schuch-Chung-Hartmann Equation**

Now, let us to study the dynamics of the Schuch-Chung-Hartmann equation. To do is let us perform the following correspondences:[5]

$$\rho(x,\ t)\ =\ \phi^2(x,\ t)\ , \qquad (2.6) \quad \text{(quantum mass density)}$$

$$v_{qu}(x,\ t)\ =\ \frac{\hbar}{m}\,\frac{\partial S(x,\ t)}{\partial x}\ , \qquad (2.7) \quad \text{(quantum velocity)}$$

$$V_{qu}(x,\ t)\ =\ -\ \frac{\hbar^2}{2\,m}\,\frac{1}{\sqrt{\rho}}\,\frac{\partial^2 \sqrt{\rho}}{\partial x^2}\ =\ -\ \frac{\hbar^2}{2\,m\,\phi}\,\frac{\partial^2 \phi}{\partial x^2}\ . \qquad (2.8a,b) \quad \text{(Bohm quantum potential)}$$

Putting the eqs. (2.6,7) into eq. (2.4) we get [remember that $\frac{\partial}{\partial v}\ (\ell n\ u)\ =\ \frac{1}{u}\,\frac{\partial u}{\partial v}$ and $\ell n\ (u^n)\ =\ n\ \ell n\ u$]:

$$\frac{\partial}{\partial t}\ (2\ \ell n\ \phi)\ =$$

$$=\ -\ \frac{\hbar}{m}\ [\frac{\partial^2 S}{\partial x^2}\ +\ \frac{\partial S}{\partial x}\ \frac{\partial}{\partial x}\ (2\ \ell n\ \phi)]\ -\ 2\ \nu\ (\ell n\ \phi -\ <\ell n\ \phi>)\ \ \rightarrow$$

$$\frac{\partial}{\partial t}\ (\ell n\ \phi^2)\ =$$

$$=\ -\ \frac{\hbar}{m}\ [\frac{\partial^2 S}{\partial x^2}\ +\ \frac{\partial S}{\partial x}\ \frac{\partial}{\partial x}\ (\ell n\ \phi^2)]\ -\ 2\ \nu\ (\ell n\ \phi -\ <\ell n\ \phi>)\ \ \rightarrow$$

$$\frac{\partial}{\partial t}\ (\ell n\ \rho)\ =$$

$$=\ -\ \frac{\hbar}{m}\ [\frac{\partial^2 S}{\partial x^2}\ +\ \frac{\partial S}{\partial x}\ \frac{\partial}{\partial x}\ (\ell n\ \rho)]\ -\ 2\ \nu\ (\ell n\ \sqrt{\rho} -\ <\ell n\ \sqrt{\rho}>)\ \ \rightarrow$$

$$\frac{1}{\rho}\ \frac{\partial \rho}{\partial t}\ =$$

$$=\ -\ \frac{\hbar}{m}\ (\frac{\partial^2 S}{\partial x^2}\ +\ \frac{\partial S}{\partial x}\ \frac{1}{\rho}\ \frac{\partial \rho}{\partial x})\ -\ \nu\ (\ell n\ \rho -\ <\ell n\ \rho>)\ =$$

$$=\ -\ \frac{\partial}{\partial x}\ [\frac{\hbar}{m}\ (\frac{\partial S}{\partial x})]\ -\ \frac{1}{\rho}\ \frac{\partial\ \rho}{\partial x}\ [\frac{\hbar}{m}\ (\frac{\partial S}{\partial x})]\ -\ \nu\ (\ell n\ \rho -\ <\ell n\ \rho>)\ \ \rightarrow$$

$$\frac{1}{\rho}\ \frac{\partial \rho}{\partial t}\ +\ \frac{\partial v_{qu}}{\partial x}\ +\ \frac{v_{qu}}{\rho}\ \frac{\partial \rho}{\partial x}\ =\ -\ \nu\ (\ell n\ \rho -\ <\ell n\ \rho>)\ \ \rightarrow$$

$$\frac{\partial \rho}{\partial t}\ +\ \rho\ \frac{\partial v_{qu}}{\partial x}\ +\ v_{qu}\ \frac{\partial \rho}{\partial x}\ =\ -\ \nu\ \rho\ (\ell n\ \rho -\ <\ell n\ \rho>)\ \ \rightarrow$$

$$\frac{\partial \rho}{\partial t}\ +\ \frac{\partial (\rho\ v_{qu})}{\partial x}\ =\ -\ \nu\ \rho\ (\ell n\ \rho -\ <\ell n\ \rho>)\ . \qquad (2.9)$$

We must note that the presence of the second member in expression (2.9), indicates descoerence of the considered physical system represented by (2.1).

Now, taking the eq. (2.5) and using the eqs. (2.7,8b), will be:

$$-\ \hbar\ \frac{\partial S}{\partial t}\ =\ -\ (\frac{\hbar^2}{2\ m\ \phi})\ \frac{\partial^2 \phi}{\partial x^2}\ +\ \frac{1}{2}\ m\ (\frac{\hbar}{m}\ \frac{\partial S}{\partial x})^2\ +$$

$$+\ V(x,\ t)\ +\ \hbar\ \nu\ (S - <S>)\ \ \rightarrow$$

$$\hbar\ [\frac{\partial S}{\partial t}\ +\ \nu\ (S -\ <S>)]\ +\ \frac{1}{2}\ m\ v_{qu}^2\ +\ V\ +\ V_{qu}\ =\ 0\ . \qquad (2.10)$$

Considering that:

$$<f(x,\ t)> =\ \int_{-\ \infty}^{+\ \infty} \rho(x,\ t)\ f(x,\ t) dx\ =\ g(t)\ , \qquad (2.11)$$

then:

$$\frac{\partial <S>}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} \rho(x,\ t)\ S(x,\ t) dx = \frac{\partial g(t)}{\partial x} = 0\ . \qquad (2.12)$$

Now, differentiating the eq. (2.5) with respect x, and using the eqs. (2.7,8b,12), we have:

$$- \hbar\ \frac{\partial^2 S}{\partial x\ \partial t} = - \frac{\hbar^2}{2\ m}\ \frac{\partial}{\partial x}\ [\frac{1}{\phi}\ \frac{\partial^2 \phi}{\partial x^2} - (\frac{\partial S}{\partial x})^2] + \frac{\partial V}{\partial x} + \hbar\ \nu\ (\frac{\partial S}{\partial x} - \frac{\partial <S>}{\partial x}) \rightarrow$$

$$- \frac{\partial}{\partial t}\ (\frac{\hbar}{m}\ \frac{\partial S}{\partial x}) = \frac{1}{m}\ \frac{\partial}{\partial x}\ (- \frac{\hbar^2}{2\ m}\ \frac{1}{\phi}\ \frac{\partial^2 \phi}{\partial x^2}) +$$

$$+ \frac{1}{2}\ \frac{\partial}{\partial x}\ (\frac{\hbar}{m}\ \frac{\partial S}{\partial x})^2 + \frac{1}{m}\ \frac{\partial V}{\partial x} + \nu\ \frac{\hbar}{m}\ \frac{\partial S}{\partial x} - \frac{\hbar}{m}\ \frac{\partial <S>}{\partial x} \rightarrow$$

$$\frac{\partial v_{qu}}{\partial t} + v_{qu}\ \frac{\partial v_{qu}}{\partial x} +$$

$$+ \nu\ v_{qu} = - \frac{1}{m}\ \frac{\partial}{\partial x}\ (V + V_{qu})\ . \qquad (2.13)$$

Considering the "substantive differentiation" (local plus convective) or "hidrodynamic differention": $d/dt = \partial/\partial t + v_{qu}\ \partial/\partial x$ and that $v_{qu} = dx_{qu}/dt$, the eq. (2.13) could be written as:[5]

$$m\ d^2x/dt^2 = - \nu\ v_{qu} - \frac{1}{m}\ \frac{\partial}{\partial x}\ (V + V_{qu})\ , \qquad (2.14)$$

that has a form of the *Second Newton Law*.

As to calculated the Feynman-de Broglie-Bohm propagator of the linearized Schuch-Chung-Hartmann equation along a classical trajetory, is necessary to find the quantum wave packet (QWP) of the Schuch-Chung-Hartmann equation (see Cap. 4), then let us obtained the QWP.

3. **The Quantum Wave Packet of the Linearized Schuch-Chung-Hartmann Equation along a Classical Trajetory**

Initially, let us considerer the following *ansatz*:[6]

$$\rho\ (x,\ t) = [2\pi\ a^2(t)]^{-\ 1/2}\ exp\left(- \frac{[x - q(t)]^2}{2\ a^2(t)} \right), \qquad (3.1)$$

where $a(t)$ and $q(t)$ are auxiliary functions of time, to be determined in what follows; they represent the *width* and *center of mass of wave packet*, respectively.

Taking the eq. (3.1), let us calculated the expressions (remember that $\ell n\ e^{\alpha} = \alpha$):

$$\ell n\ \rho\ (x,\ t) = \ell n \left([2\ \pi\ a^2(t)]^{-\ 1/2}\ e^{- \frac{[x - q(t)]^2}{2\ a^2(t)}} \right) =$$

$$= \ell n\ [2\ \pi\ a^2(t)]^{-\ 1/2}\ -\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}\ . \qquad (3.2)$$

$$< \ell n\ \rho\ (x,\ t) > = < \ell n\ \Big([2\ \pi\ a^2(t)]^{-\ 1/2}\ e^{-\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}}\ \Big) > =$$

$$= \ell n\ [2\ \pi\ a^2(t)]^{-\ 1/2}\ -\ <\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}\ > . \qquad (3.3)$$

Considering that:

$$\int_{-\ \infty}^{\infty} z^2\ e^{-\ z^2}\ dz\ =\ \frac{\sqrt{\pi}}{2}\ ,$$

and the eq. (2.11), we have: [1]

$$\ell n\ \rho -\ < \ell n\ \rho > =\ -\ \frac{a^2}{2\ \rho}\ \frac{\partial^2 \rho}{\partial x^2}\ . \qquad (3.4)$$

Insering the eq. (3.4) into eq. (2.9), results:

$$\frac{\partial \rho}{\partial t}\ +\ \frac{\partial(\rho\ v_{qu})}{\partial x}\ =\ -\ \nu\ \rho\ (-\ \frac{a^2}{2\ \rho}\ \frac{\partial^2 \rho}{\partial x^2})\ \ \rightarrow$$

$$\frac{\partial \rho}{\partial t}\ +\ \frac{\partial(\rho\ v_{qu})}{\partial x}\ -\ \frac{\partial}{\partial x}\ (D\ \frac{\partial \rho}{\partial x})\ =\ 0, \qquad (3.5a)$$

where:

$$D\ =\ \frac{\nu\ a^2}{2}\ . \qquad (3.5b)$$

Defining: [7]

$$\vartheta_{qu}\ =\ v_{qu}\ -\ \frac{D}{\rho}\ \frac{\partial \rho}{\partial x}\ , \qquad (3.6a)$$

then the eq. (3.5a) will be the form:

$$\frac{\partial \rho}{\partial t}\ +\ \frac{\partial(\rho\ \vartheta_{qu})}{\partial x}\ =\ 0. \qquad (3.6b)$$

Now, substituting (3.1) into (3.5a) and integrated the result, we obtain: [1]

$$\vartheta_{qu}(x,\ t)\ =\ \frac{\dot{a}(t)}{a(t)}\ [x\ -\ q(t)]\ +\ \dot{q}(t)\ , \qquad (3.7a)$$

and:

$$v_{qu}\ (x,\ t)\ =\ [\frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}]\ [x\ -\ q(t)]\ +\ \dot{q}(t)\ . \qquad (3.7b)$$

To obtain the quantum wave packet of the linearized Schuch-Chung-Hartmann equation along a classical trajetory given by (2.1), let us expand the functions $S(x,\ t)$, $V(x,\ t)$ and $V_{qu}(x,\ t)$ around of $q(t)$ up to second Taylor order. In this way we have:

$$S(x,\ t)\ =\ S[q(t),\ t]\ +\ S'[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{S''[q(t),\ t]}{2}\ [x\ -\ q(t)]^2\ , \qquad (3.8)$$

$$V(x,\ t)\ =\ V[q(t),\ t]\ +\ V'[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{V''[q(t),\ t]}{2}\ [x\ -\ q(t)]^2\ . \qquad (3.9)$$

$$V_{qu}(x,\ t)\ =\ V_{qu}[q(t),\ t]\ +\ V'_{qu}[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{V''_{qu}[q(t),\ t]}{2}\ [x\ -\ q(t)]^2\ . \qquad (3.10)$$

Differentiating (3.8) in the variable x, multiplying the result by $\frac{\hbar}{m}$, using the eqs. (2.7) and (3.7b), taking into account the polynomial identity property and also considering the second Taylor order, we obtain:

$$\frac{\hbar}{m}\ \frac{\partial S(x,\ t)}{\partial x}\ =\ \frac{\hbar}{m}\ \Big(\ S'[q(t),\ t]\ +\ S''[q(t),\ t]\ [x\ -\ q(t)]\ \Big)\ =$$

$$=\ v_{qu}(x,\ t)\ =\ [\frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}]\ [x\ -\ q(t)]\ +\ \dot{q}(t)\quad \to$$

$$S'[q(t),\ t]\ =\ \frac{m\ \dot{q}(t)}{\hbar}\ ,\qquad S''[q(t),\ t]\ =\ \frac{m}{\hbar}\ [\frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}]\ , \qquad (3.11a,b)$$

Substituting (3.11a,b) into (3.8), results:

$$S(x,\ t)\ =\ S_o(t)\ +\ \frac{m\ \dot{q}(t)}{\hbar}\ [x\ -\ q(t)]\ +\ \frac{m}{2\ \hbar}\ [\frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}]\ [x\ -\ q(t)]^2\ , \qquad (3.12a)$$

where:

$$S_o(t) \equiv S[q(t),\ t]\ , \qquad (3.12b)$$

are the classical actions.

Now, considering that:

$$\int_{-\ \infty}^{\infty}\ z^n\ e^{-\ z^2}\ dz\ =\ \frac{\sqrt{\pi}}{2};\ 0;\ \sqrt{\pi}\ ,$$

respectivelly, for n = 2, 1, 0, and using the eqs. (2.11), (3.1) and (3.12a), we have:

$$< S > =\ \int_{-\ \infty}^{+\ \infty}\ \rho(x,\ t)\ S(x,\ t)dx\ =\ S_1\ +\ S_2\ +\ S_3, \qquad (3.13a)$$

where:

$$S_1\ =\ \int_{-\ \infty}^{+\ \infty}\ [2\pi\ a^2(t)]^{-\ 1/2}\ exp\ \Big(\ -\ \frac{[x\ -\ q(t)]^2}{2\ a^2(t)}\ \Big)\ S_0(t)\ dx\ =\ S_0(t), \qquad (3.13b)$$

$$S_2 \; = \; \int_{-\,\infty}^{+\,\infty} [2\pi \; a^2(t)]^{-\;1/2} \; exp \left(\; - \; \frac{[x \; - \; q(t)]^2}{2 \; a^2(t)} \; \right) \frac{m \; \dot{q}(t)}{\hbar} \; [x \; - \; q(t)] \; dx \; = \; 0, \qquad (3.13c)$$

$$S_3 \; = \; \int_{-\,\infty}^{+\,\infty} [2\pi \; a^2(t)]^{-\;1/2} \; exp \left(\; - \; \frac{[x \; - \; q(t)]^2}{2 \; a^2(t)} \; \right) \frac{m}{2 \; \hbar} \; [\frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2}] \; [x \; - \; q(t)]^2 \; dx =$$

$$= \; a^2(t) \times \frac{m}{2 \; \hbar} \; [\frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2}]. \qquad (3.13d)$$

Inserting the eqs. (3.13b-d) into eq. (3.13a), results:

$$< S > = \; S_0(t) \; + \; a^2(t) \times \frac{m}{2 \; \hbar} \; [\frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2}]. \qquad (3.14)$$

Differentiating the (3.12a) with respect to t, we obtain (remembering that $\frac{\partial x}{\partial t} = 0$):

$$\frac{\partial S \; (x, \; t)}{\partial t} \; = \; \dot{S}_o(t) \; + \; \frac{\partial}{\partial t} \; [\; \frac{m \; \dot{q}(t)}{\hbar} \; [x \; - \; q(t)] \;] \; + \; \frac{\partial}{\partial t} \left[\; \frac{m}{2 \; \hbar} \left([\frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2}] \right) [x \; - \; q(t)]^2 \; \right] \; \rightarrow$$

$$\frac{\partial S \; (x, \; t)}{\partial t} \; = \; \dot{S}_o(t) \; + \; \frac{m \; \ddot{q}(t)}{\hbar} \; [x \; - \; q(t)] \; - \; \frac{m \; \dot{q}^2(t)}{\hbar} \; +$$

$$+ \; \frac{m}{2 \; \hbar} \; [\frac{\ddot{a}(t)}{a(t)} \; - \; \frac{\dot{a}^2(t)}{a^2(t)}] \; [x \; - \; q(t)]^2 \; - \; \frac{m \; \dot{q}(t)}{\hbar} \left(\; \frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2} \; \right) [x \; - \; q(t)] \; . \qquad (3.15)$$

Considering the eqs. (2.6) and (3.1), let us write V_{qu} given by (2.8a,b) in terms of potencies of $[x \; - \; q(t)]$. Before, we calculate the following derivations:

$$\frac{\partial \phi \; (x, \; t)}{\partial x} \; = \; \frac{\partial}{\partial x} \left(\; [2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; a^2(t)}} \; \right) \; =$$
$$[2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; a^2(t)}} \; \frac{\partial}{\partial x} \left(\; - \; \frac{[x \; - \; q(t)]^2}{4 \; a^2(t)} \; \right) \; \rightarrow$$

$$\frac{\partial \phi \; (x, \; t)}{\partial x} \; = \; - \; [2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; ^2(t)}} \; \frac{[x \; - \; q(t)]}{2 \; a^2(t)} \; ,$$

$$\frac{\partial^2 \phi \; (x, \; t)}{\partial x^2} \; = \; \frac{\partial}{\partial x} \left(\; - \; [2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; a^2(t)}} \; \frac{[x \; - \; q(t)]}{2 \; a^2(t)} \; \right) =$$

$$= \; - \; [2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; a^2(t)}} \; \frac{\partial}{\partial x} \left(\; \frac{[x \; - \; q(t)]}{2 \; a^2(t)} \; \right) -$$

$$- \; [2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; a^2(t)}} \; \frac{[x \; - \; q(t)]}{2 \; a^2(t)} \; \frac{\partial}{\partial x} \left(\; - \; \frac{[x \; - \; q(t)]^2}{4 \; a^2(t)} \; \right) \; \rightarrow$$

$$\frac{\partial^2 \phi \; (x, \; t)}{\partial x^2} \; = \; - \; [2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; a^2(t)}} \; \frac{1}{2 \; a^2(t)} \; + \; [2 \; \pi \; a^2(t)]^{-\;1/4} \; e^{-\;\frac{[x \; - \; q(t)]^2}{4 \; a^2(t)}} \; \frac{[x \; - \; q(t)]^2}{4 \; a^4(t)} \; =$$

$$= \, - \, \phi \, (x, \, t) \, \frac{1}{2 \, a^2(t)} \, + \, \phi \, (x, \, t) \, \frac{[x \, - \, q(t)]^2}{4 \, a^4(t)} \quad \to$$

$$\frac{1}{\phi \, (x, \, t)} \, \frac{\partial^2 \phi \, (x, \, t)}{\partial x^2} \, = \, \frac{[x \, - \, q(t)]^2}{4 \, a^4(t)} \, - \, \frac{1}{2 \, a^2(t)} \, . \qquad (3.16)$$

Substituting (3.16) into (2.8b) and taking into account (3.10), results:

$$V_{qu}(x, \, t) \, = \, \frac{\hbar^2}{4 \, m \, a^2(t)} \, - \, \frac{\hbar^2}{8 \, m \, a^4(t)} \, [x \, - \, q(t)]^2 \, . \qquad (3.17)$$

$$V_{qu}[q(t), \, t] \, = \, \frac{\hbar^2}{4 \, m \, a^2(t)} \, , \qquad (3.18a)$$

$$V'_{qu}[q(t), \, t] \, = \, 0, \quad V''_{qu}[q(t), \, t] \, = \, - \, \frac{\hbar^2}{4 \, m \, a^4(t)} \, . \qquad (3.18b,c)$$

Inserting the eqs. (3.7b,8,9) and (3.12a,14,15,17), into (2.10), we obtain [remembering that $S_o(t)$, $a(t)$ and $q(t)$]:

$$\hbar \, [\frac{\partial S}{\partial t} \, + \, \nu \, (S - \, < S >)] \, + \, \frac{1}{2} \, m \, v_{qu}^2 \, + \, V \, + \, V_{qu} \, =$$

$$= \, \hbar \, \dot{S}_o(t) \, + \, m \, \ddot{q}(t) \, [x \, - \, q(t)] \, - \, m \, \dot{q}^2(t) \, + \, \frac{m}{2} \, [\, \frac{\ddot{a}(t)}{a(t)} \, - \, \frac{\dot{a}^2(t)}{a^2(t)} \,] \, [x \, - \, q(t)]^2 \, -$$

$$- \, m \, \dot{q}(t) \, [\frac{\dot{a}(t)}{a(t)} \, - \, \frac{\nu}{2} \,] \, [x \, - \, q(t)] \, + \, \nu \, \Big(\, \hbar \, S_0(t) \, + \, m \, \dot{q}(t) \, [x \, - \, q(t)] \, +$$

$$+ \, \frac{m}{2} \, [\frac{\dot{a}(t)}{a(t)} \, - \, \frac{\nu}{2}] \, [x \, - \, q(t)]^2 \, - \, \hbar \, S_0(t) \, - \, a^2(t) \, \times \, \frac{m}{2} \, [\frac{\dot{a}(t)}{a(t)} \, - \, \frac{\nu}{2}] \, \Big) \, +$$

$$+ \, \frac{m}{2} \, [\frac{\dot{a}(t)}{a(t)} \, - \, \frac{\nu}{2}]^2 \, [x \, - \, q(t)]^2 \, + \, m \, \dot{q}(t) \, [\frac{\dot{a}(t)}{a(t)} \, - \, \frac{\nu}{2}] \, [x \, - \, q(t)] \, + \, \frac{m \, \dot{q}^2(t)}{2} \, +$$

$$+ \, V[q(t), \, t] \, + \, V'[q(t), \, t] \, [x \, - \, q(t)] \, + \, \frac{1}{2} \, V''[q(t), \, t] \, [x \, - \, q(t)]^2 \, +$$

$$+ \, \frac{\hbar^2}{4 \, m \, a^2(t)} \, - \, \frac{\hbar^2}{8 \, m \, a^4(t)} \, [x \, - \, q(t)]^2 \, = \, 0 \, . \qquad (3.19)$$

Expanding the eq. (3.19) in potencies of $[x \, - \, q(t)]$, we obtain (remember that $[x \, - \, q(t)]^o \, = \, 1$):

$$\Big(\, \hbar \, \dot{S}_o(t) \, - \, \frac{1}{2} \, m \, \dot{q}^2(t) \, - \, a^2(t) \, \times \, \frac{m \, \nu}{2} \, [\frac{\dot{a}(t)}{a(t)} \, - \, \frac{\nu}{2}] \, + \, V[q(t), \, t] \, + \, \frac{\hbar^2}{4 \, m \, a^2(t)} \, \Big) \, [x \, - \, q(t)]^o \, +$$

$$+ \, \Big(\, m \, \ddot{q}(t) \, + \, \nu \, m \, \dot{q}(t) \, + \, V'[q(t), \, t] \, \Big) \, [x \, - \, q(t)] \, +$$

$$+ \, \Big(\, \frac{m}{2} \, \frac{\ddot{a}(t)}{a(t)} \, - \, \frac{\nu^2 \, m}{8} \, + \, \frac{1}{2} \, V"[q(t), \, t] \, - \, \frac{\hbar^2}{8 \, m \, a^4(t)} \, \Big) \, [x \, - \, q(t)]^2 \, = \, 0 \, . \qquad (3.22)$$

As (3.22) is an identically null polynomium, all coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) \ = \ \frac{1}{\hbar}\Big(\ \frac{1}{2} \ m \ \dot{q}^2(t) \ + \ a^2(t) \times \frac{\nu \ m}{2} \ [\frac{\dot{a}(t)}{a(t)} \ - \ \frac{\nu}{2}] \ - \ V[q(t), \ t] \ - \ \frac{\hbar^2}{4 \ m \ a^2(t)} \ \Big) \ , \qquad (3.23)$$

$$\ddot{q}(t) \ + \ \nu \ \dot{q}(t) \ + \ \frac{1}{m} \ V'[q(t), \ t] \ = \ 0, \qquad (3.24)$$

$$\ddot{a}(t) \ + \ a(t) \ \Big(\ \frac{1}{m} \ V"[q(t), \ t] \ - \ \frac{\nu^2}{4} \ \Big) \ = \ \frac{\hbar^2}{4 \ m^2 \ a^3(t)} \ . \qquad (3.25)$$

Assuming that the following initial conditions are obeyed:

$$q(0) \ = \ x_o \ , \quad \dot{q}(0) \ = \ v_o \ , \quad a(0) \ = \ a_o \ , \quad \dot{a}(0) \ = \ b_o \ , \qquad (3.26\text{a-d})$$

and that:

$$S_o(0) \ = \ \frac{m \ v_o \ x_o}{\hbar} \ , \qquad (3.27)$$

the integration of (3.23) gives:

$$S_o(t) \ = \ \frac{1}{\hbar} \int_o^t dt' \ \Big(\ \frac{1}{2} \ m \ \dot{q}^2(t') \ + \ a^2(t) \times \frac{m \ \nu}{2} \ [\frac{\dot{a}(t)}{a(t)} \ - \ \frac{\nu}{2}] \ -$$

$$- \ V[q(t'), \ t'] \ - \ \frac{\hbar^2}{4 \ m \ a^2(t')} \ \Big) \ + \ \frac{m \ v_o \ x_o}{\hbar}. \qquad (3.28)$$

Taking the eq. (3.28) in the eq. (3.12a) results:

$$S(x, \ t) \ = \ \frac{1}{\hbar} \int_o^t dt' \ \Big(\ \frac{1}{2} \ m \ \dot{q}^2(t') \ + \ a^2(t) \times \frac{m \ \nu}{2} \ [\frac{\dot{a}(t)}{a(t)} \ - \ \frac{\nu}{2}] \ - \ V[q(t'), \ t'] \ - \ \frac{\hbar^2}{4 \ m \ a^2(t')} \ \Big) \ +$$

$$+ \ \frac{m \ v_o \ x_o}{\hbar} \ + \ \frac{m \ \dot{q}(t)}{\hbar} \ [x \ - \ q(t)] \ + \ \frac{m}{2 \ \hbar} \ \Big[\ \frac{\dot{a}(t)}{a(t)} \ - \ \frac{\nu}{2} \ \Big] \ [x \ - \ q(t)]^2 \ . \qquad (3.29)$$

The above result permit us, finally, to obtain the wave packet for the linearized Schuch-Chung-Hartmann equation along a classical trajetory. Indeed, considering (2.2), (2.6), (3.1) and (3.29), we get: [6]

$$\psi(x, \ t) \ = \ [2 \ \pi \ a^2(t)]^{- \ 1/4} \ exp \ \Big[\ \Big(\ \frac{i \ m}{2 \ \hbar} \ [\frac{\dot{a}(t)}{a(t)} \ - \ \frac{\nu}{2}] \ - \ \frac{1}{4 \ a^2(t)} \ \Big) \ [x \ - \ q(t)]^2 \ \Big] \ \times$$

$$\times \ exp \ \Big[\ \frac{i \ m \ \dot{q}(t)}{\hbar} \ [x \ - \ q(t)] \ + \ \frac{i \ m \ v_o \ x_o}{\hbar} \ \Big] \ \times$$

$$\times \ exp \ \Big[\ \frac{i}{\hbar} \int_o^t dt' \ \Big(\ \frac{1}{2} \ m \ \dot{q}^2(t') \ + \ a^2(t) \times \frac{m \ \nu}{2} \ [\frac{\dot{a}(t)}{a(t)} \ - \ \frac{\nu}{2}] \ - \ V[q(t'), \ t'] \ -$$

$$- \frac{\hbar^2}{4\ m\ a^2(t')} \Big) \Big] \ . \qquad (3.30)$$

Note that putting $\nu \ = \ 0$ into (3.30) we obtain the quantum wave packet of the Schrödinger equation with the potential V(x, t). [7]

4. **The Feynman-de Broglie-Bohm Propagator of the Linearized Schuch-Chung-Hartmann Equation along a Classical Trajetory**

4.1. **Introduction**

In 1948, [8] Feynman formulated the following principle of minimum action for the Quantum Mechanics:

The transition amplitude between the states $\mid a\ >$ and $\mid b\ >$ of a quantum-mechanical system is given by the sum of the elementary contributions, one for each trajectory passing by $\mid a\ >$ at the time t_a and by $\mid b\ >$ at the time t_b. Each one of these contributions have the same modulus, but its phase is the classical action $S_{c\ell}$ for each trajectory.

This principle is represented by the following expression known as the "Feynman propagator":

$$K(b,\ a)\ =\ \int_a^b e^{\frac{i}{\hbar}\ S_{c\ell}(b,\ a)}\ D\ x(t)\ , \qquad (4.1)$$

with:

$$S_{c\ell}(b,\ a)\ =\ \int_{t_a}^{t_b} L\ (x,\ \dot{x},\ t)\ dt\ , \qquad (4.2)$$

where $L(x,\ \dot{x},\ t)$ is the Lagrangean and $D\ x(t)$ is the Feynman's Measurement. It indicates that we must perform the integration taking into account all the ways connecting the states $\mid a\ >$ and $\mid b\ >$.

Note that the integral which defines $K(b,\ a)$ is called "path integral" or "Feynman integral" and that the Schrödinger wavefunction $\psi(x,\ t)$ of any physical system is given by (we indicate the initial position and initial time by x_o and t_o, respectively): [9]

$$\psi(x,\ t)\ =\ \int_{-\ \infty}^{+\ \infty} K(x,\ x_o,\ t,\ t_o)\ \psi(x_o,\ t_o)\ dx_o\ , \qquad (4.3)$$

with the quantum causality condition:

$$\lim_{t,\ t_o\ \to\ 0} K(x,\ x_o,\ t,\ t_o)\ =\ \delta(x\ -\ x_o)\ . \qquad (4.4)$$

4.2. **Calculation of the Feynman-de Broglie-Bohm Propagator for the Linearized Schuch-Chung-Hartmann Equation along a Classical Trajetory**

According to Section 3, the wavefunction $\psi(x,\ t)$ that was named wave packet of the of the linearized Schuch-Chung-Hartmann equation along a classical trajetory, can be written as [see (3.30)]:

$$\psi(x,\ t)\ =\ [2\ \pi\ a^2(t)]^{-\ 1/4}\ exp\ \Big[\ \Big(\ \tfrac{i\ m}{2\ \hbar}\ [\tfrac{\dot{a}(t)}{a(t)}\ -\ \tfrac{\nu}{2}]\ -\ \tfrac{1}{4\ a^2(t)}\ \Big)\ [x\ -\ q(t)]^2\ \Big]\ \times$$

$$\times\ exp\ \Big[\ \tfrac{i\ m\ \dot{q}(t)}{\hbar}\ [x\ -\ q(t)]\ +\ \tfrac{i\ m\ v_o\ x_o}{\hbar}\ \Big]\ \times$$

$$\times\ exp\ \Big[\ \tfrac{i}{\hbar}\ \int_o^t\ dt'\ \Big(\ \tfrac{1}{2}\ m\ \dot{q}^2(t')\ +\ a^2(t)\ \times\ \tfrac{m\ \nu}{2}\ [\tfrac{\dot{a}(t)}{a(t)}\ -\ \tfrac{\nu}{2}]\ -\ V[q(t'),\ t']\ -$$

$$-\ \tfrac{\hbar^2}{4\ m\ a^2(t')}\ \Big)\ \Big]\ .\qquad (4.5)$$

where [see (3.24,25)]:

$$\ddot{q}(t)\ +\ \nu\ \dot{q}(t)\ +\ \tfrac{1}{m}\ V'[q(t),\ t]\ =\ 0,\qquad (4.6)$$

$$\ddot{a}(t)\ +\ a(t)\ \Big(\ \tfrac{1}{m}\ V"[q(t),\ t]\ \ -\ \tfrac{\nu^2}{4}\ \Big)\ =\ \tfrac{\hbar^2}{4\ m^2\ a^3(t)}\ .\qquad (4.7)$$

where the following initial conditions were obeyed [see (3.26a-d)]:

$$q(0)\ =\ x_o\ ,\quad \dot{q}(0)\ =\ v_o\ ,\quad a(0)\ =\ a_o\ ,\quad \dot{a}(0)\ =\ b_o\ .\qquad (4.8a\text{-}d)$$

Therefore, considering (4.3), the Feynman-de Broglie-Bohm propagator will be calculated using (4.5), in which we will put with no loss of generality, $t_o\ =\ 0$. Thus:

$$\psi(x,\ t)\ =\ \int_{-\ \infty}^{+\ \infty}\ K(x,\ x_o,\ t)\ \psi(x_o,\ 0)\ dx_o\ .\qquad (4.9)$$

Let us initially define the normalized quantity:

$$\Phi(v_o,\ x,\ t)\ =\ (2\ \pi\ a_o^2)^{1/4}\ \psi(v_o,\ x,\ t)\ ,\qquad (4.10)$$

which satisfies the following completeness relation: [10]

$$\int_{-\ \infty}^{+\ \infty}\ dv_o\ \Phi^*(v_o,\ x,\ t)\ \Phi(v_o,\ x',\ t)\ =\ (\tfrac{2\ \pi\ \hbar}{m})\ \delta(x\ -\ x')\ .\qquad (4.11)$$

Taking the eqs. (2.2,6), we have:

$$\psi^*(x,\ t)\ \psi(x,\ t)\ =\ \phi^2\ =\ \rho(x,\ t)\ .\qquad (4.12)$$

Now, using the eqs. (4.10,12), we get:

$$\Phi^*(v_o,\ x,\ t)\ \psi(v_o,\ x,\ t)\ =$$

$$=\ (2\ \pi\ a_o^2)^{1/4}\ \psi^*(v_o,\ x,\ t)\ \psi(v_o,\ x,\ t)\ =\ (2\ \pi\ a_o^2)^{1/4}\ \rho(v_o,\ x,\ t)\ \ \rightarrow$$

$$\rho(v_o,\ x,\ t)\ =\ (2\ \pi\ a_o^2)^{-\ 1/4}\ \Phi^*(v_o,\ x,\ t)\ \psi(v_o,\ x,\ t)\ . \qquad (4.13)$$

On the other side, substituting (4.13) into (3.6b), integrating the result and using (3.1) and (4.10) results [remembering that $\psi^*\ \psi(\pm\ \infty)\ \ \rightarrow\ \ 0$]:

$$\frac{\partial(\Phi^*\ \psi)}{\partial t}\ +\ \frac{\partial(\Phi^*\ \psi\ \vartheta_{qu})}{\partial x}\ =\ 0\ \ \rightarrow$$

$$\frac{\partial}{\partial t}\int_{-\ \infty}^{+\ \infty} dx\ \Phi^*\ \psi\ +\ (\Phi^*\ \psi\ \vartheta_{qu})|_{-\ \infty}^{+\ \infty}\ =$$

$$=\ \frac{\partial}{\partial t}\int_{-\ \infty}^{+\ \infty} dx\ \Phi^*\ \psi\ +\ (2\ \pi\ a_o^2)^{1/4}\ (\psi^*\ \psi\ \vartheta_{qu})|_{-\ \infty}^{+\ \infty}\ =\ 0\ \ \rightarrow$$

$$\frac{\partial}{\partial t}\int_{-\ \infty}^{+\ \infty} dx\ \Phi^*\ \psi\ =\ 0\ . \qquad (4.14)$$

The eq. (4.14) shows that the integration is time independent. Consequently:

$$\int_{-\ \infty}^{+\ \infty} dx'\ \Phi^*(v_o,\ x',\ t)\ \psi(x',\ t)\ =\ \int_{-\ \infty}^{+\ \infty} dx_o\ \Phi^*(v_o,\ x_o,\ 0)\ \psi(x_o,\ 0)\ . \qquad (4.15)$$

Multiplying (4.15) by $\Phi(v_o,\ x,\ t)$ and integrating over v_o and using (4.11), we obtain [remembering that $\int_{-\ \infty}^{+\ \infty} dx'\ f(x')\ \delta(x' -\ x)\ = f(x)$]:

$$\int_{-\ \infty}^{+\ \infty}\int_{-\ \infty}^{+\ \infty} dv_o\ dx'\ \Phi(v_o,\ x,\ t)\ \Phi^*(v_o,\ x',\ t)\ \psi(x',\ t) =$$

$$= \int_{-\ \infty}^{+\ \infty}\int_{-\ \infty}^{+\ \infty} dv_o\ dx_o\ \Phi(v_o,\ x,\ t)\ \Phi^*(v_o,\ x_o,\ 0)\ \psi(x_o,\ 0)\ \ \rightarrow$$

$$\int_{-\ \infty}^{+\ \infty} dx'\ (\tfrac{2\ \pi\ \hbar}{m})\ \delta(x'\ -\ x)\ \psi(x',\ t)\ =\ (\tfrac{2\ \pi\ \hbar}{m})\ \psi(x,\ t) =$$

$$= \int_{-\ \infty}^{+\ \infty}\int_{-\ \infty}^{+\ \infty} dv_o\ dx_o\ \Phi(v_o,\ x,\ t)\ \Phi^*(v_o,\ x_o,\ 0)\ \psi(x_o,\ 0)\ \ \rightarrow$$

$$\psi(x,\ t)\ =\ \int_{-\ \infty}^{+\ \infty}\Big[\ (\tfrac{m}{2\ \pi\ \hbar})\int_{-\ \infty}^{+\ \infty} dv_o\ \Phi(v_o,\ x,\ t)\ \times$$

$$\times\ \Phi^*(v_o,\ x_o,\ 0)\ \Big]\ \psi(x_o,\ 0)\ dx_o\ . \qquad (4.16)$$

Comparing (4.9) and (4.16), we have:

$$K(x,\ x_o,\ t)\ =\ \tfrac{m}{2\ \pi\ \hbar}\int_{-\ \infty}^{+\ \infty} dv_o\ \Phi(v_o,\ x,\ t)\ \Phi^*(v_o,\ x_o,\ 0)\ . \qquad (4.17)$$

Substituting (4.5) and (4.10) into (4.17), we finally obtain the Feynman-de Broglie-Bohm Propagator of the linearized Schuch-Chung-Hartmann equation along a classical trajetory [remembering that $\Phi^*(v_o,\ x_o,\ 0)\ =\ exp\ (-\ \frac{i\ m\ v_o\ x_o}{\hbar})$]:

$$K(x,\ x_o;\ t)\ =\ \frac{m}{2\ \pi\ \hbar}\ \int_{-\ \infty}^{+\ \infty} dv_o\ \sqrt{\frac{a_o}{a(t)}}\ \times$$

$$\times\ exp\ \Big[\ \Big(\ \frac{i\ m}{2\ \hbar}\ [\frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}]\ -\ \frac{1}{4\ a^2(t)}\ \Big)\ [x\ -\ q(t)]^2\ +\ \frac{i\ m\ \dot{q}(t)}{\hbar}\ [x\ -\ q(t)]\ \Big]\ \times$$

$$\times\ exp\ \Big[\ \frac{i}{\hbar}\ \int_o^t dt'\ \Big(\ \frac{1}{2}\ m\ \dot{q}^2(t')\ +\ a^2(t')\ \times\ \frac{m\ \nu}{2}\ [\frac{\dot{a}(t')}{a(t')}\ -\ \frac{\nu}{2}]\ -\ V[q(t'),\ t']\ -$$

$$-\ \frac{\hbar^2}{4\ m\ a^2(t')}\ \Big)\ \Big]\ , \quad (4.18)$$

where $q(t)$ and $a(t)$ are solutions of the (4.6, 7) differential equations.

Finally, it is important to note that putting $\nu\ =\ 0$ and V[q(t'), t'] = 0 into (4.6), (4.7) and (4.18) we obtain the free particle Feynman propagator. [1, 9]

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