Instituto de Física
Universidade de São Paulo

# Bohmnian Trajectories for the Schuch-Chung-Hartmann Equation

José Maria Filardo Bassalo
Antonio Boulhosa Nassar
Mauro Sérgio Dorsa Cattani

Publicação IF 1707
09/08/2017

UNIVERSIDADE DE SÃO PAULO
Instituto de Física
Cidade Universitária
Caixa Postal 66.318
05315-970 - São Paulo - Brasil

# Bohmnian Trajectories for the Schuch-Chung-Hartmann Equation

**José Maria Filardo Bassalo**

Academia Paraense de Ciências (www.apaci.net.org)

jmfbassalo@gmail.com

**Antonio Boulhosa Nassar**

Extension Program-Department of Sciences, University of California, Los Angeles, California 90024

nassar@ucla.edu

**Mauro Sérgio Dorsa Cattani**

Instituto de Física da USP, 05389-970, São Paulo, SP

mcattani@if.usp.br

**Abstract**: In this paper we study the Bohmnian Trajectories for the Schuch-Chung-Hartmann Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm.

**Keywords**: De Broglie-Bohm Quantum Mechanics; Bohmnian Trajectories of the Schuch-Chung-Hartmann Equation

PACS 03.65 - Quantum Mechanics

## 1. Introduction: The Bohmnian Trajectories

In this article, we calculated the *Bohmnian Trajectories* for the Schuch-Chung-Hartmann Equation. To obtain these trajectories we adopted the quantum mechanical formalism of de Broglie-Bohm. This was done because this formalism permits to perform essential linear approximations along the classical trajectories that are the basic ingredients of the of the Feynman's principle of minimum action of quantum mechanics. [1]

## 2. The Bohmnian Trajectories for the Schuch-Chung-Harmann Equation

Now, let us calculate the Bohmnian trajectories for the Schuch-Chung-Hartman Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm. [2]

### 2.1. The Schuch-Chung-Hartmann Equation

In 1983-1985, D. Schuch, K. M. Chung and Hermann Hartmann [3] proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\,\frac{\partial\Psi(x,\ t)}{\partial t} \;=\; -\,\frac{\hbar^2}{2\ m}\,\frac{\partial^2\ \Psi(x,\ t)}{\partial x^2}\;+$$

$$+\ \{\ V(x,\ t)\ +\ \frac{\hbar\ \nu}{i}\ [\ \ell n\Psi(x,\ t) -\ <\ell n\Psi(x,\ t)>\ ]\ \}\ \times\ \Psi(x,\ t)\ , \qquad (2.1.1)$$

where $\Psi(x,\ t)$ and $V(x,\ t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, and $\nu$ is a constant.

#### 2.1.1. The Wave Function of the Schuch-Chung-Hartmann Equation

Writting the wave function $\Psi(x,\ t)$ in the polar form defined by the Madelung-Bohm transformation [4, 5] we obtain:

$$\Psi(x,\ t)\ =\ \phi(x,\ t)\ \times\ exp\ [i\ S(x,\ t)]\ , \qquad (2.1.1.1)$$

where $\phi(x,\ t)$ will be defined in what follows.

Calculating the derivatives, temporal and spatial, of (2.1.1.1), we get [remembering that $exp\ [i\ S]$ is common factor]: [2]

$$\frac{\partial\Psi}{\partial t}\ =\ exp\ (i\ S)\ (\ \frac{\partial\phi}{\partial t}\ +\ i\ \phi\ \frac{\partial S}{\partial t}\ )\ , \qquad (2.1.1.2a)$$

$$\frac{\partial^2\Psi}{\partial x^2}\ =\ exp\ (i\ S)\ [\ \frac{\partial^2\phi}{\partial x^2}\ +\ 2\ i\ \frac{\partial S}{\partial x}\ \frac{\partial\phi}{\partial x}\ +\ i\ \phi\ \frac{\partial^2 S}{\partial x^2}\ -\ \phi\ (\ \frac{\partial S}{\partial x}\ )^2\ ]\ . \qquad (2.1.1.2b)$$

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.1.1), we have: [2]

$$i\,\hbar\left(\ \frac{\partial\phi}{\partial t}\ +\ i\ \phi\ \frac{\partial S}{\partial t}\ \right)\ =\ -\ \frac{\hbar^2}{2\ m}\left[\ \frac{\partial^2\phi}{\partial x^2}\ +\ 2\ i\ \frac{\partial S}{\partial x}\ \frac{\partial\phi}{\partial x}\ +\ i\ \phi\ \frac{\partial^2 S}{\partial x^2}\ -\ \phi\left(\ \frac{\partial S}{\partial x}\ \right)^2\ \right] +$$

$$+\ \{\ V(x,\ t)\ -\ i\ \hbar\ \nu\ [\ \ell n\phi -\ <\ell n\phi>\ +i(S-<S>)\ ]\ \}\ \times\ \phi(x,\ t)\ , \qquad (2.1.1.3)$$

Separating the real and imaginary parts of the relation (2.1.1.3), results:

a) imaginary part

$$\frac{\partial\phi}{\partial t}\ =\ -\ \frac{\hbar}{2\ m}\ (\ 2\ \frac{\partial S}{\partial x}\ \frac{\partial\phi}{\partial x}\ +\ \phi\ \frac{\partial^2 S}{\partial x^2}\ )\ -\ \nu\ (\ell n\phi -\ <\ell n\phi>)\ , \qquad (2.1.1.4)$$

b) real part

$$-\ \hbar\ \frac{\partial S}{\partial t}\ =\ -\ \frac{\hbar^2}{2\ m}\ [\ \frac{1}{\phi}\ \frac{\partial^2\phi}{\partial x^2}\ -\ (\ \frac{\partial S}{\partial x}\ )^2\ ]\ +\ V(x,\ t)\ +\ \hbar\ \nu\ (S -\ <S>)\ . \qquad (2.1.1.5)$$

### 2.1.2. Dynamics of the Schuch-Chung-Hartmann Equation

Now, let us see the correlation between the expressions (2.1.1.3-4) and the traditional equations of the Ideal Fluid Dynamics: [6] a) *Continuity Equation*, b) *Euler's equation* (for conservative systems) or b') *Navier-Stokes equation* (for non-conservative systems). To do is let us perform the following correspondences:

$$\sqrt{\rho(x,\ t)} \ = \ \phi(x,\ t) \ , \qquad (2.1.2.1) \quad \text{(quantum mass density)}$$

$$v_{qu}(x,\ t) \ = \ \frac{\hbar}{m} \ \frac{\partial S(x,\ t)}{\partial x} \ , \qquad (2.1.2.2) \qquad \text{(quantum velocity)}$$

$$V_{qu}(x,\ t) \ = \ - \ \frac{\hbar^2}{2\ m} \ \frac{1}{\sqrt{\rho}} \ \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \ = \ - \ \frac{\hbar^2}{2\ m\ \phi} \ \frac{\partial^2 \phi}{\partial x^2} \ . \qquad (2.1.2.3a,b) \qquad \text{(Bohm quantum potential)}$$

Putting eq. (2.1.2.1,2) into (2.1.1.3) we get:

$$\frac{\partial \sqrt{\rho}}{\partial t} \ = \ - \ \frac{\hbar}{2\ m} \ (\ 2 \ \frac{\partial S}{\partial x} \ \frac{\partial \sqrt{\rho}}{\partial x} \ + \ \sqrt{\rho} \ \frac{\partial^2 S}{\partial x^2}) \qquad \rightarrow$$

$$\frac{1}{2\ \sqrt{\rho}} \ \frac{\partial \rho}{\partial t} \ = \ - \ \frac{\hbar}{2\ m} \ (\ 2 \ \frac{\partial S}{\partial x} \ \frac{1}{2\ \sqrt{\rho}} \ \frac{\partial \rho}{\partial x} \ + \ \sqrt{\rho} \ \frac{\partial^2 S}{\partial x^2} \ ) \qquad \rightarrow$$

$$\frac{1}{\rho} \ \frac{\partial \rho}{\partial t} \ = \ - \ \frac{\hbar}{m} \ (\ \frac{\partial S}{\partial x} \ \frac{1}{\rho} \ \frac{\partial \rho}{\partial x} \ + \ \frac{\partial^2 S}{\partial x^2} \ ) \qquad \rightarrow$$

$$\frac{1}{\rho} \ \frac{\partial \rho}{\partial t} \ = \ - \ \frac{\partial}{\partial x} \ (\ \frac{\hbar}{m} \ \frac{\partial S}{\partial x} \ ) \ - \ \frac{1}{\rho} \ (\ \frac{\hbar}{m} \ \frac{\partial S}{\partial x} \ ) \ \frac{\partial \rho}{\partial x} \qquad \rightarrow$$

$$\frac{1}{\rho} \ \frac{\partial \rho}{\partial t} \ = \ - \ \frac{\partial v_{qu}}{\partial x} \ - \ \frac{v_{qu}}{\rho} \ \frac{\partial \rho}{\partial x} \qquad \rightarrow$$

$$\frac{\partial \rho}{\partial t} \ + \ \rho \ \frac{\partial v_{qu}}{\partial x} \ + \ v_{qu} \ \frac{\partial \rho}{\partial x} \ = \ 0 \qquad \rightarrow$$

$$\frac{\partial \rho}{\partial t} \ + \ \frac{\partial (\rho \ v_{qu})}{\partial x} \ = \ 0 \ , \qquad (2.1.2.4)$$

which represents the continuity equation of the mass conservation law of the Fluid Dynamics. [6] We must note that this expression also indicates <u>coerence</u> of the considered physical system represented by (2.1.1).

Now, let us see the correlation between the expressions (2.1.1.4-5) and the traditional equations of the Ideal Fluid Dynamics. [6] Thus, putting the eqs. (2.1.2.3a,b) into the eq. (2.1.1.4), we obtain: [2]

$$\frac{\partial \rho}{\partial t} \ + \ \frac{\partial (\rho v_{qu})}{\partial x} \ = \ - \ \nu \ \rho \ (\ell n \rho - \ < \ell n \rho >), \qquad (2.1.2.5)$$

expression that indicates decoherence of the considered physical system represented by the Schuch-Chung-Hartmann Equation ($SCH - E$) [eq. (2.1.1)]; then the *Continuity Equation* it is not preserved.

Now, let us obtained another dynamic equation of the $SCH - E$. Considering the eqs. (2.1.2.1-3b) and (2.1.1.5), we obtain: [2]

$$\hbar \ [\tfrac{\partial S}{\partial t} \ + \ \nu \ (S - \ - \ < S >) \ + \ \tfrac{1}{2} \ m \ v_{qu}^2 \ + \ V \ + \ V_{qu} \ = \ 0 \ . \qquad (2.1.2.6)$$

Considering that:

$$< f(x, \ t) > = \ \int_{-\ \infty}^{+\ \infty} \rho(x, \ t) \ f(x, \ t) \ dx \ = \ g(t) \ , \qquad (2.1.2.7)$$

then:

$$\tfrac{\partial}{\partial x} \ < S(x, \ t) > = \ \tfrac{\partial}{\partial x} \int_{-\ \infty}^{+\ \infty} \rho(x, \ t) \ S(x, \ t) \ dx \ = \ \tfrac{\partial g(t)}{\partial x} \ = \ 0. \qquad (2.1.2.8)$$

Now, differentiating the eq. (2.1.1.3) with respect $x$, and using the eqs. (2.1.2.1-3b) and (2.1.2.4), we have: [2]

$$\tfrac{\partial v_{qu}}{\partial t} \ + \ v_{qu} \ \tfrac{\partial v_{qu}}{\partial x} \ + \ \nu \ v_{qu} \ = \ - \ \tfrac{1}{m} \ \tfrac{\partial}{\partial x} \ (V \ + \ V_{qu}) \ , \qquad (2.1.2.9a)$$

where:

$$V_{qu}(x, \ t) \ = \ \tfrac{\hbar^2}{4 \ m \ a^2(t)} \ - \ \tfrac{\hbar^2}{8 \ m \ a^4(t)} \ \times \ [x \ - \ q(t)]^2 \ . \qquad (2.1.2.9b)$$

We observe that the eq. (2.1.2.9a) has the aspect of the *Navier-Stokes Equation* [6] for a real fluid in movement.

Considering the "substantive differentiation" (local plus convective) or "hidrodynamic differention": $d/dt \ = \ \partial/\partial t \ + v_{qu} \ \partial/\partial x$ and that $v_{qu} \ = \ dx_{qu}/dt$, the eqs. (2.1.2.9a,b) could be written as:

$$\tfrac{d^2 x}{dt^2} \ + \ \nu \ \tfrac{dx}{dt} \ = \ - \ \tfrac{1}{m} \ \tfrac{\partial}{\partial x} \ [ \ V(x, \ t) \ + \ V_{qu}(x, \ t) \ ], \qquad (2.1.2.10)$$

what has a form of the *Dissipative Second Newton Law*, being the terms of the second member, respectively, the *classical newtonian force* and the *quantum bohmnian force.*

### 2.1.3. The Quantum Wave Packet of the Linearized Schuch-Chung-Hartmann Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Schuch-Chung-Hartmann Equation ($SCH - E$) along a classical trajetory, we must calculate one *modified quantum velocity* ($v_{qum}$) that satisfating the *Continuity Equation.* Considering the *ansatz*: [7]

$$\rho \ (x, \ t) = \ [2\pi \ a^2(t)]^{- \ 1/2} \ exp \ \Big( \ - \ \tfrac{[x \ - \ q(t)]^2}{2 \ a^2(t)} \ \Big), \qquad (2.1.3.1)$$

where $a(t)$ and $q(t)$ are auxiliary functions of time, to be determined in what follows; they represent the *width* and *center of mass of wave packet*, respectively, we can proven that: [2]

$$\frac{\partial \rho}{\partial x} \; = \; - \; \rho \; \frac{[x \; - \; q(t)]}{a^2}, \qquad (2.1.3.2)$$

and:

$$\ell n\rho \; - \; < \ell n\rho > \; = \; - \; \frac{a^2(t)}{2 \; \rho} \; \frac{\partial^2 \rho}{\partial x}. \qquad (2.1.3.3)$$

Putting the eq. (2.1.3.3) into eq. (2.1.2.5), results:

$$\frac{\partial \rho}{\partial t} \; + \; \frac{\partial(\rho v_{qu})}{\partial x} \; = \; - \; \nu \; \rho \; ( - \; \frac{a^2}{2 \; \rho} \; \frac{\partial^2 \rho}{\partial x} \; ) \; = \; \frac{\nu \; a^2}{2} \; \frac{\partial}{\partial x} \; ( \; \frac{\partial \rho}{\partial x} \; ) \; \rightarrow$$

$$\frac{\partial \rho}{\partial t} \; + \; \frac{\partial}{\partial x} \; [ \; \rho \; ( \; v_{qu} \; - \; \frac{\nu \; a^2}{2 \; \rho} \; \frac{\partial \rho}{\partial x} \; ) \; ] \; = \; 0. \qquad (2.1.3.4)$$

Defining: [7]

$$v_{qum} \; = \; v_{qu} \; - \; \frac{\nu}{2} \; \frac{a^2}{\rho} \; \frac{\partial \rho}{\partial x} \; , \qquad (2.1.3.5)$$

then the eq. (2.1.3.4) will be the form:

$$\frac{\partial \rho}{\partial t} \; + \; \frac{\partial(\rho v_{qum})}{\partial x} \; = \; 0, \qquad (2.1.3.6)$$

expression that indicates <u>coherence</u> of the $SCH - E$.

So, using the same operational protocol of the [8], results:

$$v_{qum}(x, \; t) \; = \; \frac{\dot{a}(t)}{a(t)} \; [x \; - \; q(t)] \; + \; \dot{q}(t). \qquad (2.1.3.7)$$

Considering the eqs. (2.1.3.2,5,7), we have:

$$v_{qu}(x, \; t) \equiv \frac{dx(t)}{dt} \; = \; [ \; \frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2} \; ] \times [x \; - \; q(t)] \; + \; \dot{q}(t). \qquad (2.1.3.8)$$

We observe that the integration of the eq. (2.1.3.8) give us the *bohmnian quantum trajectory* of the physical system represented by $SCH - E$.

To obtain the quantum wave packet $[\Psi(x \; , \; t)]$ of the $SCH - E$ given by eq. (2.5.1), let us expand the functions S(x, t), V(x t), and $V_{qu}$(x, t) around of $q(t) \; = < x >$ up to *second Taylor order*. In this way, we have:

$$S(x, \; t) \; = \; S[q(t), \; t] \; + \; S'[q(t), \; t] \times [x \; - \; q(t)] \; + \; \frac{S"[q(t) \; t]}{2} \times [x \; - \; q(t)]^2, \qquad (2.1.3.9)$$

$$V(x,\ t)\ =\ V[q(t),\ t]\ +\ V'[q(t),\ t]\times[x\ -\ q(t)]\ +\ \frac{V"[q(t),\ t]}{2}\times[x\ -\ q(t)]^2, \qquad (2.1.3.10)$$

$$V_{qu}(x,\ t)\ =\ V_{qu}[q(t),\ t]\ +\ V'_{qu}[q(t),\ t]\times[x\ -\ q(t)]\ +$$

$$+\ \tfrac{1}{2}\ V"_{qu}[q(t),\ t]\times[x\ -\ q(t)]^2, \qquad (2.1.3.11)$$

where (') and (") means, respectively: $\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial q^2}$.

Differenting the eq. (2.1.3.9) in the variable $x$, multiplying the result by $\hbar/m$ and using the eqs. (2.1.2.2) and (2.1.3.8), results:

$$\frac{\hbar}{m}\ \frac{\partial S(x,\ t)}{\partial x}\ =\ \frac{\hbar}{m}\ \{\ S'[q(t),\ t]\ +\ S"[q(t),\ t]\times[x\ -\ q(t)]\ \}\ =\ v_{qu}(x,\ t)\ =$$

$$=\ [\ \frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}\ ]\times[x\ -\ q(t)]\ +\ \dot{q}(t)\ \ \rightarrow$$

$$S'[q(t),\ t]\ =\ \frac{m\ \dot{q}(t)}{\hbar}\ ,\quad S"[q(t),\ t]\ =\ \frac{m}{\hbar}\ [\ \frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}\ ]\ . \qquad (2.1.3.12a,b)$$

Substituting the eqs. (2.1.3.12a,b) into eq. (2.1.3.9), we have:

$$S(x,\ t)\ =\ S_0(t)\ +\ \frac{m\ \dot{q}(t)}{\hbar}\times[x\ -\ q(t)]\ +\ \frac{m}{2\ \hbar}\ [\ \frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}\ ]\times[x\ -\ q(t)]^2\ , \qquad (2.1.3.13a)$$

where:

$$S_o(t)\equiv S[q(t),\ t]\ . \qquad (2.1.3.13b)$$

Now, considering that:

$$\textstyle\int_{-\ \infty}^{+\ \infty}\ z^n\ exp(-\ z^2)\ dz\ =\ \frac{1}{2}\ \sqrt{\pi};\ 0;\ \sqrt{\pi},(n\ =\ 2;\ 1;\ 0) \qquad (2.1.3.14a\text{-}c)$$

and using the eqs. (2.1.2.7), (2.1.3.1) and (2.1.3.13a), results: [7]

$$<S>\ =\ S_0(t)\ +\ a^2(t)\times\frac{m}{2\ \hbar}\ [\ \frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}\ ]. \qquad (2.1.3.15)$$

Differenting the eq. (2.1.3.13a) in relation to the time $t$ and using the eqs. (2.1.3.2) and (2.1.3.13b), results (remember that $\partial x/\partial t\ =\ 0$): [8]

$$\frac{\partial S(x,\ t)}{\partial t}\ =\ \dot{S}_o(t)\ -\ \frac{m\ \dot{q}^2(t)}{\hbar}\ +\ \frac{m}{\hbar}\ \{\ \ddot{q}(t)\ -\ \dot{q}(t)\ [\ \frac{\dot{a}(t)}{a(t)}\ -\ \frac{\nu}{2}\ ]\ \}\times[x\ -\ q(t)]\ +$$

$$+\ \frac{m}{2\ \hbar}\ [\ \frac{\ddot{a}(t)}{a(t)}\ -\ \frac{\dot{a}^2(t)}{a^2(t)}\ ]\times[x\ -\ q(t)]^2\ . \qquad (2.1.3.16)$$

Using the eqs. (2.1.1.5), (2.1.2.2) and (2.1.2.3b), we obtain:

$$- \hbar \, \frac{\partial S}{\partial t} \; = \; [- \, \frac{\hbar^2}{2 \; m} \, \frac{1}{\phi} \, \frac{\partial^2 \, \phi}{\partial x^2} \; + \; \frac{m}{2} \, ( \, \frac{\hbar}{m} \, \frac{\partial S}{\partial x})^2] \; + \; V(x \; , \; t) \; + \; \hbar \, \nu \, (S - \; < S >) \quad \rightarrow$$

$$\hbar \, \frac{\partial S}{\partial t} \; + \; \frac{m}{2} \, v_{qu}^2 \; + \; V(x, \; t) \; + \; V_{qu}(x, \; t) \; + \; \hbar \, \nu \, (S - \; < S >) \; = \; 0 \; . \qquad (2.1.3.17)$$

Inserting the eqs. (2.1.2.9b) and (2.1.3.8,10,11,13a,15) into eq. (2.1.3.17), ordering the result in potencies of $[x \; , \; q(t)]$, and considering that $(x \; - \; q)^o \; = \; 1$, we have: [8]

$$\{\hbar \, \dot{S}_o(t) \; - \; \frac{m}{2} \, \dot{q}^2(t) \; - \; \frac{m \; \nu}{2} \, a^2(t) \times [ \, \frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2} \, ] \; + \; V[q(t), \; t] \; + \; \frac{\hbar^2}{4 \; m \; a^2(t)} \, \} \times [x \; - \; q(t)]^0 \; +$$

$$+ \; \{m \, \ddot{q}(t) \; + \; m \, \nu \, \dot{q}(t) \; + \; V'[q(t), \; t]\} \times [x \; - \; q(t)] \; +$$

$$+ \, \{\frac{m}{2} \, \frac{\ddot{a}(t)}{a(t)} \; - \; \frac{m \; \nu^2}{8} \; + \; \frac{1}{2} \, V"[q(t), \; t] \; - \; \frac{\hbar^2}{8 \; m \; a^4(t)} \, \} \times [x \; - \; (q)t)]^2 \; = \; 0 \; . \qquad (2.1.3.18)$$

As the above relation [eq. (2.1.3.18)] is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) \; = \; \frac{1}{\hbar} \, \{ \, \frac{m}{2} \, \dot{q}^2(t) \; - \; V[q(t), \; t] \; + \; \frac{m \; \nu}{2} \, a^2(t) \times [ \, \frac{\dot{a}(t)}{a(t)} \; - \; \frac{\nu}{2} \, ] \; - \; \frac{\hbar^2}{4 \; m \; a^2(t)} \, \} \; , \qquad (2.1.3.19)$$

$$\ddot{q}(t) \; + \; \nu \, \dot{q}(t) \; + \; \frac{1}{m} \, V'[q(t), \; t] \; = \; 0 \; , \qquad (2.1.3.20)$$

$$\ddot{a}(t) \; + \; \nu \, \dot{a}(t) \; + \; \frac{1}{m} \, \{ \, V"[q(t), \; t] \; - \; \frac{\nu^2}{4} \, \} \times a(t) \; = \; \frac{\hbar^2}{4 \; m^2 \; a^3(t)} \; . \qquad (2.1.3.21)$$

Now, let us consider that V[q(t), t] is given by:

$$V[q(t), \; t] \; = \; \frac{1}{2} \, m \, \omega^2(t) \, q^2(t), \qquad (2.1.3.22)$$

which is the Time Dependent Harmonic Oscillator Potencial).

In this case, we have:

$$V'[q(t), \; t] \; = \; m \, \omega^2(t) \, q(t), \quad V"[q(t), \; t] \; = \; m \, \omega^2(t) \; . \qquad (2.1.3.23a,b)$$

Putting the eqs. (2.1.3.23a,b) into eqs. (2.1.3.20,21), results:

$$\ddot{q}(t) \; + \; \nu \, \dot{q}(t) \; + \; \omega^2(t) \, q(t) \; = \; 0 \; , \qquad (2.1.3.24)$$

$$\ddot{a}(t) \; + \; \nu \, \dot{a}(t) \; + \; [\omega^2(t) \; - \; \frac{\nu^2}{4 \; m}] \, a(t) \; = \; \frac{\hbar^2}{4 \; m^2 \; a^3(t)} \; . \qquad (2.1.3.25)$$

Assuming that the following initial conditions are obeyed:

$$q(0) \ = \ x_o \ , \quad \dot{q}(0) \ = \ v_o \ , \quad a(0) \ = \ a_o \ , \quad \dot{a}(0) \ = \ b_o \ , \qquad (2.1.3.26\text{a-d})$$

and that:

$$S_o(0) \ = \ \tfrac{m \ v_o \ x_o}{\hbar} \ , \qquad (2.1.3.27)$$

the integration of the expression (2.1.3.19) and considering the eq. (2.1.3.22) will be given by:

$$S_o(t) \ = \ \tfrac{1}{\hbar} \int_o^t dt' \ \{ \ \tfrac{m}{2} \ \dot{q}^2(t') \ + \ \tfrac{m \ \nu}{2} \ [ \ \tfrac{\dot{a}(t')}{a(t')} \ - \ \tfrac{\nu}{2} \ ] \ - \ \tfrac{1}{2} \ m \ \omega^2(t') \ q^2(t') \ - \ \tfrac{\hbar^2}{4 \ m \ a^2(t')} \ \} \ +$$

$$+ \ \tfrac{m \ v_o \ x_o}{\hbar} \ . \qquad (2.1.3.28)$$

Taking into account the eq. (2.1.3.28) in the eq. (2.1.3.13a) and considering the eq. (2.1.3.13b), results:

$$S(x, \ t) \ = \ \tfrac{1}{\hbar} \int_o^t dt' \ \{ \ \tfrac{m}{2} \ \dot{q}^2(t') \ + \ \tfrac{m \ \nu}{2} \ a^2(t') \times [ \ \tfrac{\dot{a}(t')}{a(t')} \ - \ \tfrac{\nu}{2} \ ] \ -$$

$$- \ \tfrac{1}{2} \ m \ \omega^2(t') \ q^2(t') \ - \ \tfrac{\hbar^2}{4 \ m \ a^2(t')} \ \} \ + \ \tfrac{m \ v_o \ x_o}{\hbar} \ +$$

$$+ \ \tfrac{m \ \dot{q}(t)}{\hbar} \times [x \ - \ q(t)] \ + \ \tfrac{m}{2 \ \hbar} \ [ \ \tfrac{\dot{a}(t)}{a(t)} \ - \ \tfrac{\nu}{2} \ ] \times [x \ - q(t)]^2 \ . \qquad (2.1.3.29)$$

The eq. (2.1.3.29) permit us, finally, to obtain the wave packet for the linearized Schuch-Chung-Hartmann Equation ($SCH - E$) along a classical trajectory. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b), (2.1.2.1), (2.1.3.1) and (2.1.3.29), we get: [8]

$$\Psi(x, \ t) \ = \ [2 \ \pi \ a^2(t)]^{- \ 1/4} \times exp \Big( \ \{ \ \tfrac{i \ m}{2 \ \hbar} \ [ \ \tfrac{\dot{a}(t)}{a(t)} \ - \ \tfrac{\nu}{2} \ ] \ - - \ \tfrac{1}{4 \ a^2(t)} \ \} \times [x \ - \ q(t)]^2 \ \Big) \times$$

$$\times \ exp \ \{ \ \tfrac{i \ m \ \dot{q}(t)}{\hbar} \times [x \ - \ q(t)] \ + \ \tfrac{i \ m \ v_o \ x_o}{\hbar} \ \} \times$$

$$\times \ exp \ \Big( \ \tfrac{i}{\hbar} \int_o^t dt' \ \{ \ \tfrac{m}{2} \ m \ \dot{q}^2(t') \ + \ \tfrac{m \ \nu}{2} \ a^2(t') \times [ \ \tfrac{\dot{a}'(t)}{a(t')} \ - \tfrac{\nu}{2} \ ] \ -$$

$$- \ \tfrac{1}{2} \ m \ \omega^2(t') \ q^2(t') \ - \ \tfrac{\hbar^2}{4 \ m \ a^2(t')} \ \} \ \Big). \qquad (2.1.3.30)$$

### 2.1.4. The Bohmnian Trajectories for the Schuch-Chung-Hartmann Equation

The associated Bohmnian Trajectories, [9]-[13] for the Schuch-Chung-Hartmann Equation ($SCH - E$) of an evolving *ith* particle of the ensemble with an initial position $x_{0i}$ can be calculated by considering that:

$$\dot{x}_i(t) \ = \ v_{qu}[x_i(t), \ t]. \qquad (2.1.4.1)$$

Then substituting the eq. (2.1.4.1) into eq. (2.1.3.8), results:

$$\dot{x}_i(t) \ = \ \Big[ \ \Big( \ \frac{\dot{a}(t)}{a(t)} \ - \ \frac{\nu}{2} \ \Big) \times [x \ - \ q(t)] \ + \ \dot{q}(t) \ \Big] \ \ \rightarrow$$

$$x_i(t) \ = \ \int_{t_0}^{t} \ \{ \ \Big[ \ \Big( \ \frac{\dot{a}(t')}{a(t')} \ - \ \frac{\nu}{2} \ \Big) \times [x \ - \ q(t')] \ + \ \dot{q}(t') \ \Big] \ \} \ dt' \ . \qquad (2.1.4.2)$$

The eqs. (2.1.3.24,25) show that a continuous measurement of a quantum wave packet gives specific features to its evolution: the appearance of distinct classical and quantum elements, respectively. This measurement consist of monitoring the position of quantum systems and the result is the measured classical path $q(t)$ for $t$ within a quantum uncertainty $a(t)$.

From the eq. (2.1.3.25), we note that for $\nu \neq 0$ a stationary regime can be reached and that the width $[a(t)]$ of the wave packet can be related to the resolution of measurement as follows. Then considering that $a(t) \ = \ cte \ [\dot{a}(t) \ = \ 0; \, a(t_0) \ = \ a_0]$ in the eq. (2.1.3.25) and considering the $t_0$ the *initial time*, we have:

$$\omega_0^2(t_0) \ = \ \frac{\nu^2}{4 \ m} \ + \ (\frac{1}{\tau_B})^2 \ , \qquad (2.1.4.3a)$$

where [9, 14]:

$$\tau_B \ = \ \Big( \ \frac{2 \ m \ a_0^2}{\hbar} \ \Big) \ = \ 6,8 \times \ 10^{-26} s, \qquad (2.1.4.1.3b)$$

is the *Bohmtime constant* which determines the time resolution of the quantum measurement. By the eqs. (2.1.4.3a,b) we observe that, as $\nu \neq 0$, then: $t_0 \neq 0$.

## 3. Conclusion

The eqs. (2.1.4.3a,b) means that if an initially free wave packet is kept under a certain continuous measurement, its $(a_0)$ may not spread in time. Then, the associated *Bohmnian Trajectories* of an evolving *ith* particle of the ensemble with an initial position $x_{0i}$ is giving by the eq. (2.1.4.2).

## NOTES AND REFERENCES

1. R. P. FEYNMAN and A. R. HIBBS. **Quantum Mechanics and Path Integrals**, McGraw-Hill Book Company, USA (1965).

2. J. M. F. BASSALO, P. T. S. ALENCAR, M. S. D. CATTANI e A. B. NASS AR, **Tópicos da Mecânica Quântica de de Broglie-Bohm**, EDUFPA (2002); E-Book (http://publica-sbi.if.usp.br/PDFs/pd1655.pdf. (2010)

3. SCHUCH, D., CHUNG, K. M. and HARTMANN, H. 1983. *Journal of Mathematical Physics* **24**: 1652; —–. 1984. *Journal of Mathematical Physics* **25**: 3086; —–. 1985. *Berichte der Bunsen-Gesellschaft für Physikalische Chemie* **89**: 589.

4. E. MADELUNG. *Zeitschrift für Physik* **40** (3,4): 322 (1926).

5. D. BOHM. *Physical Review* **85**: 166 (1952).

6. See books on the Fluid Mechanics, for instance:

6.1. V. L. STREETER and W. R. DEBLER. **Fluid Mechanics**, McGraw-Hill Book Company, Incorporation (1966).

6.2. A. L. COIMBRA. **Mecânica dos Meios Contínuos**, Ao Livro Técnico S. A. (1967).

6.3. L. LANDAU et E. LIFSHITZ. **Mécanique des Fluides**. Éditions Mir (1969).

6.4. J. M. F. BASSALO. **Introdução à Mecânica dos Meios Contínuos**, EDUFPA (1973).

6.5. M. S. D. CATTANI. **Elementos de Mecânica dos Fluidos**, Edgard Blücher (1990/2005).

7. A. B. NASSAR, J. M. F. BASSALO, P. T. S. ALENCAR, L. S. CANCELA and M. S. D. CATTANI. 1977. *Physical Review* **E56**: 1230.

8. J. M. F. BASSALO, A. B. NASSAR and M. S. D. CATTANI. 2017. *http://publica-sbi.if.usp.br/PDFs/pd***1706**, 21 July.

9. A. B. NASSAR. 2013. *Annals of Physics* **331**: 317.

10. A. S. SANZ, F. BORONDO and S. MIRET-ARTÉS. *Physical Review* **B61**: 7743 (2000).

11. A. K. PAN. *Pramana Journal of Physics* **74**: 867 (2010).

12. P. HOLLAND. *Annals of Physics* **315**: 505 (2005).

13. R. E. WYATT. **Quantum Dinamics with Trajectories**, Springer, New York (2005).

14. If the initial wave packet width ($a_0$) is taken to be equal to $2,8 \times 10^{-15}$ $m$ (the approximate size of an electron of mass $m$) then $\tau_B$ to be about $10^{-25}$ sec, for a continuous measurement. We note that (see 10.), experiments to measure the size of the electron consist on colliding two beams of electrons against each other and counting how many are scattered and altered their trajectories. By counting the collisions, and knowing how many particles we have thrown, we can estimate the average size of each particle in the beam.