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ENERGY VACUUM FLUCTUATIONS AND THE FLUCTUATION - DISSIPATION THEOREM

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Abstract. We study the semi classical motion of a non relativistic linear electric dipole oscillator in the quantum vacuum. Taking into account the energy *vacuum-fluctuations* and the *stochastic* effects in this motion can be interpreted within the context of the Fluctuation-Dissipation Theorem. This paper was written to young researchers in Physics. The process was didactically analyzed using simple physical models but in a way as rigorously as possible.

Key words: zero-point energy; atomic electron orbit; fluctuation-dissipation theorem.

(I) Introduction.

In quantum field theory[1,2] (QFT) the empty space is visualized as consisting of fields, with the field at every point in space and time being a quantum harmonic oscillator, with neighboring oscillators interacting with each other. In our particular electromagnetic case the *matter field* is made up of *fermions* and the *force field* is made up of *photons*. All these fields have *zero-point energy* (ZPE).[2] Unlike classical mechanics, quantum systems constantly fluctuate in their lowest energy state due to the Heisenberg uncertainty principle.[3,4] As well atoms and molecules, the empty space (vacuum) has these properties. A related term to ZPE is *zero-point field* (ZPF), which is the lowest energy state of the particular field.[5] The vacuum can be viewed not as an empty space, but a combination of the zero-point fields.[5] Each point in this space makes a contribution of energy E = hν/2, resulting in a contribution of infinite ZPE in any finite volume; this is one reason renormalization is needed to make sense of QFT.

The electromagnetic field is the oldest and best known quantized force field. Maxwell´s equations have been superseded by the quantum electrodynamics (QED). The QED, that can be seen in many basic text books[3,4] was the starting point to begin to better understand all quantum field theories. In QED the vector potential **A(r**,t), for a plane wave mode of the field, is given by

$$\mathbf{A}(\mathbf{r},t) = \sum_{k\lambda}[2\pi\hbar c^2/\omega_{\mathbf{k}}V]^{1/2}\,[a_{\mathbf{k}\lambda}(0)e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}.\mathbf{r})} + a^{+}_{\mathbf{k}\lambda}(0)e^{i(\omega_{\mathbf{k}} t - \mathbf{k}.\mathbf{r})}]\,\mathbf{e}_{k\lambda} \qquad (I.1),$$

where $a_{\mathbf{k}\lambda}$ and $a^{+}_{\mathbf{k}\lambda}$ are the photon annihilation and creation operators, respectively, for the wave vector **k** and polarization λ and $\mathbf{e}_{k\lambda}$ are the unit vector polarization of the field; $a_{\mathbf{k}\lambda}$ and $a^{+}_{\mathbf{k}\lambda}$ obey the bosonic commutation relations:[3,4]

$$[a_{\mathbf{k}\lambda}(t), a^{+}_{\mathbf{k}'\lambda'}(t)] = \delta^3_{\mathbf{k},\mathbf{k}'}\,\delta_{\lambda,\lambda'} \text{ and } [a_{\mathbf{k}\lambda}(t), a_{\mathbf{k}'\lambda'}(t)] = [a^{+}_{\mathbf{k}\lambda}(t),a^{+}_{\mathbf{k}'\lambda'}(t)] = 0 \quad (I.2),$$

and the "free" field Hamiltonian is given by

$$H_F = \sum_{k\lambda} \hbar\omega_{\mathbf{k}}\,(a^{+}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} + 1/2) \qquad (I.3).$$

The least eigenvalue for H_F is $H_F = \sum_{k\lambda}(1/2)\,\hbar\omega_{\mathbf{k}}$. This state describes the zero-point energy of the vacuum which clearly is highly divergent.

In **Section 1** will be analyzed, in the non relativistic limit, the motion of linear oscillator in vacuum remembering that from Maxwell´s equations, the electromagnetic energy of the "free” field", i.e. one with no sources, is described by[1,2]

$$H_F = (1/8\pi) \int d^3\mathbf{r}\,(\mathbf{E}^2 + \mathbf{B}^2) \qquad (I.4),$$

where $\mathbf{E} = \mathrm{grad}(\Phi) - (1/c)\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \mathrm{rot}(\mathbf{A})$. Finally, in **Section 2** the effect of the vacuum interaction with the oscillator can interpreted within the context of the Fluctuation-Dissipation Theorem.

(1) Non Relativistic Linear Dipole Oscillator in Vacuum.

According to Classical Electrodynamics (**CE**),[6,7] accelerated charge emits radiation and then appears on the charge, in non relativistic limit, a *reaction force* named "Abraham-Lorentz Force" given by

$$\mathbf{F}_{rad} = (2e^2/3c^3)(d\mathbf{a}/dt) = m\tau\,(d\mathbf{a}/dt) \qquad (1.1),$$

where m and e are its mass and charge, $\mathbf{a} = d\mathbf{v}/dt$ is the charge acceleration and $\tau = (2e^2/3mc^3)$. A charge submitted to an harmonic motion with frequency ω_o it will obey the following equation of motion,

$$m\,(d^2\mathbf{x}/dt^2) = -\,m\,\omega_o^2\,\mathbf{x}^2 + m\,\tau\,(d^3\mathbf{x}/dt^3) \qquad (1.2).$$

According to extensive analysis found in the literature,[6,7]this equation is useful only in the domain where the reactive term is a small correction. This is the case, for instance, of the small damping approximation when $d^2\mathbf{x}/dt^2 \approx -\omega_o^2\,\mathbf{x}^2$, that is, $d^3\mathbf{x}/dt^3 \approx -\omega_o^2\,d\mathbf{x}/dt$. In these conditions Eq.(1.2) becomes:

$$d^2\mathbf{x}/dt^2 + \tau\,\omega_o^2\,(d\mathbf{x}/dt) + \omega_o^2\,\mathbf{x}^2 \approx 0 \qquad (1.3),$$

showing that $\mathbf{x}(t)$ would be exponentially dampened.

Now, let us analyze using a semiclassical approach, the motion of a non relativistic dipole oscillator interacting with the *vacuum field* described by the Hamiltonian:[1,2,5]

$$H = (1/2m)(\mathbf{p} - e\mathbf{A}/c)^2 + (1/2)m\omega_o^2\,\mathbf{x}^2 + H_F \qquad (1.4),$$

where the linear dipole is given by $\mathbf{d} = e\mathbf{x}$, ω_o is its harmonic vibration frequency and the charge has a mass m. It has the same form as the corresponding classical Hamiltonian and the Heisenberg equations of motion for the oscillator and the field are formally the same as their classical counterparts. For instance the Heisenberg equations for the coordinates $\mathbf{x}$ and the canonical momentum $\mathbf{p} = m\,(d\mathbf{x}/dt) + e\mathbf{A}/c$ of the oscillator are given by,

$$d\mathbf{x}/dt = (i\hbar)^{-1}[\mathbf{x}, H] \quad \text{and} \quad d\mathbf{p}/dt = (i\hbar)^{-1}[\mathbf{p}, H] \qquad (1.5).$$

Taking into account (1.5) it can be shown that:[5]

$$m\,(d^2\mathbf{x}/dt^2) = d\mathbf{p}/dt - (e/c)d\mathbf{A}/dt =$$

$$= e\mathbf{E} + (e/c)(d\mathbf{x}/dt) \times \mathbf{B} - m\omega_o^2\,\mathbf{x}^2 \qquad (1.6),$$

since the convective derivative $d\mathbf{A}/dt = \partial\mathbf{A}/\partial t + [(d\mathbf{x}/dt)\bullet \text{grad}]\mathbf{A}^3$.

For non relativistic motion we may neglect the magnetic force in Eq.(1.6) obtaining:

$$d^2\mathbf{x}/dt^2 + \omega_o^2\,\mathbf{x}^2 \approx (e/m)\mathbf{E(t)} \qquad (1.7).$$

Since $\Phi = 0$ we get $\mathbf{E(t)} = -(1/c)\partial\mathbf{A}/\partial t$, using Eq.(I.1). Neglecting the spatial dependence of the field in the dipole approximation results[5]

$$\mathbf{E(t)} \approx \sum_{\mathbf{k}\lambda}[2\pi\hbar\omega_{\mathbf{k}}/V]^{1/2}\,[a_{\mathbf{k}\lambda}(t) + a^{+}_{\mathbf{k}\lambda}(t)]\,\mathbf{e}_{k\lambda} \qquad (1.8).$$

Similarly, using the Heisenberg equations $da_{\mathbf{k}\lambda}(t)/dt = (i\hbar)^{-1}[a_{\mathbf{k}\lambda}(t), H]$ we have

$$da_{\mathbf{k}\lambda}/dt = i\omega_{\mathbf{k}}a_{\mathbf{k}\lambda} + ie[2\pi/\hbar\omega_{\mathbf{k}}V]^{1/2}\,\mathbf{v} \bullet \mathbf{e}_{k\lambda} \qquad (1.9),$$

where $\mathbf{v} = d\mathbf{x}/dt$.

Note that in deriving the equations for $\mathbf{x}$, $\mathbf{p}$ and $a_{\mathbf{k}\lambda}$ it was assumed the fact that equal-time particle and field operators commute at all times t that was valid at t = 0 when the matter-field interpretation is presumed to begin.[5]

The formal solution of the field equation (1.9) is:

$$a_{\mathbf{k}\lambda}(t) = a_{\mathbf{k}\lambda}(0)\, e^{-i\omega_{\mathbf{k}} \mathbf{t}} + ie[2\pi/\hbar\omega_{\mathbf{k}}V]^{1/2} \int_o^t dt'\, \mathbf{e}_{k\lambda} \bullet \mathbf{v}(t')\, e^{-i\omega_{\mathbf{k}}(\mathbf{t}-t')} \qquad (1.10).$$

Substituting Eq.(1.10) into Eq.(1.8) results $\mathbf{E}(t) = \mathbf{E_o}(t) + \mathbf{E}_{RR}(t)$, where

$$\mathbf{E_o}(t) = i \sum_{k\lambda} [2\pi\hbar\omega_{\mathbf{k}}/V]^{1/2} \left[a_{\mathbf{k}\lambda}(0)\, e^{-i\omega_{\mathbf{k}}\mathbf{t}} - a^{+}_{\mathbf{k}\lambda}(0)\, e^{i\omega_{\mathbf{k}}\mathbf{t}}\right] \mathbf{e}_{k\lambda}$$

and

$$\mathbf{E}_{RR}(t) = -(4\pi e/V) \sum_{k\lambda} \int_o^t dt' [\, \mathbf{e}_{k\lambda} \bullet \mathbf{v}(t')] \cos \omega_{\mathbf{k}}(t - t') \qquad (1.11).$$

That is, the total electric field $\mathbf{E}(t)$ acting on the dipole has two parts: $\mathbf{E_o}(t)$ is the free "zero-point field" or "vacuum field" and $\mathbf{E_{RR}}(t)$ is the source field, the field *generated* by the dipole and *acting on* the dipole. In quantum theory there is always an "external" field, the source-free or vacuum field $\mathbf{E_o}(t)$. One can see[1,2] that the expectation value of the free field is $< \mathbf{E_o}(t) > = 0$ and that the energy density associated with the free field is infinite: $(1/4\pi) < \mathbf{E_o}^2(t) > = (1/V) \sum_{k\lambda} (1/2)\, \hbar\omega_{\mathbf{k}}$.

As the *radiation reaction field* is given by[5] $\mathbf{E}_{RR}(t) = (2e/3c^3)\, d^3\mathbf{x}/dt^3$, we verify that Eq.(1.4) becomes written as,

$$d^2\mathbf{x}/dt^2 + \omega_o^2\, \mathbf{x}^2 - \tau\, (d^3\mathbf{x}/dt^3) \approx (e/m)\mathbf{E_o(t)} \qquad (1.12),$$

where $\tau = 2e^2/3mc^3$.

According to Eq.(1.2), classically the dipole in the vacuum is not acted upon by any "external" field if there are no sources other than the dipole itself. However, according to Eq.(1.12), in the quantum mechanical approach, the dipole is acted by an "external" field or vacuum field $\mathbf{E}_o(t)$.

For *small damping* $d^2\mathbf{x}/dt^2 \approx - \omega_o^2\, \mathbf{x}^2$, that is, $d^2\mathbf{x}/dt^2 \sim - \omega_o^2\, (d\mathbf{x}/dt)$, Eq.(1.12) becomes,

$$d^2\mathbf{x}/dt^2 + \tau\, \omega_o^2 (d\mathbf{x}/dt) + \omega_o^2\, \mathbf{x}^2 \approx (e/m)\mathbf{E_o(t)} \qquad (1.13),$$

showing that without the *free field* $\mathbf{E_o}(t)$ this equation is equal to Eq(1.3).

(2)Fluctuation-Dissipation Theorem.

Let us remember now the classical "Brownian motion" when a particle perform a random motion.[8] According to Langevin a particle that performs a random motion is submitted to two kinds of forces. One dissipative proportional to its velocity (Stokes force), and another f(t) which has a random (or stochastic) character due to myriads impacts of the particles with molecules of medium. Supposing the simplest case of the motion of a particle with mass m along the x axis we have

$$m\,(d^2x/dt^2) = -\,\alpha(dx/dt) + f(t) \qquad (2.1).$$

The first term of (2.1), **$-\,\alpha(dx/dt)$,** is a **dissipative force** being α the dissipative coefficient and **f(t)** a **stochastic force**, *impossible to be represented analytically*, characterized by the following time average properties

$$< f(t) > = 0 \qquad (2.2),$$

because in average the value of the stochastic forces due to collisions is expected to be zero and

$$< f(t)f(t') > = B\,\delta(t - t') \qquad (2.3),$$

supposing that the molecular impacts are independent , where B is a constant. Eq.(2.1) supplemented by properties (2.2) and (2.3) is named **Langevin Equation**. Dividing (2.1) by m results,

$$dv/dt = -\,v/\tau_r + \zeta(t) \qquad (2.4),$$

where $\tau_r = m/\alpha$ is the " **relaxation time**" and $\zeta(t) = f(t)/m$.

This case can be generalized involving all processes where there are simultaneously dissipative and stochastic effects. This was clarified by the Fluctuation-Dissipation Theorem (FDT)[9] proven by H. Callen and T. Welton[10] and expanded by R. Kubo.[9] The FDT applies both to classical systems and quantum mechanical fluctuations.[11] Generally speaking if a system is coupled to a bath that can take energy from the system in an effectively irreversible way, then the bath must also cause fluctuations. Fluctuations and dissipations cannot exist one without the other.

Eq.(1.13) can be written as

$$d^2\mathbf{x}/dt^2 + \omega_o^2\,\mathbf{x}^2 \approx -\,(d\mathbf{x}/dt)/\tau_r + (e/m)\mathbf{E_o(t)} \qquad (2.5),$$

where $-(d\mathbf{x}/dt)/\tau_r$ is the *dissipative* term, $\tau_r = 1/\tau\omega_o^2$ the relaxation time and $(e/m)\mathbf{E_o(t)}$ the *stochastic* term. In this way, the energy dissipated in the form of radiation by the charge is absorbed by the field ("bath") and vice versa, that is, the fluctuations of the field are absorbed by the charge that being accelerated emits radiation. See illustration in **Figure 1**.[5]

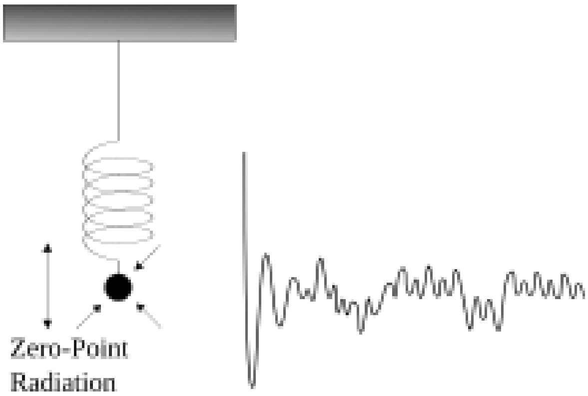

Figure 1. Zero-point radiation from the vacuum continually imparts random impulses on the oscillating charge, so that it never comes to a complete stop.

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