Black Holes and Wormholes: Brief Comments

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Abstract.
Are analyzed properties of Black Holes and Wormholes obtained by Einstein Theory of Gravitation (TGE) using the Schwarzschild, Eddington and Kruskal metrics. This article was written for undergraduate and graduate students of Physics.
Key words: *Einstein gravitation theory; black holes; wormholes.*

Resumo.
São analisadas propriedades dos Buracos Negros e dos Buracos de Minhocas obtidas pela Teoria de Gravitação de Einstein (TGE) usando as métricas de Schwarzschild, Eddington e Kruskal. Esse artigo foi escrito para alunos de graduação e pós-graduação de Física.

I. Introduction

Following the procedure adopted in our previous articles[1] we mention only a small number of articles and books. The calculations are performed with sufficient precision, leaving aside some refinements that can be found in articles. The predictions are compared with experimental results without the excessive concern of analyzing in detail the techniques used and their limitations. These aspects can be found in the references that will cited. In a previous article[1] using Einstein's Theory of Gravitation (**EGT**) we showed how to calculate the space-time metric generated in a vacuum around a spherically symmetrical distribution of mass M, without charge, at rest and not in rotation. This metric defines Schwarschild's geometry. In polar coordinates, $x_o = x_4 = ct$, $x_1 = r$, $x_2 = \theta$ and $x_3 = \varphi$, this metric, which is defined through the invariant, $ds^2 = c^2 d\tau^2$ (τ = proper time), is given by

$$ds^2 = (1-2\kappa/r)c^2dt^2 - dr^2/(1-2\kappa/r) - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2 \qquad \text{(I.1)},$$

where $\kappa = GM/c^2$, is called the Schwarschild metric (SM). Thus, according to (I.1) we have $g_{oo}(r) = g^{44}(r) = Z = (1\text{-}2\kappa/r) = N(r)$, $g^{11}(r) = 1/Z = M(r)$, $g^{22}(r) = r$ and $g^{33}(r,\theta) = r^2\sin^2\theta$.

The dimensionless quantity $\kappa/r = GM/c^2r$ can be seen as a measure of the intensity of the gravitational field.[2]. We have studied in previous articles[3] several gravitational effects when $GM/c^2r \ll 1$ such as light deflection, luminous Doppler effect, time dilation, perihelion precession of planets and time delay of radar echoes.[3] In the solar system the relativistic gravitational effects are very small. Just note that on the surface of the Sun $GM_{sun}/c^2R_{sun} \sim 10^{-6}$ since $G/c^2 = 7{,}414\ 10^{-28}$ m/kg, the sun mass $M_{sun} = 2.3\ 10^{30}$ kg an radius $R_{sun} = 6.96\ 10^5$ km. These effects start to become large in the vicinity of a very massive and very compact star when $2GM/c^2r$ approaches the unit.[4] For example, when $r = 3GM/c^2$ the deflection of light begins to become so large that the light signal may move in a closed circular orbit[2] around the star (see Appendix A).

J. Mitchell[5] in 1784 was the first to report the spectacular effect produced by the gravitational potential GM/r when it becomes very large. Using Newton's theory of gravitation (NGT) he showed that the escape velocity V_e of a body of mass m must be $V_s \geq (2GM/R)^{1/2}$ where R is the radius of the planet. As V_e is independent of the mass of the body, he argued that not even light could escape the gravitational attraction if R were less than the limit value r* given by

$$r^* = 2GM/c^2 \qquad \text{(I.2).}$$

In terms of r_s given by (I.2) the MS (I.1) can be re-written in the form

$$ds^2 = (1 - r_s/r)c^2dt^2 - dr^2/(1 - r_s/r) - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2 \qquad \text{(I.3).}$$

It can be shown (see Appendix B) using (I.3) that the curvature of space -time $R_{\alpha\beta\mu\nu}$ presents divergences at the limit of $r \to$ rs. As the component $R_{o1o1} = r_s/[r(1- r_s/ r)]$ we see that $R_{o1o1} \to \infty$ at the limit $r \to r_s$. Since the tidal forces [2] are proportional to the $R_{\alpha\beta\mu\nu}$ curvature, they would be immensely large as $r \to r_s$. The non-escape radius $r_s = 2GM/ c^2$ is called *Schwarzschild radius* or gravitational radius of mass M. The calculation made using NGT, which gives the correct value of r_s, but leads us to misinterpret what happens: light or particle emitted radially out of the region with $r \leq r_s$ does not rise, stop and then descend. In fact, as we will see below, according to TGE's previsions it falls immediately and never begins to move radially outward. The region of space-time into which a signal can enter, but from which no signal can exit, is called Schwarzschild Black Hole or, simply, **BH.** For an outside observer, the spherical surface of radius r_s constitutes what we call the event horizon **(HE)**, Schwarzschild horizon or, simply, **horizon**. Everything below HE remains invisible to that observer. For a BH with a Sun mass we have $r_s = 2GM_{sun}/c^2 = 3.41$ km.

Taking into account (I.3), it appears that the points $r = r_s$ and $r = 0$ are singularities of the SM. When $r \rightarrow r_s$ we have $g_{oo} = (1 - r_s/r) \rightarrow 0$ and $g_{11} = 1/(1 - r_s/ r) \rightarrow - \infty$. When $r \rightarrow 0$ we have $g_{oo} = (1 - r_s /r) \rightarrow - \infty$ and $g_{11} = 1/(1 - r_s/r) \rightarrow 0$. As seen in Appendix B and as will be shown in Section 2, the singularities of the metric coefficients $g_{\mu\nu}$ and the curvature $R_{\alpha\beta\mu\nu}$ at point rs can be eliminated with an appropriate choice of the coordinates where tidal forces[2] would be finite at $r = r_s$. As the physical effects, which are the tidal forces, remain finite, well behaved, we can conclude that $r = r_s$ is a mathematical, an spurious singularity, or, still, a pseudo-singularity. It is not a physical singularity. However, the point $r = 0$ appears to be a physical singularity, since it cannot be removed by any transformation of coordinates within the context of the EGT (Appendix B). In the vicinity of $r = 0$, infinite tidal forces should appear, indicating that $r = 0$ is a real physical singularity. Perhaps quantum gravitational effects may inhibit the appearance of this singularity.[6,11]

Another important fact is that the singularity at the point $r = r_s$ generates a critical difference in space-time outside and inside BH. For $r > r_s$ we have $g_{oo} > 0$ and $g_{11} < 0$; for $r < r_s$, we have the opposite, $g_{oo} < 0$ and $g_{11} > 0$. Thus, if in the region $r > r_s$ a small change in t is made keeping with r = constant we will have $ds^2/c^2 = d\tau^2 = g_{oo}dt^2 > 0 \rightarrow$ separation in the temporal coordinate it is timelike. Within BH, that is, for $r < r_s$ we will have, $ds^2/c^2 = d\tau^2 = g_{oo}\, dt^2 < 0 \rightarrow$ separation in the temporal coordinate is spacelike. Similarly, if in the region $r > r_s$ a small change in r is made while maintaining t = constant we will have $ds^2/c^2 = dr^2 = g^{11}\, dr^2 > 0 \rightarrow$ separation in the spatial coordinate is timelike and for $r < r_s$, $ds^2/c^2 = dr^2 = g_{11}\, dr^2 < 0 \rightarrow$ separation in the spatial coordinate is spacelike.

In **Section 1** we will calculate the travel times of a light signal and a space probe when describing a radial trajectory in space-time described by the SM given by (I.3). In **Section 2** we will show how SM (I.3) is transformed adopting the coordinates proposed by Eddington in 1924 [6,7] and by Kruskal.[10] With these new coordinates, the singularities of $g^{\mu\nu}$ and $R_{\alpha\beta\mu\nu}$ in $r = r_s$ disappear, however, the singularity remains at $r = 0$. We will calculate the light path times in radial paths in the case of Eddington coordinates and see how from the Schwarzschild geometry emerges "wormholes" (**WH**).

(1)Travel times of radial paths of the light and probe.

The event horizon (**HE**) at $r = r_s$ plays a fundamental role in the BH. The region within that horizon is strictly isolated from the rest of the Universe. Let's see how HE affects physical phenomena.

The first important result was obtained[3] using the SM, taking into account (I.1), the time dilation $d\tau$ of a clock with coordinate r measured by an observer far from BH:

$$d\tau = (1 - r_s/r)\, dt \qquad (1.1).$$

Thus, a clock in $r \approx r_s$ travels infinitely slower than a clock in infinity. This means that if a space probe in the vicinity of a black hole, with r ≈ rs, sends light signals separated by time intervals of 1 s (measured on your watch) an observer very distant from him will receive these pulses of light separated by time intervals much greater than 1s (measured on the observer's watch).

Eq.(1.1) shows an infinite time delay (infinite "redshift") for a clock at $r = r_s$. Indeed, as we will see below, a clock (a material body) cannot remain at rest on the surface of events. Only a light signal can remain at rest at $r = r_s$.

(1.1) Light travel time.

Let us consider a light ray traveling radially in a region described by the SM. Putting $ds^2 = d\theta^2 = d\varphi^2 = 0$ in (I.3) we get:

$$0 = (1 - r_s/r)\, c^2 dt^2 - dr^2 / (1 - r_s/r) \qquad (1.2),$$

obtaining the *speed coordinate* (or velocity) dr/dt, measured by an observer very far from the black hole,

$$r/dt = \pm\, c\, (1 - r_s/r) \qquad (1.3),$$

where the ± sign of dr/dt means that the light is moving, respectively, in the positive radial direction (**S+**) or in the negative radial direction (S-). Note that according to (1.3), dr/dt = 0 in HE, that is, at $r = r_s$.

(1.1a) Light motion towards the BH center (S-).

The time $t_1(r)$ that the light takes (measured by an observer away from BH) starting from a point with initial coordinate $r = R >> r_s$ and approaching the BH, arriving at a point $r \geq r_s$ is, given by (1.3):

$$t_1(r) = -\int_R^r dr/c(1 - r_s/r) = (R - r)/c + (r_s/c)\ln[(R - r_s)/(r - r_s)] \qquad (1.4),$$

where $T = (R - r)/c$ would be the light path time of $R \rightarrow r$ in the absence of a gravitational field. According to (1.4) the light would take an infinite time to reach the point $r = r_s$. For a very distant observer the speed of the light $dr/dt \rightarrow 0$ (see 1.3) when $r \rightarrow r_s$; consequently, the light signal would take an infinite time to reach $r = r_s$.

On the other hand, the travel time $t_2(r)$ that the light takes to go from a point $r \leq r_s$ to r = 0 is given by, using (1.3):

$$t_2(r) = -\int_r^0 dr/c(1 - r_s/r) = r/c + (r_s/c)\ln[(r_s/(r_s - r)] \qquad (1.5).$$

According to (1.5) the travel time $t_2(r)$ of a light signal departing from a point with $r_s > r$ is finite. However, it would never reach $r = 0$ if it started from $r = r_s$, because at that point the light would have zero speed.

These results show that a light signal sent from outside BH towards the center of BH takes an infinite time to reach HE. If the signal is sent towards $r = 0$ from a point $r < r_s$ it will take a finite time to reach the point $r = 0$. It will never reach the center if it is sent from a point with $r = r_s$.

(1.1b) Light signal with positive radial motion (S+).

Integrating (1.3), with the + sign, from $0 \rightarrow r_s$ one can see that the travel time from $0 \rightarrow r_s$ would be infinite, because the speed of the signal tends to zero in the HE, that is, at $r = r_s$. The light would never exceed the distance $r = r_s$. However, integrating (1.3), with the + sign, of $r \rightarrow R$ with $r > r_s$, we have a finite travel time, that is, the light would always reach the observer who is at point R.

Thus, if light is sent **(S +)** from inside BH in the direction $0 \rightarrow r_s$ it would never exceed HE. However, if it is sent **(S+)**from outside the BH it would always arrive at an external point R.

(1.2) Probe fall time in a radial path.

Let us now calculate the proper travel time measured by a clock placed on a space probe that moves radially towards the black hole **(S-).** According to the calculations shown in Appendix C the equation (B.5) that gives the radial path of a particle in an SM is given by

$$(dr/cd\tau)^2 = r_s/r + 1 - B^2 \qquad (1.6),$$

where B is a constant of motion. If in the initial state the particle has zero velocity, very far from the BH and falls towards it (S-), from (1.6) we have (see Appendix A)

$$d\tau = - c(r/r_s)^{1/2}\, dr \qquad (1.7).$$

Integrating (1.7) from a generic point r to r = 0, we have,

$$\tau(r) = \tau_o - (2/3)\,(r_s/c)\,(r/r_s)^{3/2} \qquad (1.8).$$

where $\tau_o = \tau(r = 0)$ is the proper time that probe takes to reach $r = 0$. Note that the proper time measured by a clock on the probe varies uniformly when crossing the horizon at $r = r_s$. Once the probe reaches the horizon, it takes a time $\tau = \tau(r_s) - \tau_o = (2/3)(r_s/c)$ to reach the center of BH. In the case of a BH with a mass of 10 M_{sun} we would have $\Delta\tau \sim 10^{-4}$ s.

Let us now calculate the time t, which the probe would take, measured by an observer far from BH, to go from a very distant point to the event horizon $r = r_s$. Now, from (1.1) and (1.7) we obtain[6]

$$dt = -r^{3/2}dr / (r - r_s)\, r_s^{1/2} \quad (1.9).$$

Integrating (1.9):

$$t(r) = t_o + (r_s/c) [- (2/3) (r/r_s)^{3/2} -2(r/r_s)^{1/2} + \ln|((r/r_s)^{1/2}+1)/((r/r_s)^{1/2}-1)| \quad (1.10).$$

For long distances $r >> r_s$ we have,

$$t(r) \approx to - (2/3) (r_s/c) (r/r_s)^{3/2} -2 (r_s/c)(r/r_s)^{1/2} \quad (1.11).$$

As for long distances t(r) and τ(r) must coincide (see 1.7 and 1.9), that is, $d\tau \approx -dt \approx c\,(r/r_s)^{1/2}\, dr$, the constant t_o must be chosen accordingly, that is, $t_o \equiv \tau_o + 2(r_s/c)(r/r_s)^{1/2}$. With this choice we obtain

$$t(r) \approx \tau_o - (r_s/c)(2/3)(r/r_s)^{3/2} \quad (1.12).$$

In **Figure 1**are shown [6] the times t(r) and τ(r) as functions of r, represented by dashed and continuous lines, respectively. Remembering that $t(r_s) = \tau_o - (2/3)(r_s/c)^{3/2}$ and $t(r =0) = \tau_o$ we have $\Delta t = \Delta\tau = (2/3)\,(r_s/c)$.

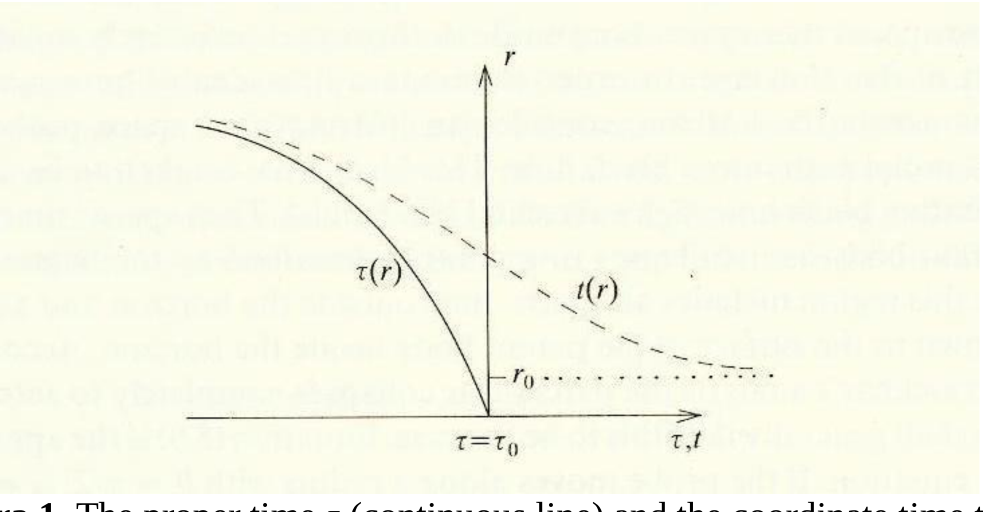

Figura 1. The proper time τ (continuous line) and the coordinate time t(r) (dashed line) plotted as functions of r and t, for a probe falling radially into a BH.[6]

It is important to note that the probe takes an infinite time t to reach the horizon. On the contrary, the proper time τ measured by a clock on the probe takes a finite time passing through the horizon and reaching r = 0. In practice, the instruments on the probe would not survive this trip, as they would be destroyed by the immense gravitational tidal forces.

The behaviors of $t(r)$ and $\tau(r)$ as a function of r are completely different when the probe approaches and crosses the horizon, clearly illustrating how the curvature of the space-time with SM makes it impossible to cover all space-time with a single set of Cartesian coordinates.

(1.3)Light signal emitted by a probe escaping from the BH.

In Section (1.2) we calculated the falling time of a probe. Now suppose that a light source on the probe, escaping from the BH, emits a light signal with a proper frequency f_o. The energy emitted by the source, per unit of proper time, is the luminance given $L_o \propto f_o/\Delta\tau$. For an observer at a point r, the luminance $L \propto f/\Delta t$ is given, taking into account that $f = f_o(1- r_s/r)^{1/2}$ and that $\Delta t = \Delta\tau/(1 - r_s/r)^{1/2}$:

$$L(r) \propto (1- r_s/r)f_o \qquad (1.12).$$

Considering that the arrival time of the signal at an observer located at point r = R as the time t = 0, the light would be emitted by the probe at the time $T(r) = - t_1(r)$ calculated with a (1.4). Now, according to (1.4) taking into account that the light is sent in $r \approx rs$ we will have

$$T(r) \approx - (r_s/c) \ln(r - r_s) \qquad (1.13).$$

From (1.12) and (1.13) we obtain,

$$L(T) \propto f_o \exp (- cT/r_s) \qquad (1.14).$$

This shows that the luminous signal emitted by the probe decays exponentially as it approaches the HE, that is, when $r \rightarrow r_s$. In the case of a BH with a mass of 10 M_{sun} the time constant r_s/c is of the order of 10^{-4} s. So, a white light emitted from the surface of a collapsing star, turning into the HE, quickly turns red and disappears in a r_s/c time scale.

(2) Eddington and Kruskal coordinates.

(2.a) Eddington coordinates.

In order to eliminate the singularities that appear in the SM (I.1) or (I.3) when using the polar coordinates $x_o = x_4 = ct$, $x_1 = r$, $x_2 = \theta$ and $x_3 = \varphi$, Eddington [7] proposed in 1924 a set of coordinates that would be more appropriate to study a BH defining a time t*:

$$t^* = t + (r_s/c) \ln|(r/r_s - 1)| \qquad (2.1).$$

With this change in variables, the SM (I.3) is replaced by

$$ds^2 = (1 - r_s/r)\, c^2 dt^{*2} - 2c(r_s/r)drdt^* - (1 + r_s/r)dr^2 + r^2 d\Omega^2 \qquad (2.2).$$

Now, the new metric coefficients $g_{\mu\nu}$ of (2.2) have no singularities at $r = r_s$. They also present singularities at r = 0. The Eddington coordinates that were rediscovered by Finkelstein[8] in 1958 are also called Eddington - Finkelstein coordinates. These coordinates allow for a physically clearer interpretation of what occurs in the vicinity of a **White Hole** (**WH**) (see Section 3), but, not so obvious for regions further away from a BH. It can be shown (**Appendix B**) that with the new coordinates there are no singularities in the curvature $R_{\alpha\beta\mu\nu}$ at $r = r_s$ and, consequently, the tidal forces are finite at this point. They tend to infinity when $r \to 0$ (see **Appendix B**). In these new coordinates, the radial trajectory of a light signal is determined from

$$0 = (1 - r_s/r)\, c^2 dt^{*2} - 2c(r_s/r)drdt^* - (1 + r_s/r)dr^2$$

that has as solutions

$$dr/dt^* = -c \qquad \text{and} \qquad dr/dt^* = c\,(1 - rs/r)/(1 + r_s/r) \qquad (2.3).$$

In the absence of a BH, (2.3) are given by $dr/dt^* = -c$ and $dr/dt^* = c$, respectively. The first equation would describe a light signal going towards r = 0(**S-)** and the second one going in the opposite direction to r =0 (S+) Thus, in the general case, the first equation of (2.3) would describe a light signal that moves towards r = 0 with a constant speed −c for any value of r, that is, $0 \le r < \infty$. In other words, the light that comes from outside will always enter BH. The second solution that would describe the path of light in **S +** shows that for $0 \le r < r_s$ the speed is negative, for $r = r_s$ the speed is zero and that from r > rs it becomes positive and grows until it reaches the maximum value + c at infinity. Thus, if the probe is within the horizon (HE), the light it emits will always have a speed turned to r = 0 or S-. In this way the light that comes from outside (S-) or the one that is emitted from inside BH, either in the positive + or negative direction - always goes inexorably towards r = 0.

There is an alternative form of Eddington coordinates, that is, instead of (2.1), defining

$$t^* = t - (r_s/c) \ln |(r/r_s - 1)| \qquad (2.4).$$

With this choice, (2.3) are replaced by

$$dr/dt^* = c \qquad \text{and} \qquad dr/dt^* = -c(1 - r_s/r)/(1 + r_s/r) \qquad (2.5).$$

In these conditions, it can be seen that the light will always leave the **horizon** and instead of a BH we have a **White Hole** (**WH**). In this way, matter will always be ejected from the singularity r = 0. According to current theories of stellar evolution [2,6,9] only BH must exist and never a white hole, although mathematically this was possible according to (2.5). This shows that coordinate choices in a space-time described by the SM can generate very subtle properties, as we will see below analyzing Kruskal's coordinates.[10]

(2.b) Kruskal coordinates.

Another coordinate system (u, v, θ, φ) convenient for studying BH was proposed by Kruskal.[10]. For (a) $r < r_s$ and (b) $r > r_s$ we have,

(a) $r < r_s$

$$u = (1- r/r_s)^{1/2} \exp(r/2r_s) \sinh(ct/2r_s)$$
$$v = (1- r/r_s)^{1/2} \exp(r/2r_s) \cosh(ct/2r_s) \qquad (2.6)$$

(b) $r > r_s$

$$u = (r/r_s -1)^{1/2} \exp(r/2r_s) \cosh(ct/2r_s)$$
$$v = (r/r_s -1)^{1/2} \exp(r/2r_s) \sinh(ct/2r_s) \qquad (2.7).$$

The inverse transformations of (a) are given by

$$(r/r_s -1) \exp(r/r_s) = u^2 - v^2 \quad \text{and} \quad ct = 2r_s \tanh^{-1}(u/v) \qquad (2.8),$$

and the inverse of (b) by,

$$(r/r_s -1) \exp(r/r_s) = u^2 - v^2 \quad \text{and} \quad ct = 2r_s \tanh^{-1}(v/u) \qquad (2.9).$$

With these Kruskal coordinates ds^2 defined by (I.3) is given by

$$ds^2 = (4r_s{}^3/r) \exp(-r/r_s) (dv^2 - du^2) - r^2 d\theta^2 - r^2\sin^2\theta\, d\varphi^2 \qquad (2.10),$$

where the coordinate r is a function of u and v, according to (2.8) or (2.9).

As in the case of the Eddington metric, in the Kruskal metric (2.10) there is only the inevitable singularity r = 0. Eq.(2.10) is also known as the “maximal extension of the Schwarzschild metric”. The variety defined by the Kruskal metric is said *maximal* because the geodesics have an infinite length in both directions of a singularity; they do not begin or end in a singularity. In the book of Misner,Thorne and Wheeler [10] we see in detail how to obtain the radial geodesics of massive particles and photons using

Kruskal's (u, v) plane compared to those obtained using Schwarzschild's coordinates (ct, r) (see Ohanian[2]). The photon radial geodesics ("lightlike") are obtained from (2.10) making $ds^2 = 0$, $\theta = \pi/22$ and φ = constant, obtaining $dv^2 = du^2$. Showing that the geodesics of the photons are straight lines u = v in the plane (u,v). Differentiating (2.6) and (2.7) it appears that along the geodesics we have dr/dt = ± c. In **Figure 2** we show one of the representations of the plane (u,v) seen in Ohanian's book [2] [Fig. (9.4), pag.319]. In the plane (u, v) appear the regions I and IV of the space-time (u,v) that represent, respectively, BH and WH, which have the point u = v = 0 in common. Regions I and III are outside the light cone: communication between them is impossible.

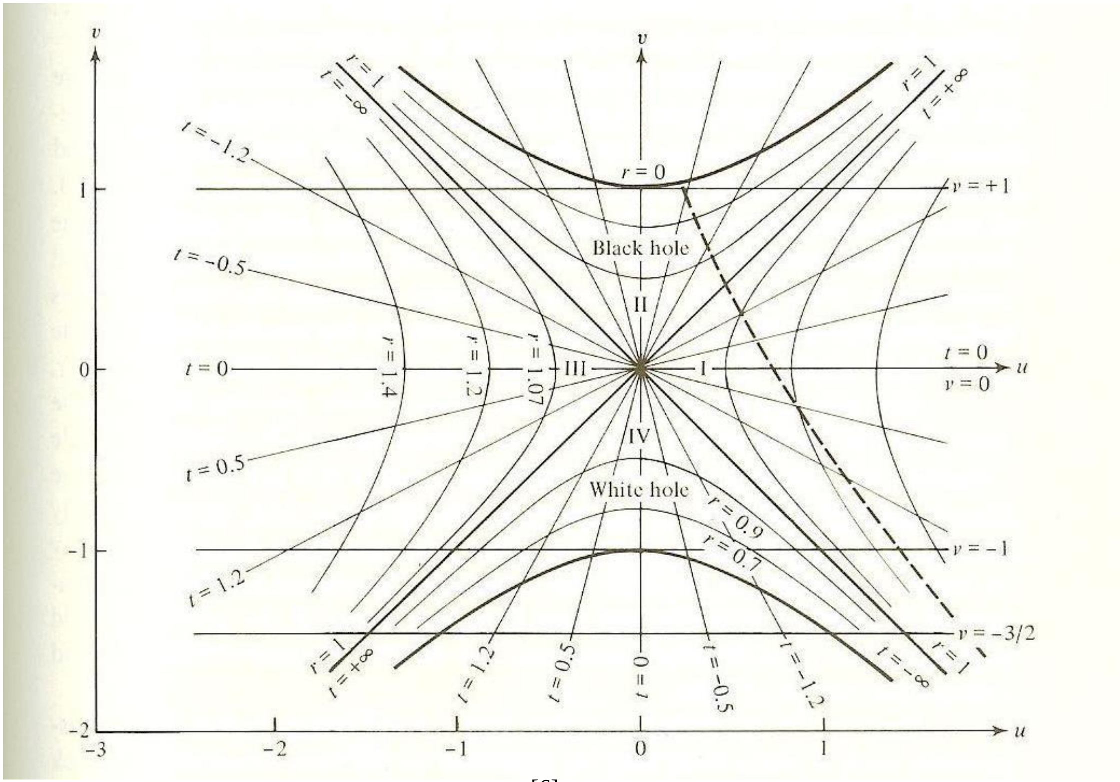

Figure 2. The "maximal geometry"[6] of Schwarzschild in the (u,v) Kruskal coordinates.

(2.c)Wormholes (WoH).

It is fascinating to analyze the geometry of space-time involved with physical phenomena. We suggest that students read section 23.8 of the book Gravitation[11]and the articles by Fuller, Misner and Wheeler [16,17] on this topic, "Geometrodynamics". For very relativistic static stars, the geometry of space-time deviates strongly from plane Lorentz -Euclidean geometry. Indeed, taking into account (I.3), with t = constant, that the radial distance $\ell(r)$ of a coordinate point r measured from $r = r_s$ is given by

$$\ell(r) = \int_{r_s}^{r} dr/(1- r_s/r)^{1/2}.$$

The area A(r) of a sphere with radius r is given by $A(r) = 4\pi r^2$ and the length s(r) of the circumference of radius r measured in the equatorial plane, where $\theta = \pi/2$, is given by $s(r) = 2\pi r$. Since $(1- r_s/r)^{1/2} <1$ we find that $d\ell(r)/dr$ varies very quickly for r around r_s and remains constant at the limit of r >> rs. This "strange" behavior can be visualized more easily using a process called "geometric immersion" as follows.

Doing $\theta = \pi/2$ in (I.3) we will have an arc element dl^2 in the equatorial plane given by

$$dl^2 = dr^2/(1- r_s/r) + r^2 d\varphi^2 \qquad (2.11),$$

which obeys a geometry of a curved 2-dim space (r, φ). What is done is to emerge this curved space in a flat 3-dim space (r, φ, z). In this space, the z coordinate is an "artificial coordinate" that has nothing to do with the z coordinate of real space. The distances dl in the flat 3-dim space with cylindrical coordinates (r, φ, z) where we will emerge the 2-dim curve described by (2.11) are given by

$$dl^2 = dr^2 + r^2 d\varphi^2 + dz^2 = dr^2 [1 + (dz/dr)^2] + r^2 d\varphi^2 \qquad (2.12).$$

The immersion is done in such a way that the distances dl along the surface described by (2.12) coincide with the distances dl given by (2.11). With this condition, from (2.11) and (2.12), we obtain

$$(dz/dr)^2 = 1/(1- r_s/r) - 1 \qquad (2.13).$$

Integrating (2.13), for z > 0 results,

$$z(r) = 2r_s(r/r_s - 1)^{1/2} + C \qquad (2.14).$$

Assuming that at z = 0 we have $r = r_s$ we obtain the surface of a paraboloid of revolution given by

$$z(r) = 2r_s\,(r/r_s - 1)^{1/2} \qquad (2.15).$$

The bottom part of the surface, that is, the part with z < 0, can be thought of as a deformation similar to that of a "second universe". [2,11,16,17] In **Figure 3** (see fig.9.5 (a) of Ohanian[2],pag.322) we see the parabolic surface z(r) connecting the "two Euclidean universes" (two "flat spaces"). It widens, opening at both ends like a funnel and narrowing in the middle at z = 0. This surface is called "**wormhole**" (**WoH**).[2,11,16,17] The top and bottom parts of the funnel are flat surfaces that represent two Euclidean flat spaces that are very far from the BH with center in z = 0, at the neck of the funnel. At this narrower point r = s, in the vicinity of the BH, the greatest space-time deformation occurs.

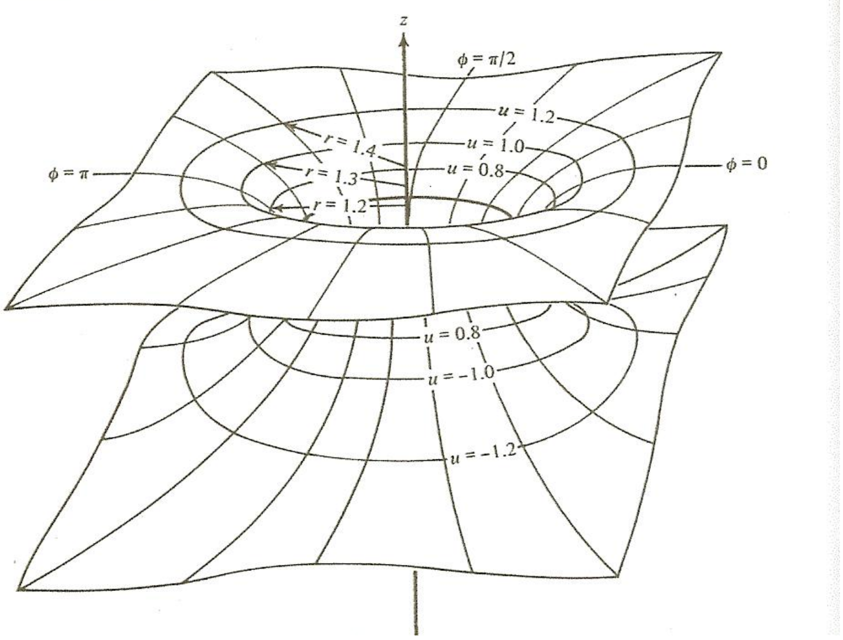

Figure 3. **Wormhole Geometry (WoH).** The surface z(r, φ) was calculated for the case t = 0 and θ = π/2 (in Kruskal's metric we have v = 0 and θ = π/2). The r coordinate is measured in units of r_s and the parameter u is given, putting t = 0 in (2.7), by $u = \pm (r/r_s - 1)^{1/2} \exp(r/2r_s)$.

On the 2-dim z(r) surface the measured distances dℓ between any two nearby points (r, φ) and (r + dr, φ + dφ) are correctly reproduced. The circles of radii r have their own circumferences s(r) equal to 2πr. Distances measured off the surface have no physical significance; points outside the surface have no physical significance; the 3-dim Euclidean space has no physical meaning. Only the 2-dim curved surface has meaning. The 3-dim regions inside and outside the funnel have no physical meaning, that is, the Euclidean “immersor” space (r, φ, z) has no physical meaning. It only allows you to view the geometry of the space around the star in a convenient way: with it we can see how quickly the distances ℓ increase as a function of the coordinates (r, φ) and how the circumferences (straight sections of the funnel) s(r) vary with r.

WoH is also interpreted as being a connection between two flat Euclidean spaces[10,11,16] in the limit case in which the mouths of the funnels are very distant from each other compared to the dimensions of the bottlenecks of the WoH.

Kruskal's metric depends on the time t that appears in the functions u and v as we see in (2.6)-(2.9). In **Figure 3** we show the case of WoH for the time t = 0 or v = 0 (v > 0 when t > 0 and v < 0 when t < 0). The coordinate v plays the role of “time” in the Kruskal metric.[2] If t changes, u and v also change. It can be shown[17] that the WoH geometry the varies with v (or t), as schematized[2] in **Figure.3**. For v < −1 there is no WoH, only two Euclidean spaces, upper and lower, disconnected, each with a cusp: the WoH is collapsed.

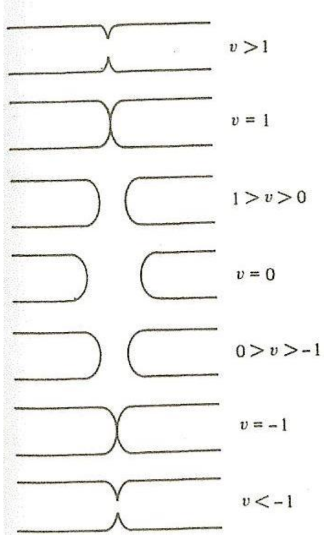

Figura 3. Schematic[2] temporal evolution of the WoH geometry.

For v = −1 the WoH comes into existence, but the bottleneck is closed. For 0 > v> −1 the neck is open and for v = 0, which is the case seen in **Figure 4**, it has a maximum diameter. For positive times 1> v> 0 the neck begins to close, closing at v = 1. For v > 1 the WoH collapses leaving the lower and upper Euclidean spaces disconnected, each one with a cusp. When the bottleneck is open it is presumed that matter absorbed by the BH in "our Universe" passing by the bottleneck, is captured by the WH in the "other Universe".

Finally, it is important to remember that the Schwarzschild and Kruskal metrics are valid in a region where there is no matter, that is, they are solutions of the TGE equations in a region where $T_{\nu\mu} = 0$. Thus, they are not relevant to the problem of gravitational collapse that should give rise to BH. The interpretations deduced using these metrics, including WoH, could be applied only to the case of BH already existing in the Universe.

(3)Comments.

Probably the observed BHs have little or no resemblance to those predicted by SC and Kerr.[2] However, their mathematical descriptions are beautiful: "Se non è vero, é bene trovato".

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APPENDIX A.

Circular orbit of light around a very massive and compact star.

According to previous article[1,3] to calculate the trajectory (geodesic) of a particle that moves in a vacuum we have to take into account the following equations

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

and (A.1)

$$d^2x^\alpha/ds^2 \;+\; \Gamma_{\tau\,\nu}{}^\alpha\, (dx^\nu/ds)\, (dx^\tau/ds) \;= 0,$$

where the symbols Christoffel $\Gamma_{\mu\nu}{}^\alpha = \{_\mu{}^\alpha{}_\nu\}$ are defined by

$$\Gamma_{\mu\nu}{}^\alpha = \{_\mu{}^\alpha{}_\nu\} = (g^{\alpha\lambda}/2)(\,\partial_\nu\, g_{\lambda\mu} + \partial_\mu\, g_{\lambda\nu} - \partial_\lambda\, g_{\mu\nu}\,) \qquad \text{(A.2)}$$

and the tensors $g_{\mu\nu}$ are defined, according to the Schwarzschild metric,

$$ds^2 = e^{N(r)}\, c^2dt^2 \;-\, (e^{M(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\; d\varphi^2) \qquad \text{(A.3)},$$

where $e^{N(r)} = g_{oo}(r) = g_{44}(r) = Z = (1- 2\kappa/r)$, $e^{M\,(r)} = g_{11}(r) = 1/Z$, $g_{22}(r) = r^2$ and g_{33} (4) $= r^2\sin^2\theta$, taking into account that $x_o = x_4 = ct$, $x_1 = r$, $x_2 = \theta$ and $x_3 = \varphi$. With these values we can calculate Christoffel's symbols. Remembering[4] that $g^{\mu\nu} = M_{\mu\nu}/|g|$ where $|g|$ is the determinant of $g_{\mu\nu}$ and $M_{\mu\nu}$ is the minor determinant of $g_{\mu\nu}$ in g. Since the elements of g are diagonal, we have $|g|= |g_{11}\, g_{22}\, g_{33}\, g_{44}\,| = c^2e^{2(N\,+\,M)}\, r^4\sin^2\theta$.Thus, $g^{oo} = c^{-2}e^{-N}$, $g^{11}= e^{-M}$, $g^{22} = r^{-2}$ and $g^{33} = r^{-2}\sin^{-2}\theta$. As $g_{\mu\nu}$ only depend on $r = x_1$ in (A.2), there are only derivatives of the type $\partial_r g_{\mu\nu}. = \partial_{x1} g_{\mu\nu}$. Indicating by $N' = \partial N/\partial\, r$ and $M' = \partial M/\partial r$ we obtain $\Gamma_{\mu\nu}{}^\alpha = \{_\mu{}^\alpha{}_\nu\}$, following the same procedure seen in detail in a previous article.[1,3,12,13]

In the particular case of the trajectory of a light signal, we have a "null geodesic"[13,14], that is, we must assume that $ds^2=0$. In this case, is defined a non-zero scalar parameter λ that varies along this geodesic. Thus, the equations shown in (A.1) and (A.3) are replaced, respectively, by

$$g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda) = 0 \qquad \text{(A.4)}$$

$$d^2x^\alpha/d\lambda^2 \;+\; \Gamma_{\tau\,\nu}{}^\alpha\, (dx^\nu/d\lambda)\, (dx^\tau/d\lambda) \;= 0, \qquad \text{(A.5)}$$

$$0 \;= Z\, c^2\, (dt/d\lambda)^2 \;-\, [Z^{-1}\, (dr/d\lambda)^{\,2} + r^2\, (d\theta/d\lambda)^2 + r^2\sin^2\theta\, (d\varphi/d\lambda)^2]. \qquad \text{(A.6)}$$

Using (A.5) and the $\Gamma_{\mu\nu}{}^\alpha$ calculated according to was mentioned above, we obtain the following equations [12]

$$d\{Z(dt/d\lambda)\}/d\lambda = 0, \qquad (A.7)$$

$$d\{r^2 (d\theta/d\lambda)\}/d\lambda - r^2 \sin\theta \cos\theta \, (d\varphi/ d\lambda)^2 = 0 \qquad (A.8)$$

$$d\{r^2 \sin^2\theta \, (d\varphi/ d\lambda)\}/d\lambda = 0 \qquad (A.9).$$

Supposing that the light moves in a plane,[3] putting $d\theta/d\lambda = 0$ and $\theta = \pi/2$ in (A.7) and (A.9), we obtain, respectively,

$$dt/d\lambda = \beta^*/Z \qquad \text{e} \qquad r^2 (d\varphi/d\lambda) = h\,\beta^* \qquad (A.10),$$

where h e β^* are integration constants.[3] Substituting (A.10) into (A.6) we get, putting $u = 1/r$,

$$(du/d\varphi)^2 = 1/h^2 - u^2 + 2\kappa u^3 \qquad (A.11).$$

Thus, from (A.11) we obtain,[2,6]

$$d^2u/d\varphi^2 + u = 3\kappa u^2 \qquad (A.12).$$

Eq.(A.12) admits as a solution r = constant = $3\kappa = 3GM/c^2$. This implies that the light has a circular orbit with a radius = $3GM/c^2$ around the star. If u is slightly larger (smaller) than $(3GM/c^2)^{-1}$ the orbit is unstable.

APPENDIX B.
Schwarzschild metric singularities and the tidal forces.

The curvature $R^{\alpha}{}_{\beta\mu\nu}$ of the Riemann space-time or *Riemann-Christoffel tensor*, is defined by [1,2,6,12,13]

$$R^{\sigma}{}_{\lambda\mu\nu} = \partial^{\mu}\Gamma_{\lambda\nu}{}^{\sigma} - \partial^{\nu}\Gamma_{\lambda\mu}{}^{\sigma} + \Gamma_{\lambda\nu}{}^{\tau}\Gamma_{\mu\tau}{}^{\sigma} - \Gamma_{\lambda\mu}{}^{\tau}\Gamma_{\tau\nu}{}^{\sigma} \qquad (B.1),$$

where $\Gamma_{\mu\nu}{}^{\alpha} = \{{}_{\mu}{}^{\alpha}{}_{\nu}\}$ are the *Christoffel* symbols given by

$$\Gamma_{\mu\nu}{}^{\alpha} = \{{}_{\mu}{}^{\alpha}{}_{\nu}\} = (g^{\alpha\lambda}/2)(\partial_{\nu} g_{\lambda\mu} + \partial_{\mu} g_{\lambda\nu} - \partial_{\lambda} g_{\mu\nu}) \qquad (B.2).$$

The EGT field-equations of the are,[2,6,12,13]

$$R_{\mu\nu} - (1/2)g_{\mu\nu} R = \kappa\, T_{\mu\nu}{}^{(m)} \qquad (B.3),$$

where $\kappa = 8\pi G/c^4$, $R_{\mu\nu}$ is the Ricci curvature tensor:

$$R_{\mu\nu} = R_{\nu\mu} = \partial_{\mu}\Gamma_{\nu\sigma}{}^{\sigma} - \partial_{\sigma}\Gamma_{\nu\mu}{}^{\sigma} + \Gamma_{\mu\sigma}{}^{\tau}\Gamma_{\nu\mu}{}^{\sigma} - \Gamma_{\nu\mu}{}^{\tau}\Gamma_{\tau\sigma}{}^{\sigma} \qquad (B.4),$$

and the scalar $R = g^{\lambda\sigma} R^{\sigma\lambda}$ is known as scalar curvature or space-time curvature invariant. It can be shown [2] that the tidal forces, which are physical effects, are directly proportional to $R^{\sigma}_{\lambda\mu\nu}$.. Suppose that in a given coordinate system there are singular points at which $R^{\sigma}_{\lambda\mu\nu} \rightarrow \infty$. If there is an adequate transformation of coordinates that allows to eliminate these singularities, that is, in such a way that the physical effects, which are the tidal forces, remain finite, well behaved, we can conclude that the singularities are spurious, mathematical or, still, pseudo -singularities.

For example, it appears that in SM (I.3) the component R^{o}_{101} is given by $R^{o}_{101} = (r_s/r)/(1 - r_s/r)$ which tends to infinity at the limit $r \rightarrow r_s$. Using, for example, Eddington-Filkenstein coordinates (see Chapter 2) and (B.1) - (B.4), it can be shown that the tidal forces are finite at the point $r = r_s$. Only at point $r = 0$ they diverge. Another coordinate system for analyzing the BH would be the "*geodesic coordinates*" (see Ohanian,[2] pag.309) which are those that present at a certain point P a space-plane metric. It is always possible[1,2] to find a coordinate system that obeys this condition. This can be seen, for example, in the Ohanian book[2] (pag. 231−233). In other words, given the coordinates x^{μ} with a metric $g^{\mu\nu}(x)$ we can always find a linear transformation for new coordinates $x'^{\mu} = b^{\mu}_{\nu} x^{\nu}$, where b^{μ}_{ν} are constants, such that at a certain point P we have $g'_{\mu\nu}(P) = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$, = (−1,1,1,1), characteristic of a Minkowski plane space. In the vicinity of point P the space is locally flat where we have a local *inertial frame*. For a particle at point P we have $d^2x'^{\mu}/d\tau = 0$, that is, it moves with constant velocity or remains at rest observed in the system of geodesic coordinates, which would be instantaneously in *free fall* with the same acceleration of the particle. Point P is taken as the origin of the *geodesic referential* or in *free fall*. We emphasize that the coordinates are geodesic *only* for a certain instant of timer. The derivatives of $g'_{\mu\nu}(P)$ are zero only at a point P of the space-time. That is, in a *given place* and at a *given time*. If we want to obtain geodesic coordinates at another point P´, it is necessary to perform a new transformation of coordinates for this point P´of the space-time.

The space-time becomes flat in an infinitesimal neighborhood of point P when the first derivatives of $g'_{\mu\nu}(x')$ becomes equal to zero. On the other hand, the second derivatives of $g'_{\mu\nu}(x')$ cannot *all* be canceled by no one transformation of coordinates.[2] This implies that the tidal forces do not cancel each other in the neighborhood of P. These forces could be measured which would allow to discriminate between the effect created by a gravitational field and the effect generated by a pseudo force of an acceleration field. In this respect, the effects of a gravitation field are not indistinguishable from the effects observed in an accelerated framework. This could occur only when the gravitational field is uniform where we have a null tidal force. See comments about this (which involves the *Equivalence Principle*),for instance, in Ohanian´s book[2](pgs.38-41).

To be sure that in a given point there is physical singularity it is necessary to quantitatively verify their values in different coordinate systems. Para podermos afirmar que um ponto é uma singularidade física precisamos verificar quantitativamente os seus valores em diferentes coordenadas. However, according to the *Kretschmann invariant* [18] that for a Schwarschild BH is given by

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 12\, r_s^2/r^6 \qquad (B.5),$$

the singularity at $r = 0$ must always exist independently of the adopted coordinate system. Perhaps quantum gravitational effects could inhibit this singularity.[6,11]

In many respects the singularity $r = r_s$ that appears in the SM (I.3) is similar[2,13] to that found in a rotating coordinate system in the case of a flat space-time. Let us consider a Lorentz plane space defined by the line element ds^2 written, respectively, in Cartesian coordinates ($x_o = x_4 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$) or in polar cylindrical coordinates ($x_o = x_4 = ct$, $x_1 = r$, $x_2 = \varphi$ and $x_3 = z$):

$$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 = c^2\, dt^2 - dr^2 - r^2\, d\varphi^2 - dz^2 \qquad (B.6).$$

In the Cartesian case $g_{oo} = g_{44} = 1$ and $g_{11} = g_{22} = g_{33} = -1$ and in the polar case $g_{oo} = g_{44} = 1$, $g_{11} = -1$, $g_{22}(r) = - r^2$ and $g^{33} = -1$. Thus, in Cartesian coordinates we have $R^{\sigma}{}_{\lambda\mu\nu} = 0$ and, consequently, also in polar coordinates, this tensor is null because the polar reference is obtained from the Cartesian through a transformation defined by $ct = ct$, $x = r \cos\varphi$, $y = r \sin\varphi$ and $z = z$.

One can pass from an inertial to a non-inertial frame in rotation with constant angular velocity ω around the z axis using (B.6) with the coordinate transformation: $x_4 = ct$, $x_1 = r$, $x_2 = \varphi + \omega t$ and $x_3 = z$, obtaining:

$$ds^2 = (1 - \omega^2 r^2/c^2)\, c^2dt^2 - dr^2 - r^2d\varphi^2 - dz^2 - 2\omega r^2 d\varphi\, dt \qquad (B.7),$$

where $g_{oo}= (1 - \omega^2 r^2/c^2)$, $g_{11} = g_{33} = -1$, $g_{13} = g_{31} = - r^2$ and $g_{02} = g_{20} = - 2\omega r^2$. Obviously g_{oo} is singular at $r = c/\omega$ which defines a surface with infinite "redshift". However, the curvature tensor $R^{\sigma}{}_{\lambda\mu\nu}= 0$ also in the case of a rotating frame, since (B.7) was obtained by a coordinate transformation from (B.6).

APPENDIX C.
Trajectory of a Massive Particle in a Gravitational Field.

Taking into account that in the SM(I.1) we have $g_{oo}(r) = g_{44}(r) = Z = (1- 2\kappa / r)$, $g_{11}(r) = 1/Z$, $g_{22}(r) = r^2$ and $g^{33}(r) = r^2\sin^2\theta$ the invariant [14,15]

$$g_{\mu\nu}\,(dx^{\mu}/ds)\,(dx^{\nu}/ds) = 1 \qquad (C.1)$$

which is associated with the equation of the trajectory (geodesic) of a particle with mass ≠ 0 is written as,

$$Z(dt/cd\tau)^2 - (dr/cd\tau)^2/Z - r^2\,(d\theta/cd\tau)^2 - r^2\sin^2\theta\,(d\varphi/cd\tau)^2 = 1 \qquad (C.2).$$

Performing calculations analogous to those used to obtain the trajectory of a planet around the Sun we can show,[3] for a movement that takes place in a plane with $\theta = \pi / 2 =$ constant, that

$$r^2 d\varphi/d\tau = A \qquad \text{and} \qquad dt/d\tau = B/Z \qquad (C.3),$$

where A = aerial velocity and B are integration constants. Under these conditions (C.2) is written as

$$B^2/Z^2 - (dr/cd\tau)^2/Z^2 - A^2/r^2 = 1 \qquad (C.4).$$

When the particle moves radially A = 0, (C.4) gives

$$(dr/cd\tau)^2 = (r_s/r) - 1 + B^2 \qquad (C.5).$$

When the particle initially is very distant, $r >> r_s$ and at rest, from (C.5), we verify that B = 1.

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