# Gravitational Waves from Micro Black Hole Binaries and Quantum Mechanics

**M.Cattani**
**Instituto de Física, Universidade of São Paulo, Brasil**
**(mcattani@if.usp.br)**

## Abstract.

This paper was written to graduate and postgraduate students of Physics. We study the emission of gravitational waves by binaries composed by micro non-charged black holes (mBH). It is assumed that the mBHb dynamics obeys General Relativity and that its inspiral motion can also be described by a quantum approach given by the Schrödinger-Newton equation, for large quantum numbers.

***Key words.*** micro black hole binary; gravitational quantum effects; gravitational waves.

## (I) Introduction.

This is a paper written to graduate and postgraduate students of physics. Our intention is to investigate only basic aspects about emission of gravitational waves (**GW**) by binaries composed by two non-charged micro black holes (**mBHb**). We use classical mechanics[1], classical electrodynamics,[2] quantum mechanics (**QM**),[3,4] special relativity **(SR)** and general relativity(**GR**).[5] In Section 1 are given significant parameters associated with micro black holes (**mBH**). In Section 2 with the GR are estimated the gravitational luminosity $L_{GW}$ and the "spiral time" $\tau$ of a mBBb. In Section 3,we suppose that the mBBH is a *microscopic* system that obeys a Schrödinger-Newton equation. So, in this context, we show to how calculate the gravitational energy per unit of time dE/dt emitted by the mBHb using an "hybrid" GR and QM approach. It is also shown that the dE/dt and the "spiral time" $\tau$ of the mBHb calculated with the hybrid approach is in good agreement with the $L_{GW}$ and $\tau$ estimated with the GR theory. In Section 4 are presented conclusions and discussions of our analysis. In Appendix A is shown how to calculate the emission of gravitational waves emitted by a BHb. In Appendix B and C is briefly shown how to calculate the electromagnetic radiation in Classical and Quantum Electrodynamics. Finally, in Appendix D are done comments on a possible Gravitation Quantum Field Theory .

## (1) Significant Parameters associated with mBH.

In Figure 1 is shown a binary (mBHb) composed by two non-charged micro black holes (mBH).[5,7]

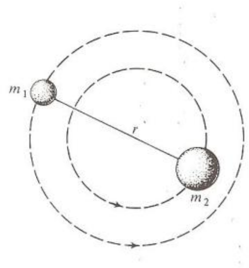


**Figure 1**. Binary system (mBHb) formed by non-charged mini black holes(mBH).

The BH mass M according to the classical GR[5] can be arbitrarily small, however, the smallest M is estimated by Planck mass[8] $M_P = (\hbar c/G)^{1/2}$. The BH Schwarzschild radius[9] $r_s$ and its lifetime $\tau_H$ [10], due to the Hawking radiation, are estimated, by $r_s = 2GM/c^2$ and $\tau_H = 5120\pi G^2M^3/(\hbar c^4)$, respectively. Here the Planck mass $M_P$, the radius $r_s$, the lifetime $\tau_H$, the "gravitational Bohr radius" $(a_o)_g$, the metric tensor component $g_{oo}(r)$ and a Plank length $\ell_P = \hbar/cM_P$ are written in terms of the constants c, G and ħ, in the MKS system,

$$M_P = (\hbar c/G)^{1/2} \sim 2\ 10^{-8}\ \ (Kg) \qquad (1.1),$$

$$r_s = 2GM/c^2 \sim 1.5\ 10^{-27}\ M \quad (m) \qquad (1.2),$$

$$\tau_H = 5120\pi G^2M^3/(\hbar c^4) \sim 4\ 10^{-18}\ M^3\ \ (s) \qquad (1.3),$$

$$(a_o)_g = \hbar^2/GM^3 \sim\ 10^{-58}/M^3 \quad (m) \qquad (1.4),$$

$$g_{oo}(r) = -1- 2GM/rc^2 \qquad (1.5),$$

and

$$\ell_P = \hbar/cM_P \sim 1.616\ 10^{-35}\ m \qquad (1.6).$$

## (2)Gravitational mBHb luminosity according to the "classical"GR.

Gravitational waves emitted by a black hole binary (BHb) formed by BH with total mass $M_+ = M_1 + M_2 \sim 20 - 30$ solar masses have been recently detected by Abbott et al.[11,12] The BHb motion is unstable; this unstable motion can be divided into three stages:[11-13] "inspiral", "merger" (or "plunge") and "ringdown". During this motion the BHb emits GW. The "inspiral" is the first stage of the BHb life which resembles a gradually shrinking orbit and take a longer time; the emitted GW are weak when BH are distant from each other. During the "inspiral" motion of a BH binary with $M_1= M_2 = M$ the gravitational luminosity $L_{GW}$ would be given by [5,13-16] (see **Appendix A**)

$$L_{GW} = dE/dt = - (8G/5c^5)\, M^2r^4\omega^6 \qquad (2.1),$$

where r is distance between the BH and ω is the **orbital rotational frequency**. With Kepler´s law[1,5] $r(t)^3\omega(t)^2 = GM_+$ the luminosity given by Eq.(2.1) becomes,

$$|L_{GW}(\omega)| = (8G/5c^5)\, M^2 r^4 \omega^6 \sim 10^{-192/3}\, (M\omega)^{10/3} \qquad (2.2),$$

or

$$|L_{GW}(r)| = -\, (8G/5c^5)\, M^2 r^4 \omega^6 \sim 10^{-84}\, (M/r)^5 \qquad (2.3).$$

In addition, as $r_s = 2GM/c^2 \sim 1.5\ 10^{-27}$ M we get

$$\omega_{max} \sim 10^{26} M^{1/2} \qquad (2.4).$$

The "**spiral time**" $\tau$ [5,16] of the BBH is estimated writing the total mechanical energy E of the BBH as $E = I\omega^2/2 - GM^2/r$ that can be written, using the "virial" theorem,[1] as $E = -\, GM^2/2r$. Taking this equation and Eq.(2.1) we verify that[5]

$$dr/dt = -\, (128/5c^5)\, G^3M^3/r^3 \qquad \text{that is,}$$

$$r^3\, dr/dt = (1/4) d(r^4)/dt = -(128/5c^5)\, G^3M^3 \qquad (2.6).$$

Integrating Eq.(2.6) from $r_o$ up to $2r_s$ we get

$$r_o^4 = (2r_s)^4 - (128/5c^5)\, G^3M^3\, \tau \qquad (2.7),$$

where $\tau$, that is also called **"time to fall"** from a generic orbit $r = r_o$ to the closest distance $2r_s$ between two BH, is given by :

$$\tau = [5c^5/(128\, G^3M^3)]\, (r_o^4 - 16r_s^4) \qquad (2.8).$$

In this paper we will suppose that the lifetime of the mBH is $\tau_H \sim 60$ s. So, to satisfy this condition we see, using Eq.(1.3), that the mBH mass must be **M ~ $10^6$ kg**. For this mass, using Eqs.(1.1)-(1.4) the Schwarzschild radius $r_s \sim 1.5\ 10^{-27}\, M \sim 10^{-21}$ m. For these masses the mBHb has *microscopic* dimensions...

**(2.1) Estimations of $L_{GW}$ and $\tau$ for M = $10^6$ kg.**

Kepler´s law, in non relativistic classical mechanics, for a binary is given by $\omega^2 r^3 = 2MG$, where $M_1 = M_2 = M$, establishes a constraint between $\omega(t)$ and $r(t)$. The maximum values of $\omega(t)$ occurs for the minimum value of $r(t)$ and vice-versa. So, putting $M = 10^6$ kg in Eq.(1.2) and Eqs.(2.2)-(2.4) we get $r_s \sim 10^{-21}$ m, $\omega_{max} \sim 10^{29}$ rad/s and the maximum luminosity

$$|L_{GW}|_{max} = |L_{GW}(\omega_{max})| = |L_{GW}(r_s)| \sim 10^{41}\ \text{J/s} = 10^{41}\text{W} \qquad (2.9).$$

The time $\tau$ to fall from $r_o \sim 100\, r_s$ m up to $2r_s \sim 10^{-21}$ m given by Eq.(2.8) is

$$\tau = [5c^5/(128\, G^3M^3)]\, (r_o^4 - 16r_s^4) \sim 3\ 10^{53}\ 10^{-76} \sim 10^{-17}\ \text{s} \qquad (2.10),$$

that is, the gravitational energy would be "instantaneously" emitted , like a "flash".

In recent GW observations[11,12] known as GW150914 and GW151226 the BBH were composed by BH with masses $M \sim 10^{30}$ kg. The measured GW frequencies are in the range 30-500 Hz, the peaked luminosities $L_{GW} \sim 10^{49}$ W and *spiral times* $\tau \sim 1$ s.

## (3)mBHb described by Schrödinger-Newton Equation.

According to Section 2, depending on the BH masses, mBHb systems can have microscopic dimensions. In this way, let us suppose that they can be taken as *small* systems in Dirac´s[6] sense and so, could be described in the *spiral stage*, for very large quantum numbers, when GR gravitational effects are small, by a Schrödinger-Newton equation[17]

$$H = \{(\hbar^2/2\mu)\Delta - GM^2/r\}\Psi(r,\theta,\varphi) = E\Psi(r,\theta,\varphi) \qquad (3.1),$$

taking into account that the BH masses are concentrated in a very small region of the space. In Eq.(3.1), r is distance between the mBH, $\Delta$ the Laplacian operador in spherical coordinates and $\mu = M_1M_2/(M_1+M_2) = M/2$ is the reduced mass of the system. In semi-classical limit [3,4], that is, for large quantum numbers, we will suppose that Eq.(3.1) can give a good description of the mBHb orbits obtained by the GR. Thus, solving Eq.(3.1)[3,4] the *gravitational energies* $E^g_n$ of the mBHb are given by

$$E^g_n = - \Theta_{grav}/n^2, \qquad (3.2),$$

where n = 1,2,3,...and $\Theta_{grav} = (M/2)(GM^2)^2/2\hbar^2 = G^2M^5/4\hbar^2$. Since $G \sim 10^{-10}$ MKS and $\hbar \sim 10^{-34}$ MKS we have

$$\Theta_{grav} = G^2M^5/4\hbar^2 \sim 10^{47} M^5 \quad J \qquad (3.3).$$

For the hydrogen-like-atom (**HLA**) with charge Z we have,[3,4]

$$E^{elet}_n = - \Theta_{eletr}/n^2 \qquad (3.4),$$

where $\Theta_{eletr} = Z^2m_ee^4/2\hbar^2$ . That is,[3,4]

$$\Theta_{eletr} = Z^2\ 13.6\ eV \sim Z^2\ 10^{-18} \quad J \qquad (3.5).$$

and the normalized energy eigenfunctions $u_{n\ell m}(r,\theta,\varphi)$ given by

$$u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\ |\ell m> \qquad (3.6),$$

where $R_{n\ell}(r)$ and $|\ell m> = Y_{\ell m}(\theta,\varphi)$ are shown in references,[3,4] remembering that $n = 1,2,...$, $\ell = 0,1,2,..,n-1$ and $m = -\ell, -\ell+1, ..., \ell-1, \ell$.

For the HLA the "**electromagnetic Bohr radius**" $(a_o)_{elet}$ is given by[3,4]

$$(a_o)_{elet} = \hbar^2/me^2 \sim 0.5\ 10^{-10}\ m \qquad (3.7)$$

Similarly, for the mBHb the "**gravitational Bohr radius**" is given by,

$$(a_o)_g = \hbar^2/G^2M^3 \qquad (3.8).$$

From Eqs.(3.2) - (3.8) we verify that the energies $E^g_n = E^{elet}_n$ if $M \sim 10^{-13}$ kg; in this case the mBHb would be *small* in Dirac´s sense. The orbit radius $r_n$ are given by $(r_n)_{elet} = n^2(a_o)_{Bohr} = n^2 (\hbar^2/me^2) \sim n^2\ 0.5\ 10^{-10}$ m and $(r_n)_g = n^2(a_o)_g = n^2(\hbar^2/G^2M^3)$. Let us remember that for the HLA the Kepler´s law is written as $\omega(t)^2r(t)^3 = Ze^2/\mu$ .

Since $v = \omega r$, the orbital relativistic parameter $\beta = (v/c)$ for the mBHb will be given $\beta = (1/r)^{1/2}(2GM/c^2)^{1/2}$. For HLA the fundamental state n = 1 is stable.[3,4] We suppose that this also occurs in the gravitational case. So, gravitational waves (GW) would be emitted in "spontaneous" decay transitions between the quantum states $u_{n\ell m}(r,\theta,\varphi) \rightarrow u_{n'\ell'm'}(r,\theta,\varphi)$ for $n \rightarrow n-1$, when $n > 1$. At this point we could ask**:"what kind of interaction field would be responsible for these transitions?"** This question will be answered in Section(3.3).

**(3.1)mBHb Stability.**

The HLA ground state $u_{n\ell m}(r,\theta,\varphi)$ (n = 1) is stable.[3,4] In this state the atomic radius $r \sim 10^{-10}$ m is much larger the nuclear radius $\sim 10^{-15}$ m. That is, the electron can be thought as moving in a orbit very far from nucleus. Supposing that this condition is essential to the stability of the HLA we "take for granted" that mBHb ground state cannot be stable if inside the sphere with radius $(a_o)_g = \hbar^2/G^2M^3$ there is "contact" between the mBH, which one with radius $r_s$. Let us suppose that mBHb system is unstable if, for instance, $4r_s > (a_o)_g$. Using (3.1) and (3.4) this condition is written as $8GM/c^2 > \hbar^2/G^2M^3$. Thus, $M^4 > (\hbar^2/G^3c^2)/8$, that is, $M > 0.5\ (\hbar/c)G^{-3/2} \sim 10^{-27}$ kg. So, we see that the mBHb would be **unstable** if

$$M > 10^{-14}\,\text{kg} \qquad (3.7).$$

Thus, for mBH masses $M > 10^{-14}$ kg our mBHb would be **unstable**. In these conditions, the mBHb **unstable motion** can be divided into three stages:[11-14] "inspiral", "merger" (or "plunge") and "ringdown". During this motion the system emits GW. The "inspiral" is the first stage of the mBBH life which resembles a gradually shrinking orbit and take a longer time; the emitted GW are weak when the mBH are distant from each other, that is, $r >> r_s$. As the mBHb orbit shrinks, the speeds of the mBH increase, and the GW emission increases. When the mBH are close ($r \sim r_s$) the GW cause the orbit to shrink rapidly. In the final fraction of a second the mBH can reach extremely high velocities. This is followed by a plunging orbit and the mBH will "merge" once they are close enough, that is, $r \leq r_s$. At this instant the GW amplitude reaches its peak. Once merged, the single hole settles down to a stable form, via a stage called "ringdown", where any distortion in the shape is dissipated as more gravitational waves.

**(3.2) Inspiral motion.**

For $M = 10^6$ kg, by Eq.(1.2) the Schwarzschild radius $r_s \sim 1.5\ 10^{-27} M \sim 10^{-21}$ m, $(a_o)_g = \hbar^2/G^2M^3 \sim 10^{-66}$ m and the binary "quantum radius" would be $(r)_g = n^2\ 10^{-66}$ m. The energies $E_n$ (see Eqs.(3.2) - (3.3)) are given by $E^g{}_n = -10^{77}/n^2$ J $\sim -10^{96}/n^2$ eV. As $r_s \sim 10^{-21}$ m the mBH would be distant when $(r)_g > 10^{-21}$ m, that is, only when $n > 10^{22}$. For $r \geq 10^{-20}$ m the binary is still a ***microscopic system***, about $10^7$ times smaller than the hydrogen-like atom(HLA). For $r \geq 10^{-20}$ m we get, using Eq.(1.5), that $g_{oo}(r) \sim -1$ showing that gravitational distortions of the metric are negligible.[5] If $r_s \sim 10^{-21}$ m and the mBBH radius $r = r_n = n^2 10^{-66}$ m we see that $r/r_s \sim 3\ 10^{27}/n^2$. For $n > 10^{22}$ we verify that $r/r_s < 1$ and relativistic effects are negligible. So, we can say that the inspiral motion is restricted to distances $r > r_s$, that is, for $n > 10^{22}$. Higher energy GW would be generated by transitions for distances $(r)_g \sim r_s$. Let us suppose that the inspiral motion occurs for n values in the range $n \sim 10^{21}$ -$10^{24}$. For these large n values we see that energies $\hbar\omega$ in the transitions $n \rightarrow n + 1$ are given by

$$\hbar\omega = E^g{}_{n+1} - E^g{}_n = -10^{77}[1/(n+1)^2 - 1/n^2] \approx 10^{77}/n^4\ \text{J} \approx 10^{96}/n^4\ \text{eV}. \qquad (3.8).$$

So, in the **inspiral** region, for $n \sim 10^{22}$ we have frequencies $\omega \sim 10^{24}$ rad/s. Note that the recently observed GW frequencies[11,12] are $\omega \sim 150\pi - 170\pi$ rad/s.

**(3.3) mBHb gravitational luminosity emitted in Schrödinger approach.**

Let us estimate the gravitational luminosity emitted in the inspiral motion by the mBHH. Let us consider GW with gravitational energies $\hbar\omega = E^g{}_{n+1} - E^g{}_n$ given by Eq. (3.8), emitted in transitions $n \to n + 1$. To do this we suppose (*without proving*) that there is *some kind* of interaction (*what kind?)* that induces transitions between the quantum states $|\, n >$. It will be done using the perturbation theory derived from Schrödinger´s equation. Let us represent by $W(t)$ this interaction harmonically depend on the time[4]

$$W^{\pm}(t) = w^{\pm} \exp[\pm i\omega t] \qquad (3.9),$$

where $w^{\pm}$ is time independent. It can be shown[4] that the transition probability $m \to n$ per unit of time $P^{\pm}{}_{nm}$ is written as

$$P^{\pm}{}_{nm} = (2\pi/\hbar)\, |< n \,|\, w^{\pm} \,|\, m >|^2\, \delta(E_n - E_m \pm \hbar\omega) \qquad (3.10),$$

where the + and - correspond to the signs in the exponential in Eq.(3.9). Thus, under this perturbation, transitions take place to states with energies satisfying the condition $E_m = E_n \pm \hbar\omega$. If the perturbation is of the form $W^+(t) = w^+ \exp(i\omega t)$ the system loses an energy $\hbar\omega$ (energy is emitted), since $E_n = E_m - \hbar\omega$ in the transition, while if it is of the form $W^-(t) = w^- \exp(-i\omega t)$ it gains an energy $\hbar\omega$, since $E_n = E_m + \hbar\omega$. Our main problem is to determine the function $W^{\pm}(t)$. The gravitational "luminosity" $(L_{GW})_{nm}$ in the *inspiral* stage would estimated by $(L_{GW})_{nm} = \hbar\omega\, P^+{}_{nm}$ for very large quantum numbers.

Before to propose a model to obtain $W^+(t)$ let us remember that according to Bohr correspondence principle (CP)[3] for very large quantum numbers, classical and quantum physics are expected to give the same answer, at least in average. The probabilistic interpretation of the phenomenon obtained with the Schrödinger´s equation will give, in average the same results obtained by classical laws. Ehrenfest, for instance, showed that Newton's laws hold on average: the quantum statistical expectation value of the position and momentum obey Newton's laws. Thus, we expect that in the inspiral stage mBHb properties estimations given by the "classical" GR and QM laws agree *in average*. In addition, as seen in **Appendix B** and **C,** in Classical Electrodynamics the luminosities $L_\omega$ , emitted by **dipolar** and **quadrupolar** radiation are given, respectively, by

$$L_\omega = dE/dt = (ck^4/3)\, |\mathbf{D}|^2 = (\omega^4/3c^3)\, |\mathbf{D}|^2 \qquad \text{and}$$
$$L_\omega = dE/dt = (\omega^6/360c^5)\textstyle\sum_{\alpha\beta} |Q_{\alpha\beta}|^2 .$$

In Quantum Electrodynamics these are given, respectively, by $L_\omega = (4\omega^4/3c^3)\, |\boldsymbol{D}_{nm}|^2$ and $L_\omega \approx (\omega^6/2\pi c^5)|Q_{nm}|^2$ , where $\omega = \omega_{nm}$ , $\boldsymbol{D}_{nm} = < n\, |\mathbf{D}\, |m >$ and $Q_{nm} = < n \,|\, Q \,|\, m >$.

Finally, according to the "classical" GR estimations, the luminosity, in the inspiral stage, $L_{GW}$ is given by the **quadrupolar radiation**[11-14] according to Eq.(2.1):

$$L_{GW} = (32\mu^2 G/5c^5) r^4\omega^6 = (8G\omega^6/5c^5)\, M^2 r^4 = (8G\omega^6/5c^5) Q^2 \qquad (3.11),$$

where $Q = Mr^2$ is the mass quadrupole of the mBHb. Thus, by analogy with the predicted electromagnetic radiation and based in the CP we admit that the QM gravitational luminosity $(L_{GW})_{nm}$ can be estimated by

$$(L_{GW})_{nm} = \hbar\omega P^{+}_{nm} \approx (8G\omega^6/5c^5) \,|< n \mid Q \mid m >|^2 \qquad (3.12).$$

In **Appendix D** is shown a different approach of Weinberg[15] to calculate $(L_{GW})_{nm}$ .

Now**,** let us give a reasonable justification for Eq.(3.12). Thus, let us suppose that $W^+$ (t) is proportional to the small perturbations $h_{\mu\upsilon}$ of the tensor metric $g_{\mu\upsilon}$ created by the quadrupole temporal oscillations $Q_{\alpha\beta}(t)$[16,19] of the mBBH that are written as

$$Q_{xx}(t) = 3\mu r^2[1 + \cos(2\omega t)]/2 \quad \text{and} \quad Q_{yy}(t) = 3\mu r^2[1- \cos(2\omega t)]/2 \qquad (3.13).$$

where $\mu = m_1m_2/(m_1 + m_2)$ and $\omega$ is the orbital angular frequency (see **Appendix A).** That is, $g_{\mu\nu}$ is slightly modified, $g_{\mu\upsilon} \approx g_{\mu\upsilon}^{(o)} + h_{\mu\upsilon}$, where $h_{\mu\upsilon}$ is due to quadrupolar effects pointed above. Taking into account that [14,19] $h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2r)(\partial^2Q_{\alpha\beta}/\partial t^2)$ the "classical" gravitational luminosity $L_{GW}$ is given by (see **Appendix A**)

$$L_{GW} = (G/45c^5) < (\partial^3Q_{\alpha\beta}/\partial t^3)^2 > = (G/45c^5) [< (\partial^3Q_{xx}/\partial t^3)^2 > + [< (\partial^3Q_{yy}/\partial t^3)^2 > ] =$$

$$= (32\mu^2G/5c^5)r^4\omega^6 = (8G\omega^6/5c^5)Q^2 \qquad (3.14),$$

where $Q = Mr^2$ is the mBBH mass quadrupole. So, admitting that $(L_{GW})_{nm} = \hbar\omega P^{+}_{nm}$ , $w^+(t) \sim h_{\alpha\beta}(t)$ and using Eq.(3.10) we will assume that the QM the *gravitational luminosity* $(L_{GW})_{nm}$ can be estimated by

$$(L_{GW})_{nm} = \hbar\omega P^{+}_{nm} \approx (8G\omega^6/5c^5) \,|< n \mid Q \mid m >|^2 \qquad (3.15),$$

in agreement with Eq.(3.12). At this point it is important to analyze this proposed mechanism to explain the decay transitions in mBBH. Indeed, as seen in **Appendix A**, the amplitude of the emitted GW are given by $\Psi_{\alpha\beta}(t,\mathbf{x}) = h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2R)(\partial^2Q_{\alpha\beta}/\partial t^2)$. That is, GW are emitted due to the "*metric perturbation*" $h_{\alpha\beta}(t)$. To obtain Eq.(3.15) a similar hypothesis is assumed: the time dependent metric modification is responsible for a potential interaction $W^+$ that induces transitions $n \rightarrow m$ between quantum states. The gravitational luminosity would now be given by $(L_{GW})_{nm} = \hbar\omega P^{+}_{nm}$. That is, gravitational quantum transitions are induced by metric perturbations due to mass quadrupolar effects. In the electromagnetic quantum field theory transitions are induced by "vacuum" fluctuations due to electric quadrupoles.

**(3.4)Estimation of the quantum luminosity $(L_{GW})_{nm}$.**

Let us compare the $L_{GW}$ emitted in the **inspiral stage** given by Eq.(2.1), using the "classical" GR, with our hybrid GR&QM approach given by Eq.(3.12). So, putting in Eq.(3.12) $M = 10^6$ kg and taking $| n > \rightarrow | m > = | n +1 >$, $\omega = \omega_{nm} = (E_n - E_m)/\hbar$ and

$$|< n \mid Q \mid m >|^2 \sim [2M < n |r^2| m > ]^2 = 4 M^2 |< n |r^2| n +1 >|^2 = 4 M^2 \,|(r^2)_{n,n+1}|^2$$

we have

$$(L_{GW})_{nm} \sim 10^{-41} \omega_{n,n+1}{}^6 \,|(r^2)_{n,n+1}|^2 \qquad (3.16).$$

As in the inspiral stage, following Eq.(3.8), $\hbar\omega = \hbar\omega_{n,n+1} = E^g{}_{n+1} - E^g{}_n \approx 10^{77}/n^4$ J, the most significant contributions to the luminosity occurs when n is the range $n \sim 10^{21}$-$10^{23}$

with frequencies in the range $\omega \sim 10^{29}$ - $10^{19}$ rad/s.

Taking, e.g., $\omega \sim 5\ 10^{27}$ rad/s and $|r_{n,n+1}| \sim 10^{-20}$m the gravitational luminosity $\mathbf{(L_{GW})_{nm}}$ estimated with the **QM approach**, using Eq.(3.16), is given by

$$\mathbf{(L_{GW})_{nm} \sim 10^{41}\ W},$$

showing a fair agreement with the luminosity $\mathbf{|L_{GW}|_{max} \sim 10^{41}\ W}$ calculated with the **GR** theory using Eq.(2.9). This agreement is not at all surprising because according to Bohr correspondence principle (CP)[3] for very large quantum numbers, e.g. n >> 1, classical and quantum physics are expected to give the same answer, at least in average.

**(3.5)Evaluation of the spiral time.**

To evaluate the QM "spiral time" $\tau$ we must remember that in this stage, according to Eqs.(3.2) and (3.3) the energy levels $E^g{}_n = - \Theta_{grav}/n^2$ are very close since quantum numbers are very large,e.g. $n > 10^{24}$. As there is a "continuum of levels" it is expected, according to the CP, the mBBH description given by quantum mechanics approaches asymptotically a state of motion obtained with the "classical" GR. Indeed, for the inspiral stage Eq.(3.11) can be written as

$$(L_{GW})_{ab} = (dE/dt)_{ab} \approx (8G\omega^6/5c^5)\ M^2r^4 = (8\ M^2G\omega^6/5c^5)\ r^4 \qquad (3.17).$$

which is similar to Eq.(2.1) given by the "classical" GR. Integrating Eq.(3.17) as was done in Section 2 we get for the spiral time $\tau$ the same result predicted by Eq.(2.8).

# (4)Conclusions and Discussions.

**(4.1)**A good agreement between the estimated luminosity and inspiral time is obtained with the GR and the quantum approach. So, it seems reasonable that in the mBHb spiral motion the effects of the gravitation interaction can be quantized in a non relativistic limit of Schrödinger - Newton equation.

**(4.2)** As, in the inspiral motion, according to Appendix C, quantum states |a > and |b > of the mBBH are represented by $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\ |\ell m>$ the quadrupole matrix elements are written as

$$Q_{ab} = \int dr\ r^4\ R_a(r)\ R_b(r) < \ell_b m_b|Y_{2m}{}^*(\theta,\varphi)|\ \ell_a m_a > \qquad (4.2.1).$$

Eq.(4.2.1) shows that, according to the Wigner-Eckart Theorem,[4] quadrupole **t**ransitions $a \rightarrow b$ are allowed only if $\ell_b = \ell_a \pm 2$ and $m_b = m_a + 2$ . So, if GW are composed by "gravitons", selection rules dictated by the matrix elements in Eq.(4.2.1) suggest that "gravitons" have spin 2.

**(4.3)**According to **Appendix (A.1)** the gravitational luminosity $L_{GW}$ emitted by a BH binary with black holes with equal mass M, is given by $L_{GW} = (32M^2G/5c^5)r^4\omega^6$ . In the radiation zone the gravitational energy is transported by a **plane wave** with amplitude $h(\omega)$ given by Eq.(A.17)[13,14]

$$h(\omega) = (4^{2/3}/\sqrt{36})\ [(GM)^{5/3}/Rc^4]\ \omega^{2/3} \qquad (4.3.1)$$

where R is distance from the BHb and the observer at the radiation zone.

Gravitational waves have been detected,[11-13] from black hole binaries (BHb) distant $R \sim 1.3\ 10^9$ light years from the Earth and with $M \sim 20$ solar masses. Using Eq.(4.3.1) and the BHb parameters given above we verify that

$$h(\omega) \sim 5.6\ 10^{-52}\ M^{5/3}\ \omega^{2/3} \sim 10^{-23}\ \omega^{2/3} \qquad (4.3.2).$$

The average measured amplitude[11-13] $< h >$ for frequencies $\omega \sim 160\ \pi$ rad/s was found to be $< h > \sim 10^{-21}$, in good agreement with $h(\omega)$ predicted by Eq.(4.3.2).

**(4.4)** Now, let us consider a mBHb and a BHb both distant $R \sim 1.3\ 10^9$ light-years $\sim 1.2\ 10^{25}$ m from the Earth. The mBH with mass m and the BH with mass M. Using Eq.(4.3.1) we have, respectively,

$$h_m(\omega) \sim 5.6\ 10^{-52}\ m^{5/3}\ \omega^{2/3} \qquad (4.4.1)$$

and

$$h_M(W) \sim 5.6\ 10^{-52}\ M^{5/3}\ W^{2/3} \qquad (4.4.2).$$

From Eqs.(4.4.1) and (4.4.2) results,

$$h_m(\omega)/\ h_M(W) = (m/M)^{5/3}\ (\omega/W)^{\ 2/3} \qquad (4.4.3).$$

According to Kepler´s law, the highest frequencies $\omega_{max}$ and $W_{max}$ are given by $\omega_{max}{}^2 = 2Gm/r_s{}^3$ and $W_{max}{}^2 = 2GM/r'_s{}^3$, where $r_s = 2Gm/c^2$ and $r'_s = 2GM/c^2$ are the Schwarzschild radius of the masses m and M, respectively. From these relations obtain

$$\omega_{max}/\ W_{max} = M/m \qquad (4.4.4),$$

showing that $\omega_{max} >> W_{max}$ if $M >> m$. That is, frequencies emitted by mBHb can be much higher than those emitted by BHb. With Eqs.(4.4.3) and (4.4.4) we verify that

$$h_M(W_{max}) = (M/m)^{5/3}\ h_m(\omega_{max}) \qquad (4.4.3),$$

showing, on the other hand, that $h_M(W_{max}) >> h_m(\omega_{max})$ when $M >> m$.

This implies that energies emitted by the BHb can be much higher than that emitted by mBHb. Considering the now days detection techniques[11,12] it seems to be easier to detect GW from BHb than those emitted by mBHb.

***Acknowledgements***. The author thanks the librarian Virginia de Paiva for his invaluable assistance in the pursuit of various texts used as references in this article.

## Appendix A. Emission of gravitational waves by BBH.

In GR [5,14-16], assuming that the gravitation field is *weak* and that the bodies have small velocities compared with the light velocity, the space-time metric tensor $g_{\mu\upsilon}$ we can put $g_{\mu\upsilon} \approx g_{\mu\upsilon}{}^{(o)} + h_{\mu\upsilon}$, where $h_{\mu\upsilon}$ is as mall perturbation of $g_{\mu\upsilon}{}^{(o)}$.[5,14-16] In the Newtonian limit we have $g_{oo} = -1 - 2\varphi/c^2$, where $\varphi = GM/r$.[5] In these conditions the Ricci tensor $R_{ik}$ can be written as

$$R_{ik} = -(1/2)\Box h_{\mu\upsilon} \qquad (A.1).$$

Defining the gravitational field as $\Psi_{\mu\upsilon} = h_{\mu\upsilon} - (1/2)\delta_{\mu\upsilon} h$, where $h = h_{\alpha}{}^{\alpha}$, in *weak field* limit the field $\Psi_{\mu\upsilon}$ obeys the equations[5, 14-16]

$$\Box\Psi_{\mu\upsilon} = - (16\pi G/c^4)\tau_{\mu\upsilon} \quad \text{and} \quad \partial_{\mu}\Psi^{\mu\upsilon} = 0 \quad (\textit{gauge condition}) \qquad \text{(A.2)},$$

where $\tau_{\mu\upsilon}$ is a pseudo-tensor mass-energy momentum.

The solution of (A.2) for retarded times is given by[5,18]

$$\Psi_{\mu\upsilon}(\mathbf{x},t) = - (4G/c^4) \int\tau_{\mu\upsilon}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x})\, d^3\mathbf{x}'/ |\mathbf{x} - \mathbf{x}'| \qquad \text{(A.3)},$$

where the integration is over the volume V of the system.

Supposing that gravitational effects are observed very far from the origin O ("wave zone") where they are produced, that is, $|\mathbf{x}| = R >> |\mathbf{x}'|$ we get from (A.3), remembering that we have a retarded time function $\tau_{\mu\upsilon}$:

$$\Psi_{\mu\upsilon}(x,t) \approx - (4G/c^4R) \int\tau_{\mu\upsilon}\, d^3\mathbf{x}' \qquad \text{(A.4)}.$$

Integrating Eq.(A.4) over the volume V we obtain the gravitational field[5,13]

$$\Psi_{\alpha\beta}(x,t) = (2G/c^2R)\, (\partial^2 Q_{\alpha\beta}/\partial t^2) \qquad \text{(A.5)},$$

where $Q_{\alpha\beta}$ is the *mass quadrupole moment* of the emitting system defined by

$$Q_{\alpha\beta} = \int\rho_o(\mathbf{x}')(3x'_{\alpha}x'_{\beta} - r'^2\delta_{\alpha\beta})\, d^3\mathbf{x}'$$

where $\rho_o$ is the mass density. At this point it opportune to remember that gravitational multipoles are defined by the potential expansion [14]

$$\varphi(\mathbf{x}) = -G \int\rho_o(\mathbf{x}')d^3\mathbf{x}'/|\mathbf{x} - \mathbf{x}'| \approx - Gm/r - (G/r^3)\, \mathbf{x}.\mathbf{D} - (G/2r^5)\Sigma_{\alpha\beta}\, Q^{\alpha\beta}\, x^{\alpha}x^{\beta} + .... \qquad \text{(A.6)},$$

where $m = \int\rho_o(\mathbf{x}')\, d^3\mathbf{x}'$, $\mathbf{D} = \int\rho_o(\mathbf{x}')\, \mathbf{x}' d^3\mathbf{x}'$ and $Q_{\alpha\beta} = \int\rho_o(\mathbf{x}')(3x'_{\alpha}x'_{\beta} - r'^2\delta_{\alpha\beta})\, d^3\mathbf{x}'$.

The *mass dipole moment* is null ($\mathbf{D} = 0$) since the origin of coordinates O is chosen to coincide with the center of mass.

In vacuum we have the *traditional wave equations*

$$\Box\Psi_{\mu\upsilon} = \Box h_{\mu\upsilon} = 0 \qquad \text{with the "gauge "} \qquad \partial(h^{\mu}{}_{\nu})/\partial x^{\mu} = 0 \qquad \text{(A.7)}$$

showing that the gravitational field propagates with the light velocity. Note that the tensor field $h_{\mu\upsilon}$ is obtained integrating Eq.(A.4) as will be seen later.

At this point we find a fruitful analogy with the electromagnetism. The Maxwell equations in *Lorentz gauge* in empty space are $\Box A_{\mu} = 0$ and $\partial A^{\mu}/\partial x^{\mu} = 0$.

Let us consider a plane GW, that is, a field that changes only in one direction z of the space. Choosing $z > 0$ as the direction of propagation of the wave we can write $h_{ik} = h_{ik}(t - z/c)$. So, the wave equation Eq.(A.7) becomes

$$[\partial^2/\partial z^2 - (1/c^2)\, (\partial^2/\partial t^2)]\, h_{ik} = 0 \qquad \text{(A.8)}$$

that has the familiar solution with the gauge condition,

$$h_{ik}(z,t) = A_{ik} \cos(k_\mu x_\mu) \quad \text{(A.9)},$$

where $k_\mu = (0,0,k,\omega)$, $k = k_z = |\mathbf{k}| = \omega/c$ is the wave vector and $\omega$ is the frequency of the wave. As $h_{ik}(z)$ obey (A.8) the following conditions are obeyed: $A_{\beta\alpha}k^\alpha = 0$ and $k_\alpha k^\alpha = 0$. Under these conditions the **amplitude tensor $A_{ik}$** has only 4 non-null components $A_{11} = - A_{22}$, $A_{12} = A_{21}$ with the condition $Tr(A_{ik}) = A_i^{\,i} = 0$ and only the following **transversal components** to the z-direction of propagation: $A_{xx} = -A_{yy}$ and $A_{xy} = A_{yx}$.

$$A_{ik} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The transversal fields $h_{xx}$, $h_{yy}$ and $h_{xy}$ are represented using (2x2) matrices called polarization matrices $(\varepsilon_+)_{ik}$ and $(\varepsilon_x)_{ik}$ :

$$(\varepsilon_+)_{ik} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad (\varepsilon_x)_{ik} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{(A.10)}$$

The general solution of Eq.(A.8) can be written as a linear combination of the fields $h_{ik}$, with polarizations (+) and (x), respectively:

$$h_{ik}^{(+)} = h_+ (\varepsilon_+)_{ik} \cos(\omega t - kz) \quad \text{and} \quad h_{ik}^{(x)} = h_x (\varepsilon_+)_{ik} \cos(\omega t - kz + \alpha) \quad \text{(A.11)},$$

where $h_+ = A_{11}$, $h_x = A_{12}$ and $\alpha$ is an arbitrary phase. The tensorial polarization of the GW creates an effect much more complicate than the linear polarization of the electromagnetic waves. These fields deform the space-time creating tidal (shear) on the matter . The line forces due to the polarizations (X) and (+) are shown in Figure 2.

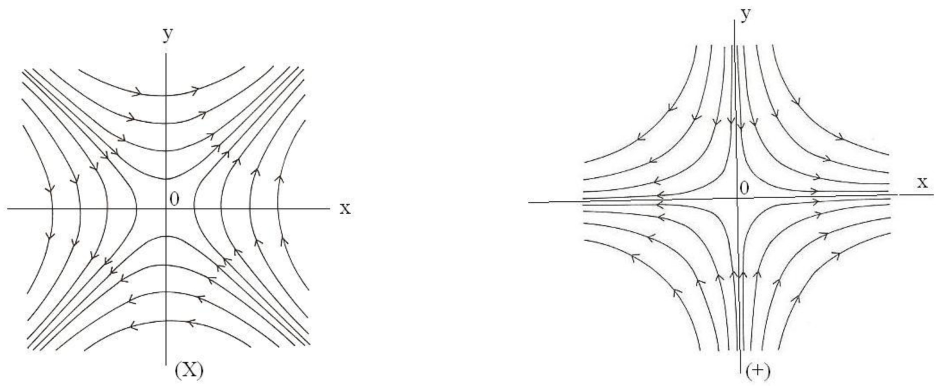


**Figure 2**. Line forces due to the polarizations (X) and (+).

The total energy emitted per unit of time dE/dt or "gravitational luminosity" $L_{GW}$ is given by[5,14]

$$L_{GW} = dE/dt = - (G/45c^5) < (\partial^3 Q_{\alpha\beta}/\partial t^3)^2 > \quad \text{(A.12)},$$

where the brackets indicates a time average and are taken into account the effect of all components of the quadrupole tensor. Note that the GW is a tensor function not a scalar function like an electromagnetic wave.

**(A.1)GW emitted by BBH.**

For a binary system (see **Fig.1**) composed by stars with masses $m_1$ and $m_2$ separated by a distance r one can show[14,19] that

$$Q_{xx} = 3\mu r^2[1 + \cos(2\omega t)]/2 \quad \text{and} \quad Q_{yy} = 3\mu r^2[1- \cos(2\omega t)]/2 \qquad \text{(A.13)},$$

where $\mu = m_1m_2/(m_1 + m_2)$ and $\omega$ is the **orbital angular frequency.** In these conditions one see that $h_{\alpha\beta}(t,\mathbf{x})$, using Eqs.(A.11) and (A.13), would be given by

$$\Psi_{\alpha\beta}(t,\mathbf{x}) = h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2R)(\partial^2Q_{\alpha\beta}/\partial t^2) \sim h\cos(2\omega t) \qquad \text{(A.14)},$$

where $h = 6\mu Gr^2/Rc^2$. Showing that the GW frequency is $\boldsymbol{\omega_g = 2\omega}$.

Using Eqs.(A.12) and (A.13) we obtain

$$L_{GW} = (G/45c^5) < (\partial^3Q_{\alpha\beta}/\partial t^3)^2 > = (G/45c^5)\,[<(\partial^3Q_{xx}/\partial t^3)^2 > + [<(\partial^3Q_{yy}/\partial t^3)^2 > ] =$$

$$= (32\mu^2G/5c^5)r^4\omega^6 \qquad \text{(A.15)}.$$

As the energy of the GW in the radiation zone is transported by a **plane wave** with amplitude h and rotation frequency $\omega$ one can show that[13,14]

$$h^2 = (8\pi G/\omega^2c^3)\,(L_{GW}/4\pi R^2) \qquad \text{(A.16)}.$$

As Kepler´s law for a binary [1,5] says that $\omega^2r^3 = G(M_1 + M_2)$ and $M_1= M_2 = M$ we get $r = (2GM/\omega^2)^{1/3}$. Substituting this r value in Eq.(A.16) we obtain h as a function of the orbital angular frequency $\omega$ (rad/s):[11,12]

$$h(\omega) = (4GM/Rc^4\sqrt{36})(2GM/\omega^2)^{2/3}\omega^2 = (4^{2/3}/\sqrt{36})\,[(GM)^{5/3}/Rc^4]\,\omega^{2/3} \qquad \text{(A.17)}.$$

Recently [11-13] gravitational waves have been detected, with frequencies $\omega \sim 160\,\pi$ rad/s. They have been emitted by a black hole binary (BHb). The BHb, that was distant $R \sim 1.3\ 10^9$ light years $\sim 1.2\ 10^{25}$ m from the Earth had $M \sim 20$ solar masses. Using Eq.(A.17) and taking into account the BHb parameters given above we see that

$$h(\omega) \sim 10^{-23}\,\omega^{2/3} \qquad \text{(A.18)}.$$

The measured average amplitude $< h >$ for frequencies $\omega \sim 160\,\pi$ rad/s was found to be $< h > \sim 10^{-21}$, in good agreement with the experimental results.

# Appendix B. Classical electromagnetic radiation.

According to classical Electrodynamics[2]

$$\Box \mathbf{A}(\mathbf{x},t) = -\mu_o \mathbf{J}(\mathbf{x},t) \qquad \text{(B.1)},$$

where $\Box$ is the d´Alembertian operator $\Box = \partial_\mu \partial^\mu$. The solution of (A.1)is given by[2]

$$\mathbf{A}(\mathbf{x},t) = \mu_o \int d^3\mathbf{x}' \int dt' [\mathbf{J}(\mathbf{x}',t')/|\mathbf{x} - \mathbf{x}'|]\, \delta(t' + |\mathbf{x} - \mathbf{x}'|/c - t) \qquad \text{(B.2)}.$$

With the sinusoidal time dependence $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x}) \exp(-i\omega t)$ (A.1) becomes given by

$$\mathbf{A}(\mathbf{x},t) = \mu_o \int \mathbf{J}(\mathbf{x}') \exp(ik|\mathbf{x} - \mathbf{x}'|)/|\mathbf{x} - \mathbf{x}'|\, d^3\mathbf{x}' \qquad \text{(B.3)},$$

that can be expanded in series taking into account that the fields are very far from the source, that is, $r >> d$ and that $d << \lambda$, where d is the dimension of the source and $\lambda$ the wavelength of the emitted radiation. The rate of the emitted electromagnetic radiation dE/dt can be calculated expanding $\mathbf{A}(\mathbf{x},t)$ using *electric and magnetic multipoles.* [2]

In vacuum (A.1) obeys the equation

$$\Box \mathbf{A}(\mathbf{x},t) = 0 \qquad \text{(B.4)}.$$

The general solutions of the above equations for **A** is formed by superposing transverse waves[2] of the field $\mathbf{A}(x_\mu)$. In *second quantization* context [4,21] planes waves **A** are written as (omitting details of normalization constant, wave polarization,...) where $k_\mu = (\mathbf{k}, i\omega/c)$,

$$\mathbf{A}(x_\mu) = \sum_{k\omega} [\mathbf{a}_{k\omega} \exp(ik_\mu x_\mu) + \mathbf{a}^*_{k\omega} \exp(-ik_\mu x_\mu)] \qquad \text{(B.5)},$$

**(B.1) Emitted electromagnetic energy per unitof time dE/dt.**

If the emitted radiation is mainly due to the electric dipole $\mathbf{D} = \int \mathbf{x}' \rho_e(\mathbf{x}')\, d^3\mathbf{x}'$ we have [2]

$$dE/dt = (ck^4/3)\, |\mathbf{D}|^2 = (\omega^4/3c^3)\, |\mathbf{D}|^2 \qquad \text{(B.6)},$$

where $\rho_e(\mathbf{x}')$ is the electric charge density and $k = 2\pi/\lambda = \omega/c$.

If the energy is mainly emitted by electric quadrupole $Q_{\alpha\beta}$ and by magnetic dipole **m** we can show that **[2]**

$$dE/dt = (ck^6/360) \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 \qquad \text{(B.7)},$$

where $Q_{\alpha\beta} = \int \rho_e(\mathbf{x}')(3x'_\alpha x'_\beta - r'^2 \delta_{\alpha\beta})\, d^3\mathbf{x}'$ and $\mathbf{m} = \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}'$.

**(B.2)Larmor Acceleration Formula.**

According to the classical electrodynamics accelerated charges emit radiation and the dominant energy loss is from electric dipole which obeys the Larmor formula (in Gaussian units),[2,17]

$$dE/dt = (2/3c^3)|d^2\mathbf{D}/dt^2| \qquad \text{(B.8)}.$$

This formula can be used to estimate the classical lifetime of the **Bohr atom**.[17] For very large quantum numbers n, Bohr's *correspondence principle* **(CP)** demands that classical physics and quantum physics give the same answer, at least in average. In these conditions as the energy levels are very close the radiate energy is estimated using the classical electrodynamics.[17] So, putting **D** = e**r** it is assumed that the electron moves in circular orbits around the nucleus emits continuously radiating energy according to,

$$dE/dt = (2/3c^3)e^2\mathbf{a}(t)^2 \qquad \text{(B.9)},$$

where **a** the electron acceleration, which is essentially the radial one $a_r = r\omega^2$. In this *adiabatic approximation* the electronic orbit remains nearly circular at all times whith $\omega \approx$ constant. According to reference [17] the electron will fall to the origin, following a spiral motion, after a time $t_{fall} \sim 10^{-11}$ s. The observed lifetime of the $2p^{1/2}$ state of the hydrogen is $\sim 10^{-9}$ s (see Appendix C). In quantum mechanics the ground state, however, "appears" to have infinite lifetime. The accelerated electron along a radius r(t) with a tangential speed $v_\Theta(t)$ and angular speed $\omega = d\Theta/dt = v_\Theta(t)/r$ emits a wave with frequency ω called *synchrotron radiation*.

Taking into account that $|a| \sim a_r = r\omega^2$ Eq.(B.9) becomes written as

$$dE/dt \approx (2e^2\omega^4/3c^3)\,\mathbf{r}(t)^2 \qquad \text{(B.10)}.$$

# Appendix C. Quantum electromagnetic radiation.

In Special Relativity (**SR**) [2,4] the generalized vector potential is defined by $A_\mu = (\mathbf{A}, iA_o) = (\mathbf{A}, i\varphi)$. A free particle with a mass m has a 4-momentum $p_\mu = (\mathbf{p}, iE)$ where E is the total energy $E = (m^2c^2 + p^2c^2)^{1/2}$ . The 4-momentum a charged particle submitted to an electromagnetic field becomes given by $p_\mu \to p_\mu - (e/c)\,A_\mu$. That is, $E \to E - e\varphi$ and $\mathbf{p} \to \mathbf{p} - (e/c)\mathbf{A}$.

The relativistic wave equation [4] for a charged spin zero particle submitted to an external electromagnetic field is obtained through the transformation

$$p_\mu - (e/c)\,A_\mu \to -i\hbar\,\partial/\partial_{x\mu} - (e/c)\,A_\mu \qquad \text{(C.1)},$$

that is

$$\left\{ \Sigma_\mu (-i\hbar\,\partial/\partial_{x\mu} - (e/c)\,A_\mu)^2 + m^2c^2 \right\}\Psi = 0 \qquad \text{(C.2)},$$

or

$$(1/c^2)[i\hbar\,\partial/\partial t - e\varphi]^2\,\Psi = [(i\hbar\,\mathrm{grad} - (e/c)\mathbf{A})^2 + m^2c^2\,]\Psi \qquad \text{(C.3)}.$$

According to quantum mechanics[4] the interaction of a charged spinless particle with the electromagnetic radiation is given by the operator, putting $\boldsymbol{p} = -i\hbar$ grad,

$$W(t) = -(e/mc)(\mathbf{A}.\boldsymbol{p}) + (e^2/2mc^2)\mathbf{A}^2 \qquad \text{(C.4)},$$

where the vector potential **A** is written in the form of a plane wave with wave vector **k** and frequency $\omega$, $\mathbf{A}(\mathbf{r},t) = A_o\, \mathbf{u} \cos[\mathbf{k.r} - \omega t]$, with **u** the unit vector determining the polarization of the radiation (direction of the electric field vector). With the perturbation theory to evaluate the transitions probabilities, in a first order approximation, we neglect the term $(e^2/2mc^2)\mathbf{A}^2$ since it is gives a small contribution, of the order of $\alpha = e^2/hc \sim 1/137$.[4] In this way we retain only the first term of (C.4),

$$W(t) = - (e/mc)(\mathbf{A}.\boldsymbol{p}) \qquad (C.5).$$

The amplitude $a_o$ will be determined in such a way that there are an average N photons of energy $\hbar\omega$ and polarization **u** in a volume V . So, from

$$\mathbf{E} = - (1/c)\partial\mathbf{A}/\partial t = A_o\, \mathbf{u}\, (\omega/c) \sin[\mathbf{k.r} - \omega t] \qquad \text{and}$$

from the condition

$$N\hbar\omega/V = < \mathbf{E}^2(t)>/4\pi = (A_o^2\omega^2/4\pi c^2) < \sin^2[\mathbf{k.r} - \omega t] > = A_o^2\omega^2/8\pi c^2$$

we see that $A_o = 2c(2\pi\hbar N/\omega V)^{1/2}$.

Writing $W(t) = w \exp(i\omega t) + w^*\exp(- i\omega t)$ where $w = A_o \exp(-i\mathbf{k.r})(\mathbf{u}.\boldsymbol{p})$ the transition probability per unit of time for a transition from a (initial) state $|b>$ to a (final)state $|a>$ with the *emission* of a quantum $\hbar\omega$ will be determined by the expression

$$P_{ab} = (2\pi/\hbar) \mid < a |w| b > |^2 \rho(E_{fin}) \qquad (C.6),$$

where the initial energy $E_{init}$ = final energy $E_{fin}$ or $E_a = E_b + \hbar\omega$ and $\rho(E_{fin}) = \rho(\hbar\omega)$ [4] is the density of final photon states $dN/d\varepsilon = \rho(\hbar\omega) = [V\omega^2/(2\pi c)^3\hbar]d\Omega$ , remembering that for photons $\varepsilon = \hbar\omega$ and $p = \varepsilon/c$. The matrix element $< a |w| b >$ is given by

$$< a |w| b > = - A_o < a \mid e^{- i\, \mathbf{k.r}} (\mathbf{u}.\boldsymbol{p}) \mid b > \qquad (C.7),$$

remembering that $\boldsymbol{p} = -i\hbar$ grad. Since the integration of matrix element is will be essentially over the region (**r**) of the size (a) of emitting system it is convenient to expand the exponential factor in a power series,

$$e^{- i\, \mathbf{k.r}} = 1 - i\, \mathbf{(k.r)} + [-i\mathbf{(k.r)}]^2/2! + ... = \qquad (C.8).$$

**(C.1) Dipole radiation**.

When $ka = 2\pi/\lambda << 1$, where $\lambda$ is the wavelength of the emitted photon, it is enough to consider only of the first term of Eq.(C.8) obtaining:[4]

$$< a |w| b > = - i\, \omega_{ab}\, A_o(\mathbf{u}.\boldsymbol{D})_{ab} \qquad (C.9),$$

where $\boldsymbol{D} = \Sigma_i\, q_i\, \boldsymbol{r}_i$ is the ***electric dipole moment operator*** of the emitting system with discrete charges $q_i$. One can show that

$$< a |w| b > = - i\, \omega_{ab}\, A_o\, \mathbf{u}.(\boldsymbol{D}_{ab}) \qquad (C.10),$$

where the vector $\boldsymbol{D}_{ab} = < a |\mathbf{D}| b >$ is called the *electrical dipole moment of the $b \to a$ transition.* In this way, using (C.6)-(C.10) we obtain the probability per unit of time $dP_{ab}^+$ that a photon with polarization **u** and frequency $\omega = |\omega_{ab}| = (E_a - E_b)/\hbar$ is emitted within a solid angle $d\Omega$ ,

$$(dP_{ab}^{+})_{dip} = N\,(\omega^3/2\pi\hbar c^3)\,|\mathbf{u}.(\boldsymbol{D}_{ab})|^2\,d\Omega \qquad (C.11).$$

The polarization **u** is perpendicular to the direction of propagation **k.** If we denote by θ the angle between **k** and the dipole moment of the transition $\boldsymbol{D}_{ab}$ we have $|\mathbf{u}.(\boldsymbol{D}_{ab})|^2 = |\boldsymbol{D}_{ab}|^2 \sin^2\theta$ . Thus,

$$(dP_{ab}^{+})_{dip} = N\,(\omega^3/2\pi\hbar c^3)\,|\boldsymbol{D}_{ab}|^2 \sin^2\theta\, d\Omega \qquad (C.12).$$

Integrating Eq.(C.12) with N =1[4] over all directions of the radiation we get the *total transition probability per unit of time* $P_{ab}$ involving the *emission of one photon*:

$$(P_{ab}^{+})_{dip} = (4\omega^3/3\hbar c^3)\,|\boldsymbol{D}_{ab}|^2 \qquad (C.13).$$

To estimate the order of magnitude of Eq.(C.13) for atomic systems with linear dimension a we put $\boldsymbol{D} = e\boldsymbol{r}$ taking $|\boldsymbol{r}_{ab}| = a \approx e^2/\hbar\omega$. Thus, $(P_{ab}^{+})_{dip}$ can be written as

$$(P_{ab}^{+})_{dip} \approx (e^2\omega/\hbar c)(\omega a/c)^2 \approx \omega/(137)^3,$$

that for optical radiation ($\omega \sim 10^{15}$/s) gives $(P_{ab})_{dip} \sim 10^9$/s. The observed lifetime $\tau \sim 1/(P_{ab})_{dip}$ of the $2p^{1/2}$ state of the hydrogen is $\tau \sim 10^{-9}$ s.[4]

Consequently, energy emitted per unit of time dE/dt will be given by $(dE_{ab})_{dip} = \hbar\omega(P_{ab}^{+})_{dip}$, that is,

$$(dE/dt)_{dip} = (4\omega^4/3c^3)\,|\boldsymbol{D}_{ab}|^2 \qquad (C.14).$$

In case of the Bohr atom with **D** = e**r** (C.14) becomes written as

$$(dE/dt)_{dip} = (4e^2\omega^4/3c^3)\,|\boldsymbol{r}_{ab}|^2 \qquad (C.15).$$

It becomes equal to Eq.(B.8) if the average energy (averaged over the time) emitted per unit of time is due to a dipole $\mathbf{D}(t) = e\mathbf{r}(t) = 2\,(|\boldsymbol{D}_{ab}|^2)^{1/2}\cos(\omega t) = 2e\,|\boldsymbol{r}_{ab}|\cos(\omega t)$.

**(C.2)Quadrupole radiation.**

If it is necessary to take into account the second term of the expansion (B.8) the matrix element $< a\,|w|\,b >$ given by Eq.(C.7) will be

$$< a\,|w|\,b > = -i\,A_o < b\,|(\mathbf{k}.\mathbf{r}')(\mathbf{u}.\mathbf{p}')|\,a > = A_o\,(\hbar k/2)\mu\omega < b|\,r'(\mathbf{n}.\mathbf{r}')|\,a > \qquad (C.16),$$

where $\omega_{ab} = \omega$, μ the electron mass and $\mathbf{n} = \mathbf{r}'/r'$. Eq.(C.16) would be responsible for *electric quadrupole* transitions involving matrix elements of the products xy, xz and yz and *dipole magnetic* transitions of matrix elements of the angular momentum operators $L_x$, $L_y$ and $L_z$. In quantum systems with spherically symmetric potential magnetic dipole transitions give no contributions to photons emission.[4] So, following the same procedure used for dipole radiation we can calculate the total emission probability per unit of time within the solid angle dΩ. The general angular distribution of the quadrupole radiation is very complicated. [2,20,21] As we only intend to obtain an order of magnitude of the quadrupole radiation we put

$$(P_{ab}^{+})_Q \approx (\omega^5/2\pi\hbar c^5)|Q_{ab}|^2 \qquad (C.17),$$

where, the quadrupole matrix element is represented by $Q_{ab}$. So, the total energy per unit of time $(dE/dt)_Q$ emitted by the quadrupole is given by

$$(dE/dt)_Q \approx (\omega^6/2\pi c^5)|Q_{ab}|^2 \qquad (C.18).$$

In classical electrodynamics we have[2]

$$(dE/dt)_{class} \approx (ck^6/240)Q_o^2 = (\omega^6/240c^5)\, Q_o^2 \qquad (C.19).$$

Let us estimate $(P_{ab}^{+})_Q$, given by Eq.(C.17), for systems emitting optical frequencies $\omega \sim 10^{15}$/s and with atomic dimensions $a \sim 10^{-7}$ cm. Taking $Q_{ab} \sim ea^2$ we verify that

$$(P_{ab}^{+})_Q \approx (\omega^5/2\pi\hbar c^5)|Q_{ab}|^2 \sim 10^5/\text{s} \qquad (C.20),$$

that is, $(P_{ab}^{+})_Q \sim 10^{-4}\, (P_{ab}^{+})_{dip}$.

**(C.3)Multipole tensor operators $T_{\ell m}(\theta,\varphi)$.**

Since calculations of quadrupole and magnetic dipole transitions and of higher order terms of the expansion (B.8) are very intricate it is convenient to use a different approach to estimate these matrix elements. In this way are used the *tensor multipole operators* $T_{\ell m}(\theta,\varphi)$ defined by [2,4,20,21]

$$T_{\ell m}(r,\theta,\varphi) = [4\pi/(2\ell+1)]^{1/2}\, r^{\ell}\, Y_{\ell m}(\theta,\varphi) = [4\pi/(2\ell+1)]^{1/2}\, r^{\ell}\, |\ell m\rangle \qquad (C.21),$$

where $\ell$ =1, 2,...correspond to dipole, quadrupole ,... and the angle $\theta$ is between **k** and **r.**

If the state functions are given by $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\,|\ell m\rangle$ the transition probabilities per unit of time $P_{ab}$ will directly proportional to $|a_E(\ell,m)|^2$ where the amplitudes $a_E(\ell,m)$ are given, for ka << 1, by[4]

$$a_E(\ell,m) = -\,[4\pi/(2\ell+1)!!](\ell+1/\ell)^{1/2}\, k^{\ell+2}\, Q_{\ell m} \qquad (C.22),$$

where

$$Q_{\ell m} = \int dr\, r^{\ell+2}\, R_a(r)\, R_b(r) \langle \ell_b m_b | Y_{\ell m}^{*}(\theta,\varphi) | \ell_a m_a \rangle.$$

The matrix element $\langle n'j'm'|T_k^q|\, n\, j\, m\rangle$ according to the Wigner-Eckart Theorem (WET)[22] is given by $\langle n'j'm'|T_k^q|\, n\, j\, m\rangle = (jkmq|j'm')\,(n'j'||Tk||nj)$, where $(jkmq|j'm') \neq 0$ only when $m + q = m'$ and $|j - k| \leq j' \leq j + k$.

For **dipole ($\ell$=1)** using Eq.(C.18) the transition probabilities per unit of time $P_{ab}$ between states $|a\rangle$ and $|b\rangle$ are proportional to $|\mathbf{D}_{ab}|^2$ where,

$$|\mathbf{D}_{ab}| = (4\pi/3)^{1/2} \int dr\, r^3\, R_a(r)\, R_b(r) \langle \ell_b m_b | Y_{10}(\theta,\varphi) | \ell_a m_a \rangle \qquad (C.23).$$

Thus, following the WET the a → b transition is allowed only if we have:

$$\ell_b = \ell_a \pm 1 \quad \text{and} \quad m_b = m_a\, .$$

This kind radiation is called *electrical dipole radiation* and is denoted by E1.

For electric **quadrupole ($\ell$=2)** $P_{ab}$ is proportional to $|Q_{ab}|^2$ where

$$Q_{ab} = \int dr\, r^4\, R_a(r)\, R_b(r) < \ell_b m_b | Y_{2m}^*(\theta,\varphi) | \ell_a m_a > \qquad (C.24),$$

showing that quadrupole **t**ransitions $a \rightarrow b$ are allowed only if

$$\ell_b = \ell_a \pm 2 \quad \text{and} \quad m_b = m_a + 2 \qquad (C.25).$$

This kind of radiation is called *electric quadrupole radiation* and is denoted by E2.

**(C.4)Second quantization approach.**

Basic ideas on the quantization of radiation can be seen in many books. In *vacuum*, with the Lorentz gauge the electromagnetic field $\mathbf{A}(x^\mu)$ is given by[4,21]

$\mathrm{div}(\mathbf{A}) = 0$, $\partial_\mu \partial^\mu = \Box \mathbf{A} = 0$, $\mu = 1,2,3,4$, $x_\mu = (\mathbf{x}, ict)$ and $A_\mu = (\mathbf{A}, i\varphi)$.

The general solutions of the above equations for **A** is formed by superposing transverse waves[2,4] of the field $\mathbf{A}(x_\mu)$. In the *second quantization* context planes waves **A** are written as (omitting details of normalization constant, wave polarization,...)

$$\boldsymbol{A}(x_\mu) = \sum_{k\omega} [\mathbf{a}_{k\omega} \exp(ik_\mu x_\mu + \mathbf{a}^*_{k\omega} \exp(-ik_\mu x_\mu)]/\sqrt{\omega} \qquad (C.26),$$

where $k_\mu = (\mathbf{k}, i\omega/c)$ , $\mathbf{a}_{k\omega}$ and $\mathbf{a}^*_{k\omega}$ are the **creation and annihilation photon operators**, respectively.

In this approach transition probabilities $P_{ab}$ are now estimated using in Eq.(C.6) the field operator **A** defined by Eq.(C.22). Taking into account transitions involving *vacuum states* and wavefunctions $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)|\ell m >$ we get the same results obtained before without the second quantization approach. The main difference now is that the electromagnetic radiation is composed by *photons*. Selection rules obeyed in *electrical dipole radiation* (E1) show that **photons** must have spin 1.

## Appendix D. Comments on the gravitation quantum field theory.

Classical electrodynamics, quantum theory and their connections are very well established. To introduce basis of a quantum field theory in GR, Weinberg[15] analyzed, for instance, the possibility to quantize the gravitational wave field $h_{\mu\nu}$ that in free obeys the equations (see **Appendix A**) $\Box h_{\rho\nu} = 0$ and $\partial h_\rho{}^\nu/\partial x^\nu = 0$. The general solutions of these equations are given by the superposition of transverse plane tensor waves $h_{\rho\nu}(x)$ which propagates with the light velocity c and helicities $\mu = \pm 2$. This would be done in order to construct, similarly to the Electromagnetic field, a *Lorentz invariant Hamiltonian* in terms of creation and annihilation operators of gravitons. That is, the Hamiltonian would be built up of quantum fields $h_{\rho\nu}(x)$ [*transverse plane waves*] that in a **second quantization** framework would be given by[15]

$$h_{\rho\nu}(x) = \sum_\mu \int d^3\mathbf{k} \{ a(\mathbf{k},\mu)\, e_{\rho\nu}(\mathbf{k},\mu) \exp(ik_\lambda x^\lambda) + a^+(\mathbf{k},\mu)\, e^*_{\rho\nu}(\mathbf{k},\mu) \exp(-ik_\lambda x^\lambda) \} \quad (D.1),$$

where $e_{\rho\nu}(\mathbf{k},\mu)$ is the polarization tensor for a graviton of momentum $h\mathbf{k}$ and **helicity** $\mu = \pm 2$, and $a(\mathbf{k},\mu)$ and $a^+(\mathbf{k},\mu)$ are the corresponding annihilation and creation operators, characterized by the commutation relations

$$[a(\mathbf{k},\mu)\ ,\ a^{+}(\mathbf{k}',\mu')] = \delta^{3}(\mathbf{k} - \mathbf{k}')\ \delta_{\mu'\mu}$$

$$[a(\mathbf{k},\mu)\ ,\ a(\mathbf{k}',\mu')] = [a^{+}(\mathbf{k},\mu)\ ,\ a^{+}(\mathbf{k}',\mu')] = 0 \qquad \text{(D.2)}$$

The difficult in this approach comes from the fact that the operator Eq.(4.1) is not a "Lorentz tensor"(*which is invariant by Lorentz group*). Remembering that $\tau_{\mu\nu}$ is a Lorentz tensor if it transforms as $\tau'_{\mu\nu} = \Lambda_{\mu}{}^{\rho}\,\Lambda_{\nu}{}^{\sigma}\,\tau_{\rho\sigma}$, where $\Lambda$ is the Lorentz matrix.[15] As shown by Weinberg[15] in Section (10.2) a "true" plane wave tensor would have helicities 0, ±1 as well ± 2. This is in contradiction with Eq.(D.1) where there are only helicities $\mu = \pm 2$. Of course, we can start with a true tensor and then subject $e_{\mu\nu}$ to a gauge transformation that will eliminate the unphysical helicities 0 and ±1, but once we choose a gauge in this way, $h_{\rho\nu}(x)$ is no longer a Lorentz tensor. This gauge condition is not Lorentz invariant. Many other attempts are mentioned by Weinberg.[15] According to him at present does not exist any complete and self-consistent quantum field theory of gravitation. In his book he presents to the reader some taste of what a quantum theory of gravitation would be like. Instead of using Lagragian or Hamiltonian formalisms he adopts a different way. In this way he proposed, for instance, that for a general system the emission rate $d\Gamma_{GW}$ of a gravitational wave ("gravitons") with frequency $\omega$ in a solid angle $d\Omega$ is given by

$$d\Gamma_{GW} = (G\omega/\hbar\pi)[T^{\lambda\nu *}(k,\omega)\ T_{\lambda\nu}(k,\omega) - (1/2)\ |T^{\lambda}{}_{\lambda}(k,\omega)|^{2}]\ d\Omega \qquad \text{(D.3)},$$

where $T_{\lambda\nu}(k,\omega)$ is the energy-momentum tensor. Using Eq.(D.1) one can show[15] that in the quadrupole approximation the total power emitted at a single discrete frequency $\omega$ is given by

$$\Gamma_{GW} = (2G\omega^{6}/5)[D^{*}_{ij}(\omega)D_{ij}(\omega) - (1/2)|D_{ij}(\omega)|^{2}] \qquad \text{(D.4)},$$

where $D_{ij}(\omega) = \int x^{i}x^{j}T^{oo}(\mathbf{x},\omega)\ d^{3}x$ which is the quadrupole matrix operator and $T^{oo}(\boldsymbol{x},\omega)$ the energy density operator written as $\rho$. In this way, $\Gamma_{GW}$ given by Eq.(D.4) could interpreted as matrix element of $\rho$ between final and initial states $\psi_a$ and $\psi_b$. That is, in a quantum transition $a \rightarrow b$ the total rate $(\Gamma_{GW})_{ab}$ would be given by

$$(\Gamma_{GW})_{ab} = (2G\omega^{5}/5\hbar)[D^{*}_{ij}(a \rightarrow b)D_{ij}(a \rightarrow b) - (1/3)|D_{ij}(a \rightarrow b)|^{2}] \qquad \text{(D.5)},$$

where $D_{ij}(a \rightarrow b) \equiv \int \psi_b{}^{*}(\mathbf{x})\ \rho\ x_i\, x_j\ \psi_a(\mathbf{x})\ d^{3}\mathbf{x}$ which is a quadrupole matrix element. He applied this formula to calculate GW emitted by 3d → 1s transition of hydrogen and concluded that there is no chance to be observe the event. Probably, he ought to have applied his formula to calculate GW emitted by mBBH.

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