Comments on Ligo -Virgo Gravitational Waves Observations

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Abstract. Many gravitation waves (GW) events have been detected from 2016 up to 2020 by the *Laser Interferometer Gravitational-Wave Observatories* (**Ligo -Virgo**). Ten emitting systems are formed by black-hole binaries and one by neutron star binary**.** In this paper we remember that besides binary systems there are many different emitting astronomical systems that could also be detected. These are, for instance, neutron star pulsations, wobbling of deformed neutron stars and cataclysmic supernovae.

Key words: *GW detection*; *astronomical GW emitting systems*.

(I)Introduction.

This paper was written to graduate and postgraduate students of Physic and Engineering. We begin remembering that solving Einstein´s equations for a static distribution of mass[1-6] one can obtain the unperturbed metric tensor $g^{(o)}_{\mu\nu}$. When this mass distribution is slightly perturbed we write $g_{\mu\nu} = g^{(o)}_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ represents a small correction of the initial gravitational field. In absence of matter $g^{(o)}_{\mu\nu}$ is given by Minkowsky tensor $g^{(o)}_{\mu\nu} = (1,1,1,-1)$. All relativistic accepted gravitation[1-6] theories predict the existence of gravitational waves (GW) that were expected to have **extremely small amplitudes.** The smallest h values able to be measured with our present experimental Ligo -Virgo techniques[7-9] are in the range $h \sim 10^{-21}$-10^{-22}.This can be seen, for instance, in recent articles[10-12] where are deduced basic equations predicting the emission of GW and experimental tests have been analyzed. During many years [3,6,7] different experimental techniques have been proposed and developed to detect GW. Interferometric detectors, for instance, were first suggested in the early 1960s and the 1970s.[6-9] After ~20 years of intense researches in many countries, in ~1990 it was arrived at the conclusion that only interferometers[6,10-12] could detect GW with certainty. Detection of GW by another systems would be fortuitous. In 2015 have been done the first GW observations, emitted by binary-star systems of black-holes (BBH).[9].These events, known as GW150914 and GW151226 have been detected with the LIGO Laser Interferometric technique.[9,13]

From 2016 up to 2020 were performed,[14] by the Ligo and Virgo, new GW detections; ten from black-hole binaries (BBH) and one from neutron star binary(NS-NS), In **Section 1** are shown the basic equations necessary to estimate the flux of gravitational energy emitted by some radiating systems, the gravitational power radiated, their angular distribution and their temporal and spatial average values. In the following sections will be studied GW emitted by four different systems. In **Section 2** by binary stars. In **Section 3** by pulsating neutron stars. In **Section 4** by wobbling neutron stars. In **Section 5** by kind II Supernova. In **Appendix** is shown that GW emitted by a very large rotating metallic bar has h $\sim 10^{-41}$.

(1)Basic Equations.

Are shown here the basic equation for the emission of gravitational waves, energy flow, irradiated gravitational power, its angular distribution and its average, temporal and spatial values.[10-12] The energy flow Φ_n emitted along a generic unit vector **n**, in units of (energy/s area), is given by $\Phi_n = F = (d^2E/dt\ dA)$, where the element of area $dA = r^2 d\Omega$:

$$F = (G/36\pi r^2c^5)\{(1/2)\ (Q^*_{\alpha\beta})^2 - (Q^*_{\alpha\beta})(Q^*_{\alpha\gamma})n_\beta n_\gamma + (1/4)(Q^*_{\alpha\beta}\ n_\alpha n_\beta)^2\} \quad (1.1),$$

where, $Q^* = d^3Q/dt^3$ and Q_{ij} is the emitter quadrupole moment tensor,[4,5]

$$Q_{ij} = \int\rho_o\ (3x_ix_j - \delta_{ij}r^2)\ dV\ .$$

The energy radiated in all directions, that is, the energy lost by the emitting system per unit of time, dE/dt, is given by[4]

$$dE/dt = -(G/45c^5)\ (Q^*_{\alpha\beta})^2.$$

The total power L_{GW} (energy/s) or gravitational luminosity of the emitting system that arrives on a spherical surface at a distance r from the emitting source is given by

$$L_{GW} = \langle dE/dt \rangle = r^2\int \langle F \rangle\ d\Omega = (G/45c^5) \langle Q^{*2}_{\alpha\beta} \rangle \qquad (1.2),$$

where the brackets < M > indicate the time average on M.

The gravitational energy is transported, in the radiation zone, by a plane wave with amplitude h_o and frequency ω that are related with the average flux value < F > by the equation[4]

$$\langle F \rangle = (c^3/32\pi G)\ h_o^2\omega^2 \qquad (1.3).$$

Taking into account Eqs.(1.2) and (1.3) we have,

$$L_{GW} = \langle dE/dt \rangle = 4\pi r^2 (c^3/32\pi G)\ h_o^2\ \omega^2 = (G/45c^5) \langle Q^{*2}_{\alpha\beta} \rangle \quad (1.4).$$

So, Eq.(1.4) permit us to establish the following protocol:
(1.A)we calculate the luminosity using (1.2),

$$L_{GW} = (G/45c^5) < Q^*_{\alpha\beta}{}^2 >.$$

(1.B) Starting from the above L_{GW}, using (1.4) we calculate the amplitude h_o, at a distance r from the emitting center

$$h_o{}^2 = (32\pi G/\omega^2 c^3)\,(L_{GW}/4\pi r^2) \qquad (1.5).$$

The total energy radiated Mc^2 in a "pulse" of very short time interval $\Delta\tau$ on a spherical surface with radius r is given by, [5]

$$\int L_{GW}\,dt = Mc^2 = 4\pi r^2 < F > \Delta\tau = (c^3/8G)\, r^2\, h_o{}^2\, \omega^2\, \Delta\tau \qquad (1.6).$$

With (1.6) one can estimate[6] the amplitude h_o, at a distance r from the emission center, in terms of the following units [r] = 10 kpc , $[Mc^2] = 10^{-3}\, M_{sun}\, c^2$, [τ] = ms and frequency [f] = kHz:

$$h_o = 10^{-18}\,(1\text{kHz}/f)(10\text{ kpc}/r)\,(M/10^{-3}\, M_{sun})(1\text{ ms}/\Delta\tau\,)^{1/2} \qquad (1.7),$$

remembering that 1 parsec = pc ≈ 3.26 ly and ly = light-year = $9.46\ 10^{15}$ m, we have pc ≈ $30.84\ 10^{15}$ m ~ $3.1\ 10^{18}$ cm.

(2)Binary Stars.

Let us consider two compact stars with masses m_1 and m_2 that move with angular velocity ω in circular orbits around its common center of mass (CM). Their distances till the CM are r_1 and r_2, respectively. One can show[5,10] that the average gravitational luminosity L_{GW} = < dE/dt > according to (1.4),is given by

$$L_{GW} = -(32G/9c^5)\mu\, r^4\omega^6 \qquad (2.1),$$

where $\mu = (m_1 + m_2)/M$ and $r = R_1 - R_2$. Once determined L_{GW} the amplitude h_o can be estimated by Eq.(1.5):

$$h_o{}^2 = (32\pi G/\omega^2 c^3)\,(L_{GW}/4\pi r^2) \qquad (2.2).$$

Simple explanations about interferometer techniques and calculations for GW emission by black hole binaries (BBH) can be seen, for instance, in our preceding paper.[10-12] Below are shown figures of confirmed detected wave amplitudes $h_o(t)$[14] for BBH and NS binary, recently detected by Ligo-Virgo. All amplitudes h_o of detected waves are in range $h_o \sim 10^{-21}$-10^{-22}.

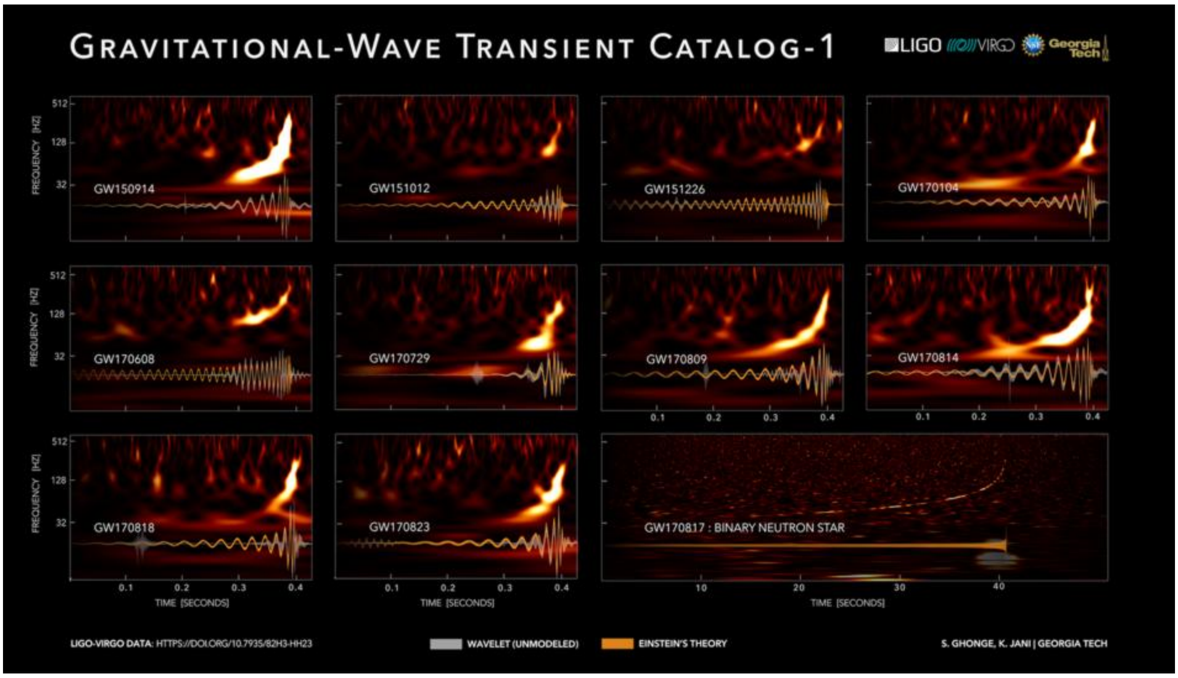

Figure 1. Gravitational waves amplitudes $h_o(t)$ as a function of the time t. The first ones are from BHH binaries and last one of a NS binary. All amplitudes are in the range $h_o \sim 10^{-21}$-10^{-22}.

Besides these eleven confirmed events[14] have also been observed *Candidate Events* and *Marginal Detections*. These possibilities are in agreement with estimations done by Misner, Thorne and Wheeler.[3] They are shown calculations of L_{GW}, dr/dt and τ for many binary star systems like, for instance, Sun and Jupiter, binaries of neutron stars and black holes.

Let us consider, for example, the NS binary W Ursae Majoris (WUMa) with $m_1 = 0.76\ M_{sun}$, $m_2 = 0.57\ M_{sun}$, $r = 1.5\ 10^{11}$ cm and $\omega = 4/\pi$ h. Using Eq.(2.1), we see that $L_{GW} \approx 10^{29}$ erg/s = 10^{22} W. Since the distance from the Earth is $r \approx 360$ ly ~110 pc the gravitational energy flux on the Earth is $\Phi = L_{GW}/4\pi r^2 \sim 10^{-13}$ erg/cm^2. From Eq.(2.2), $h_o \sim 10^{-28}$. That is, GW waves emitted by W Ursae Majoris **could not be detected.**

(2.1)PSR 1913+16

This is a binary system composed by a neutron star and by a "pulsar" (radiating neutron star). The system PSR 1913+16 was discovered by Hulse and Taylor using the radio telescope of 305 m at Arecibo.[15]

Taking into account that $m_1 \approx m_2 = 1.4\ M_{sun} \approx 2.8\ 10^{33}$ g , $r \sim 10^6$ Km and orbital period T = 7.75 h = 2.79 10^4 s the GW luminosity according to Eq.(2.1) is given by $L_{GW} \sim 10^{29}$ W. Taking into account that r ~ 5 kpc and $\omega = 4\pi/T$ we verify, using Eq.(2.2), that $h_o \sim 6.4\ 10^{-21}$, showing that the GW emitted by the "pulsar binary" **could be detected** by the Ligo-Virgo.

(3) Pulsating Neutron Stars.

Many models have been proposed [1,2,16 −19] to determine the neutron star core equation of state. A repulsive force generated by a fermionic neutron sea balances the gravitational energy of stellar matter. In a first approximation, we can describe this star as a giant nucleus composed of neutrons with a radius R ~ 10 km and mass $M \sim M_{sun}$; with a constant density $\rho \sim 10^{14} - 10^{15}$ g/cm^3, comparable to the density of nuclear matter. In this case the equation of state is calculated assuming that neutrons constitute a degenerate ideal gas [20] of fermions. More realistic equations of state are obtained by taking into account the short-range nuclear interaction between neutrons and treating neutron matter as a self-gravitating liquid that, under certain conditions, could solidify. Many articles [16−21] show that it is plausible to expect that the self-gravitating neutron matter would be governed by an equation of state that is very close to that of an elastic solid. Evidence of viscoelastic behavior is obtained by observing pulsar glitches.[22] In these conditions, let us estimate the pulsating frequencies ω of the star when its equilibrium state is disturbed taking into account that the compressibility χ of neutron matter is given by $\chi = -(\Delta\rho / \Delta p)/\rho$ where p is its pressure and ρ its density . Recalling that the speed vs propagation of sound waves is given by [23,24] $v_s = 1/\sqrt{\chi\rho} = \sqrt{\partial p/\partial\rho}$ it can be written, in a first approximation, as $v_s \sim \sqrt{p/\rho}$. Since the typical pressure p in a self - gravitating sphere [25] of radius R and mass M is $p = GM^2/R^4$ we obtain $v_s \sim (G/\rho)^{1/2}(M/R^2) \sim (GM/R)^{1/2}$. Assuming that the wavelengths λ of the oscillations are given by $\lambda \sim R$ we must have $\omega = (2\pi/\lambda)v_s \sim 2\pi\,(G\rho)^{1/2}$. As $\rho \sim 10^{14}$ g/cm^3 we find that the frequencies are $f \sim 10^3$ Hz. More accurate calculations[1] show that f are in a range from 10^{-2} to 10^4 Hz. If the non-radial pulsations that create a $\delta R(t)$ of radius R generate a quadrupole Q(t) it can be approximated by $Q(t) \sim MR^2\delta R(t)/R$. In the case of a periodic disturbance, $R(t) = \delta R \cos(\omega t)$ we will have,

$$Q(t) \sim MR^2(\delta R/R)\cos(\omega t) \qquad (3.1).$$

As $< Q^{*2} > \sim M^2R^4\omega^6(\delta R/R)^2$ the luminosity L_{GW} generated by the pulsations, using Eq.(1.2), will be given by

$$L_{GW} \sim (GM^2R^4\omega^6/c^5)(\delta R/R)^2 \qquad (3.2).$$

Assuming that $M \sim 1.4\, M_{sun}$, R ~ 10 km and $\omega \sim 10^4$ rad/s we will have $L_{GW} \sim 2\ 10^{55}(\delta R/R)^2$ W. If the amplitudes of pulsations are on the order of $\delta R/R \sim 10^{-2}$, we will have $L_{GW} \sim 2\ 10^{51}$ W. In the case of pulsars, which are neutron stars with very high rotation speeds, deformation (see Section 4) of the star due to the rotation would enlarge the quadrupole moment and, consequently, the L_{GW}.

If the pulsating neutron star is in our Galaxy, at a distance r ~ 10 kpc from Earth, the amplitude h_o of the wave, using (1.5), would be $h_o \sim 10^{-19}$. If it is in the Virgo cluster, with r ~ 15 Mpc, we would be have $h_o \sim 10^{-21}$. Thus, in both cases the GW **could be detected** by the Ligo -Virgo.

(4)Wobbling of Deformed Neutron Stars.

Let us consider a flat neutron star rotating around its axis of symmetry z with an angular velocity $\Omega = 2\pi f$, analogously to what happens with the Earth that has a spin around the North-South geographic axis. Due to this flattening, the star would have a quadrupole moment. Suppose that, for some reason, the z axis is precessing around a fixed axis ζ forming an angle θ (wobble angle) with it. If $\theta << 1$ the wobbling star emits GW with a frequency very close to f with an amplitude h_o given by [27,28]

$$h_o = 1.4\ 10^{-18}\ \varepsilon\ \theta\ I_{zz}/10^{45}\ [\mathrm{g\ cm}^2]\ f^2\ [\mathrm{kHz}]\ r^{-1}\quad \mathrm{kpc} \qquad (4.1),$$

where $\varepsilon = (I_{zz} - I_{xx})/I_{zz}$, I_{jj} are the star moments of inertia and r the distance from the star to the observer. Supposing that $M = 1.4\ M_{sun}$ and that the stellar matter obeys a state equation of the Bethe-Johnson type I, the moments of inertia I_{zz}, I_{xx} and ε have been calculated[28] as a function of $\Omega = 2\pi f$ in the interval $3000 \leq \Omega \leq 6203$ rad/s. Let us assume that the star is in our Galaxy, at a distance r ~ 20 kpc from Earth. We verify, using Eq.(4.1), that for $\Omega = 3200$ rad/s and $\Omega = 6203$ rad/s we have $(h_o)_{min}$ and $(h_o)_{max}$ given, respectively, by

$$(h_o)_{min} \sim 10^{-16}\ \theta \qquad \text{and} \qquad (h_o)_{max} \sim 10^{-14}\ \theta$$

Thus, even for very small values of θ, within a range $10^{-5} \leq \theta \leq 10^{-3}$, GW generated by the precession **could be detected** by the Ligo-Virgo.

(5) Supernova of kind II.

According to the current theory of stellar evolution,[5,6,29] stars with masses M ~ 20~30 M_{sun} are generated by the gravitational contraction of gases, mainly hydrogen. Due to this contraction, the internal pressure and temperature can increase to such an extent that they give rise to thermonuclear fusion reactions. The energy released in this process, increasing the pressure and temperature of the stellar matter, prevents the star from collapsing. Hydrogen converts to helium, but when hydrogen is exhausted, the energy produced decreases and the star begins to contract. This causes pressure and temperature to start to rise again, starting thermonuclear reactions that convert helium into carbon. Subsequent stages of burning nuclear fuel produce, finally after ~ 10^9 years, a star with a core formed by iron nuclei. Since iron nuclei have the highest binding energy per nucleus, neither fusions or fissions can produce additional energies. The outer layers of the star, as they have not reached the iron stage, continue to produce energy through fusions. The core is formed by iron and electron nuclei pulled from nuclei that moves almost freely through the volume of the star. The zero-point pressure of the degenerated electron gas [25] is responsible for most of the core pressure and the nuclei make the greatest contribution to its mass. The equations of state (pressure as a function of density) that are developed based on this model show that there can be a stable equilibrium of the star if its mass does not exceed a critical value called the limit mass M (Chandrasekar) = 1.44 M_{sun}. Thus, if M < 1.44 M_{sun} the star is in a state of equilibrium, that is, the contraction ceases and it becomes a White Dwarf. A typical white dwarf has a radius R ~ radius of the Earth and density ~ 10^2 up to 10^4 kg/m^3. The luminosity is generated in the upper layers of the star and as the nuclear fuel runs out it is due to the residual thermal energy. As the thermal energy is lost by radiation, it becomes less and less luminous, becoming, after a period of ~ 10^{10} y, an extinguished star or Black Dwarf. Due to instabilities, as they cool, they can become pulsating white dwarfs. However, if M > 1.44 M_{sun} there is no stable equilibrium, the star continues to contract, the electrons in the core are absorbed by the protons of the iron nuclei, transforming them into neutrons. Neutronization is rapid, around 1s, accompanied by a collapse of the core and a violent emission of neutrinos.[30] This phenomenon is known as the Urca Process. It was first discussed by G. Gamow and M. Schemberg while visiting Urca's casino in Rio de Janeiro. Gamow is said to have told Schemberg: "the energy disappears in the core of a supernova as quickly as the player's money disappears on the roulette table". The core gets much smaller, its density increases a lot, becoming very hard due to nuclear repulsion. The outer parts of the star implode and when they hit the neutron core they are ricocheted with such violence that a part of the star's mass is ejected giving rise to what we call a *supernova type II explosion*. A recently observed supernova is SN1987a. It is estimated that in the explosion the energy released is between 10^{-5} Mc^2 and10^{-2} Mc^2. After the explosion, if the residual mass M_r of the star [5,6,28] is M_r < 2 to 3 M_{sun} it stops contracting becoming a Neutron Star with radius R ~ 10 km and density ρ ~ 10^{14} - 5 10^{15} g/cm^3. If M_r > 3 M_{sun} the implosion cannot be stopped and it becomes a “black hole”. Assuming that the original mass of gases had a spin at the end of the star's huge contraction process, its angular velocity would be extremely high. This would accentuate the star[27,28] the asymmetry favoring the emission ofGW in the collapse. As the astrophysics of these processes is very complicated, it is only possible to estimate some limits of the parameters[1,5,6] relevant to gravitational radiation generated in the collapse. Thus, an estimate is obtained assuming that GW would be emitted in a pulse with a duration of time $\Delta\tau$ ~10^{-3} s, would carry an energy ~10^{-3} M_{sun} and would have frequencies f ~ 1/$\Delta\tau$ in the range of 10 to 10^4 Hz. Under these conditions we can calculate the amplitude h_o of the wave that would be detected on Earth using Eq. (1.7),

$$h_o = 10^{-18}\,(1\text{kHz}/f)\,(10\text{ kpc}/r)\,(M/10^{-3}\,M_{sun})\,(1\text{ ms}/\Delta\tau)^{1/2.}$$

Assuming that f = 1khz, $M = 10^{-3}\,M_{sol}$, $\Delta\tau$ = 1ms and r ~ 10 kpc, that is, that the supernova is in our Galaxy, we obtain $h_o \sim 10^{-18}$. Type II supernova explosions occur once every 30 years in our Galaxy. If the supernova is in the Virgo cluster, that is, r ~15 Mpc we have $h_o \sim 7\;10^{-22}$, that is, it **could to be detected** by Ligo-Virgo. Since in the Virgo cluster there are ~ 2000 galaxies, we would have an explosion rate every few days and, consequently, many GW detections.

APPENDIX. GW from a rotating rigid bar.

Let us consider a cylindrical bar with length L and base area A, with mass M, which rotates in a plane (x, y) with angular frequency ω around the z axis that passes through its center of mass O. Indicating by ξ the distances of the mass elements $dM = \rho_o dV = \rho_o A d\xi$ to the center O; ξ varies from -L/2 to L/2. As the coordinates (x, y) of the bar are given by x(t) = ξ cos(ωt) and y(t) = ξ sin(ωt) the quadrupole moments Q_{ij} of the bar, defined by (I.1), are given by

$$Q_{xx} = \int\rho_o\,(3x^2 - r^2)\,dV = \rho_o\,A\int[2\xi^2\cos^2(\omega t) - \xi^2\sin^2(\omega t)]\,d\xi =$$

$$= 2A\rho_o\,(L/2)^3\,[2\cos^2(\omega t) - \sin^2(\omega t)]/3 = A\rho_o\,(L/2)^3\,[1 + \text{co}\,(2\omega t)]/3 =$$

$$= ML^2\,[1 + \cos(2\omega t)]/24. \qquad \text{(A.1).}$$

Similarly, we have

$$Q_{yy} = ML^2\,[1 - \cos(2\omega t)]/24. \qquad \text{(A.2)}$$

As $z_i = 0$, we have

$$Q_{zz} = -\,ML^2/24. \qquad \text{(A.3)}$$

Thus, taking into account Q_{ij} shown by Eqs.(A.1)-(A.3) we obtain $\langle Q^{*}_{ij}{}^2\rangle = M^2L^4\omega^6/9$. In this way, the L_{GW} of the rotating bar, using Eq.(I.4), is given by

$$L_{GW} = (G/45c^5)\,\langle Q^{*}_{\alpha\beta}{}^2\rangle = (G/45c^5)\,(ML^2/3)^2\omega^6 \qquad \text{(A.4).}$$

Let us assume that the bar is made of steel with M = 500 ton = $5\;10^5$ kg, L = 20 m and that ω ~ 20 rad/s and that it does not break with the high rotation speed. In these conditions, using Eq.(A.4), we get L_{GW} ~10^{-24} W. So, at a distance r ~1m of the bar, using (1.5), we verify that h_o ~10^{-41}.

This extremely small value show that the construction of a GW generator in laboratories is an unattractive undertaking. We need to invent another more efficient method for producing GW in the laboratory. As the current techniques only allow us to detect amplitudes $h_o \geq 10^{-22}$- 10^{-21}, the unique alternative that we have is to consider astrophysical sources.

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