Negative, Complex Numbers and Gentileons.

M.Cattani (mcattani@if.com.br)
Instituto de Física da Universidade de São Paulo(USP),Brasil

Abstract.

This paper was written to graduate and postgraduate students of Physics and Mathematics. It was necessary around 2000 years to humanity to clearly understand the concepts about negative and complex numbers. This was possible only after the demonstration of the fundamental theorem of algebra done by Argand and Gauss in 1797 showing that the use of complex numbers is ***unavoidable.*** *There is also a fundamental theorem showing that quantum systems composed by N identical particles are represented by horizontal, vertical or intermediate Young Shapes. These representations are exact solutions of the quantum problem. They are not "imaginary" or "fictitious" representations of quantum systems. They cannot excluded "a priori" as happened with complex numbers in the past.*
key words: *complex numbers ;quantum systems; identical particles; Young Shapes.*

(1)Negative and Complex Numbers.

Incidentally, recently rereading some texts on the history of *negative* and *complex numbers*, I remembered how the concepts about them evolved very slowly with time. However, it is not our goal to present the history of this evolution; many articles and books can be written on the subject. [1-3]

The emergence of such numbers is mainly linked to the solving of grade 3 and grade 2 algebraic equations. Their acceptance, understanding and use took place very slowly, secularly, and gradually **.**

Solving quadratic and cubic algebraic equations has always been a subject that has fascinated mathematicians throughout history, from remote ages in Greece, Babylon, China and India, for example.

In the book "Aritmetika" by Diophantus (Sec.III AD), negative numbers often appear. There were certain problems for which the solutions were negative integer values, for example,

$$4x + 20 = 4 \qquad \text{and} \qquad x^2 + x + 1 = 0 \qquad (1.1).$$

In these situations, Diofanto limited himself to classifying the problem as **absurd**. Even in much later centuries such as 16th and 17th European mathematicians did not appreciate negative numbers and, if they appeared in their calculations, they considered them to be **false** or **impossible**. Michael Stifell (1487-1567), for example, refused to admit negative numbers as roots of an equation, calling them **numeri absurdi**. Girolamo Cardano (1501-1576) used negative numbers while calling them **numeri ficti**. English mathematicians (1758), such as Francis Maseres and William Friend, assumed that **negative numbers did not exist**.[1-3]

This scenery changed from the 18th century when the geometric interpretation of positive and negative numbers was discovered as straight line segments in opposite directions. This was thanks to Pierre Fermat and Réne Descartes who invented the Analytical Geometry.

The earliest reference to **square roots** of negative numbers seems to have appeared (Sec.I AD) in the works of a Greek mathematician Hero of Alexandria when calculating the trunk of a pyramid obtaining a value $\sqrt{81 - 144} = 3\sqrt{-7}$. Since negative values were **inconceivable** in Hellenistic mathematics, Hero simply replaced this value with $3\sqrt{7}$.

The impetus to study numbers with $\sqrt{-1}$ arose in the 16th century when algebraic solutions for the roots of cubic and quartic polynomials were discovered by the Italian mathematicians like, for instance, Nicolò Fontana Tartagia and Gerolamo Cardano. In the case of the quadratic equation $ax^2 + bx + c = 0$ we have the solutions

$$x = [-b \pm \sqrt{\Delta}]/2a , \qquad (1.2\ ),$$

where $\Delta = b^2-4ac$, that for $\Delta < 0$ has no real value for $\sqrt{\Delta}$.

For the cubic equation, for instance, $x^3 -15\,x = 4$ we would have the solution[2,3]

$$x = [\,2 + \sqrt{-121}]^{1/3} + [\,2 - \sqrt{-121}]^{1/3} \qquad (1.3).$$

These **"quantities"**, with positive numbers and with square roots of negative numbers were coined **"imaginaries"** by René Descartes in 1637 who was at pains to stress their **"unreal nature".**[1]

In spite of these troubles, in the 18th century complex numbers gained wider use, as it was noticed that formal manipulation of complex expressions could be used to simplify calculations involving, for instance, trigonometric functions. We mention, for instance, the works of Abraham de Moivre (1730), Leonhard Euler (1748) and Casper Wessel (1799).[1]

Only after the rigorous proofs of the **fundamental theorem of algebra** done by Carl Friedrich Gauss in 1797 and Caspar Wessel in 1806 it was established that the use of complex numbers is **unavoidable.** This theorem shows that with complex numbers, a solution exists to every polynomial equation of degree one or higher. However, Gauss expressed his doubts about "**the true metaphysics of the square root √-1".** It was not until 1831 that he overcame these doubts and published his treatise on complex numbers as points in a plane, establishing modern notation and terminology.[1]

(2)Indistinguishability of Identical Particles in Quantum Mechanics.

Identical particles cannot be distinguished by means of any inherent property, since otherwise they would not be identical in all respects. In classical mechanics, identical particles do not lose their "individuality", despite the identity of their physical properties: the particles at some instant can be "numbered" and we can follow the subsequent motion of each of these in its paths. So, at any instant the particles can be identified

In Quantum Mechanics(QM), there is no possibility of separately following during the motion each one of the similar particles and thereby distinguishing them. That is, in QM identical particles entirely lose their individuality, resulting in the complete indistinguishability of these particles.[5-8] This fact is called **"Principle of Indistinguishability of Identical Particles"** and plays a fundamental role in the QM of identical particles.[5-8]

(3)Quantum System of Identical Particles.

Let us consider an isolated system with total energy E composed by a constant number N of identical particles which is described by Quantum Mechanics. If H is the Hamiltonian operator of the system, the energy eigenfunction Ψ, obeys the equation $H\Psi = E\,\Psi$. The operator H and Ψ are functions of $\mathbf{x}_1,\xi_1,\ldots,\mathbf{x}_N,\xi_N$, where $\mathbf{x}_j$ and ξ_j denote the position coordinate and the intrinsic properties(spin, for instance), respectively, of the j^{th} particle. The pair $(\mathbf{x}_j,\xi_j)$ is indicated by a single number *j* (j =1,2,...,N), named particle configuration. The set of all configurations generates the *configuration space* $\varepsilon^{(N)}$. So, we can write simply $H = H(1,2,\ldots,N)$ and $\Psi = \Psi(1,2,\ldots N)$.

(4)Fundamental Theorem for Systems of Identical Quantum Particles.

We present here only a brief review of the detailed calculations, shown elsewhere,[9,10] done to demonstrate what could be thought as a *Fundamental Theorem* for systems of identical quantum particles.

Let us define by P_i the "permutation operator"(i = 1,2,...N!) which generate all possible permutations of the N particles in the space $\varepsilon^{(N)}$. Since the particles are identical the physical properties of the system must be invariant by permutations. The permutations P_i of the labels 1,2,..,N constitute a symmetric group [11-13, S_N called Permutation Groups or Symmetric Group of order n =N! Due to the identity of the particles, H and Ψ obtained by merely permuting the particles must be equivalent physically, that is, $[P_i, H] = 0$ and $|P_i\,\Psi|^2 = |\Psi_i|^2 = |\Psi|^2$. This implies that the permutations are unitary transformations and that the energy E spectrum is n times degenerate. As the functions $\{\Psi_i\}_{i=1,2..,n}$ are different and orthogonal they are associated in one-by-one correspondence an unitary operator $U(P_i)$ so that $\Psi_i = U(P_i)\Psi$. The functions $\{\Psi_i\}_{i=1,2..,n}$ form a n-dim Hilbert space $L_2(\varepsilon^{(N)})$ of all square integrable function over $\varepsilon^{(N)}$. Taking into account the orthogonal eigenfunctions $\{\Psi_i\}_{i=1,2..,n}$ and the Group Theory[11-15] we can obtain the energy eigenfunctions Ψ of systems with N particles. These can be represented by *horizontal*, *intermediate* and *vertical Young Shapes* (or *diagrams*) that can be seen elsewhere.[11-13,5]

Systems represented by *horizontal diagrams* are described by the *symmetric state functions* Ψ_S. Systems represented by *vertical diagrams* are described by the *antisymmetric state functions* Ψ_A. These functions are written, respectively, by

$$\Psi_S = (1/\sqrt{N!}) \sum_{i=1}^{N} \Psi_i \qquad \text{and} \qquad \Psi_A = (1/\sqrt{N!}) \sum_{i=1}^{N} \delta_{Pi}\,\Psi_i \qquad (4.1),$$

where, $\delta_{Pi} = \pm 1$, if P_i is an even or odd permutation, respectively.

Particles represented by the *intermediate* Young diagrams, that is, not horizontals or verticals, are described by *state functions* $\Psi(\alpha) = Y(\alpha)$ where α indicates the *partitions* of the number N:[11-13]

$$Y(\alpha) = \frac{1}{\sqrt{\tau}}\begin{pmatrix} Y_1(\alpha) \\ Y_2(\alpha) \\ \vdots \\ Y_\tau(\alpha) \end{pmatrix} \qquad (4.2),$$

where τ is the dimension of the Hilbert space which corresponds to a given partition α. In **Appendix** are shown Young diagrams for 2, 3 and 4 particles and the corresponding state functions for 2 and 3 particles.

Particles represented by horizontal diagrams, described by *symmetric state functions* Ψ_S, are called BOSONS.[5-8] On the other hand, Particles described by *antisymmetric state functions* Ψ_A and represented by vertical diagrams are called FERMIONS.[5-8] In to our preceding papers,[9,10] particles represented by intermediate diagrams and described by *intermediate state functions* Y, were named *Gentileons* or *intermediate particles.*

(5)Conclusions.

Free Bosons and Fermions are observed in nature and properties of systems composed by these particles are very well known.[5-8] On the other hand, as commented in preceding papers,[9,10] systems of Gentileons were never observed in nature. Similarly, as occurred in the past with the negative and complex numbers we could assume that Gentileons are **unreal, absurdi or "imaginaire"**. For numbers this dilemma was solved after established that the use of complex numbers is **unavoidable** with the demonstration of the *Fundamental Theorem of Algebra*. The same thing would occur with Gentileons. Indeed, as rigorously shown,[9,10] within the framework of the QM and the permutation group theory, there are state functions Y describing these new kinds of particles. Thus, as happened with the complex numbers, these states would be **unavoidable.**

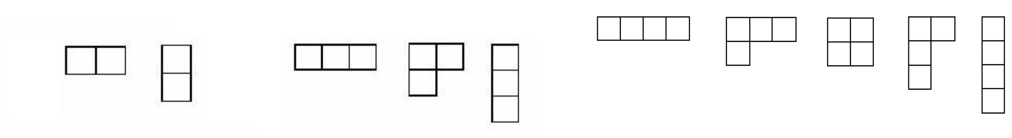
Why Gentileons have not been detected? Probably, because (A)up to now, experimental techniques are not sufficiently accurate or adequate for its detections or (B) there are unknown selection rules in nature which forbid their existence.

Acknowledgements. *The author thanks the librarian Virginia de Paiva for the invaluable assistance for publication of this paper.*

APPENDIX

(1)Young Diagrams for N= 2, 3 and 4 particles

N = 2 **N =3** **N= 4**



(2)State functions for N = 2 and 3.

Let us indicate by $e_1 = \Psi(1,2)$ and $e_2 = P(1,2)\Psi(1,2) = \Psi(2,1)$, where P(i,j) is the permutation operator, the unit vector basis of the 2-dimension Hilbert space $L_2(\varepsilon^{(2)})$. Similarly, by $e_1=\Psi(1,2,3)$,$e_2= \Psi(1,3,2)$, $e_3 = \Psi(2,1,3)$, $e_4 = \Psi(2,3,1)$, $e_5 = \Psi(3,1,2)$ and $e_6 = \Psi(3,2,1)$ the unit vector basis of the 6-dimension Hilbert space $L_2(\varepsilon^{(3)})$ obtained by the permutations P_ie_1 (i = 1,2,..,6). Detailed calculations of these functions are seen elsewhere.[9,10]

N = 2

The symmetric Ψ_S and anti-symmetric Ψ_A normalized state functions are written as:

$$\Psi_S = (e_1 + e_2)/ \sqrt{2} \quad \text{and} \quad \Psi_A = (e_1 - e_2) / \sqrt{2} \qquad (A.1).$$

N = 3

For **N= 3** we have the following state functions Ψ:

$$\Psi_S = (e_1 + e_2 + e_3 + e_4 + e_5 + e_6)/\sqrt{6}$$

$$\Psi_A = (e_1 - e_2 - e_3 - e_4 + e_5 + e_6)/\sqrt{6} \qquad (A.2)$$

Intermediate state function Y([2,1)]

$$Y([2,1]) = \frac{1}{\sqrt{4}}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} Y_+ \\ Y_- \end{pmatrix} \qquad (A.3)$$

where $Y_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $Y_- = \frac{1}{\sqrt{2}}\begin{pmatrix} Y_3 \\ Y_4 \end{pmatrix}$, and

$$Y_1 = (e_1 + e_3 - e_4 - e_6) /\sqrt{4}$$

$$Y_2 = (e_1 + 2e_2 - e_3 + e_4 - 2e_5 - e_6)/ \sqrt{12} \qquad (A.4)$$

$$Y_3 = (- e_1 + 2e_2 - e_3 - e_4 + 2e_5 - e_6)/ \sqrt{12}$$

$$Y_4 = (e_1 - e_3 - e_4 + e_6) /\sqrt{4}$$

The statefunctions Ψ_S ,Ψ_A and $\{Y_i\}_{i=1,\dots,4}$ are orthonormal, that is, $< f_n | f_m > = \delta_{nm}$, where n, m = S, A, 1,2,3 and 4. From these orthonormal properties we can easily verify that

$$|< Y | Y >|^2 = (|Y_1|^2 + |Y_2|^2 + |Y_3|^2 + |Y_4|^2)/4 \quad \text{and that}$$

$$|<Y_+|Y_+>|^2 = (|Y_1|^2 + |Y_2|^2)/2 = (|Y_3|^2 + |Y_4|^2)/2 = |<Y_-|Y_->|^2$$

From the Eqs.(A.3)-(A.4) we see that the 4-dim subspace h([2,1]), which corresponds to the *intermediate Young shape* [2,1], breaks up into two 2-dim subspaces, $h_+([2,1])$ and $h_-([2,1])$, that are spanned by the basis vectors $\{Y_1,Y_2\}$ and $\{Y_3,Y_4\}$, respectively. To these subspaces are associated the wavefunctions $Y_+([2,1])$ and $Y_-([2,1])$ defined by Eq.(A.4). There is no linear transformation which connects the vectors Y_+ and Y_-.

Quarks as Gentileons. In preceding papers[.16,17]assuming that quarks are Gentileons, we have written protons and neutrons state functions taking into account the functions Y, seen above. The confinement of quarks was also investigated.

REFERENCES.

[1]L.Rogers."The History of Negative Numbers". https://nrich.maths.org/5961

[2]https://en.wikipedia.org/wiki/Complex_number

[3]C.Cerri e M.S.Monteiro "História dos Numeros Complexos" https://www.ime.usp.br/~martha/caem/complexos.pdf

[4]C.Nery e J.Jakubovic. Curso de Matemática vol.3. Ed.Moderna (1987).

[5]A.S.Davydov."Quantum Mechanics".Pergamon Press (1965).

[6]L.D.Landau and E.M.Lifschitz . "Quantum Mechanics". Pergamon Press(1962).

[7]L. I. Schiff. "Quantum Mechanics".McGraw-Hill Book Company(1955).

[8]E. Merzbacher. "Quantum Mechanics." John-Wiley & Sons, Inc.(1961).

[9] M.Cattani."The Simplest Gentilionic Systems". Acta Phys. Pol. B20(1989). "Quantum Statistics: The Indistinguishability Principle and the Permutation Group Theory." Revista Brasileira de Ensino de Física. 29,1(2007).

[10]M.Cattani and J.M.F.Bassalo."Intermediate Statistics, Parastatistics, Fractionary Statistics and Gentileonic Statistics". arXiv:0903.4773 (2009).

[11]H. Weyl. "The Theory of Groups and Quantum Mechanics". E.P. Dutton and Company Inc.(1932).

[12]D.E. Rutherford. "Substitutional Analysis" .The Edinburgh University Press(1948)

[13]M.Hamermesh."Group Theory and its Application to Physical Problems" Addison-Wesley Publishing Company, Inc.(1962).

[14]P.Roman."Theory of Elementary Particle" North-Holland Publishing Company(1960).

[15]J.M.F.Bassalo and M.Cattani ."*Teoria de Grupo e Algumas Aplicações em Física.*" (Editora Universidade Federal do Pará(2005).

[16]M.Cattani and N.C.Fernandes. "Baryon States as Colorspinors in Gentilionic Statistics". Phys.Lett.A124,229(1987).

[17]M.Cattani."On the Confinement of Gentileons" http://publica- sbi.if.usp.br/PDFs/pd1660.pdf (2011).