Zero-Point Energy, Quantum Vacuum and Quantum Field Theory for Beginners

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Abstract. This is a didactical paper written to graduate students of Physics showing some basic aspects of the Quantum Vacuum: Zero-Point Energy, Vacuum Fluctuations, Lamb Shift and Casimir Force.

(1) Zero-Point Energy.

Graduate students of Physics learned that in 1900,[1,2] Max Planck explained the "black body radiation" showing that the average energy ε of a single energy radiator inside of a resonant cavity, vibrating with frequency ν at a absolute temperature T is given by

$$\varepsilon = h\nu/(e^{h\nu/kT} - 1) \qquad (1.1),$$

where h is the Planck constant and k the Boltzmann constant. Later, in 1912 published a modified version of the quantized oscillator introducing a residual energy factor hν/2, that is, writing

$$\varepsilon = h\nu/2 + h\nu/(e^{h\nu/kT} - 1) \qquad (1.2).$$

In this way, the term hν/2 would represent the residual energy when $T \rightarrow 0$. It is widely agreed that this Planck´s equation marks the birth of the concept of "**zero-point energy**"(**ZPE**) of a system. It would contradict the fact that in classical physics when $T \rightarrow 0$ all motion ceases and particles come completely to rest with energy tending to zero.

In addition, according to Eq.(1.2) taking into account contributions of all frequencies, from zero up to infinite, the ZPE would have an infinite energy! Many physicists have made a clear opposition to the idea of the ZPE claiming that infinite energy has no physical meaning.

Anyway, the ZPE attracted attention of many physicists and during many years and it was analyzed in different contexts like, for instance, atomic and molecular physics, condensed matter at very low temperatures and general relativity.[2] In 1916 Nernst proposed[2] that the empty space, that is, the **Vacuum** was filled with the ZPE.

(2) Vacuum Fluctuations(VF).

After the propositions of the Heisenberg matrix mechanics[3] in 1925 and the Schrödinger´s equation[3] in 1926 physicists were able to get only a rough idea of the ZPE.[2] More clear understanding was only possible after the construction of relativistic quantum equations by Klein and Gordon in 1926[4] to spin zero particles and by Dirac in 1927[5,6,] to electrons and other spin -1/2 particles. They predicted positive and negative energy states of free particles. The interference between these states create unexpected effects known as "Zitterbewegung"[7,8] and "Klein Paradox."[7] Shortly thereafter, trying to explain the "spontaneous emission" Dirac proposed[2,3,8] that the electromagnetic field in the vacuum is composed by an ensemble of harmonic oscillators introducing the concept of creation and annihilation operators of particles. So, the vector potential **A(r**,t), inside a box of volume V, would be written (see **Appendix A**):[3,9]

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda}[2\pi\hbar c^2/\omega_{\mathbf{k}}V]^{1/2}\,[a_{\mathbf{k}\lambda}\, e^{-i(\omega_{\mathbf{k}}\mathbf{t}\,-\,\mathbf{k}.\mathbf{r})} + a^{+}_{\mathbf{k}\lambda}\, e^{i(\omega_{\mathbf{k}}\mathbf{t}\,-\,\mathbf{k}.\mathbf{r})}]\,\mathbf{e}_{\mathbf{k}\lambda} \qquad (2.1),$$

where $a_{\mathbf{k}\lambda}$ and $a^{+}_{\mathbf{k}\lambda}$ are the photon annihilation and creation operators, respectively, for the wave vector **k,** polarization λ and $\mathbf{e}_{\mathbf{k}\lambda}$ the unit vector polarization of the electromagnetic field:

$$[a_{\mathbf{k}\lambda}(t),\, a^{+}_{\mathbf{k}'\lambda'}(t)] = \delta^{3}_{\mathbf{k},\mathbf{k}'}\,\delta_{\lambda,\lambda'} \text{ and } [a_{\mathbf{k}\lambda}(t),\, a_{\mathbf{k}'\lambda'}(t)] = [a^{+}_{\mathbf{k}\lambda}(t), a^{+}_{\mathbf{k}'\lambda'}(t)] = 0 \quad (2.2),$$

where $a_{\mathbf{k}\lambda}$ and $a^{+}_{\mathbf{k}\lambda}$ obey the *bosonic* commutation relations.

Thus, the spontaneous emission would be light quanta radiation induced by zero oscillations of the empty space. In this way, this theory gives reality to the ZPE and shows that the spontaneous emission is started by electromagnetic fluctuations, that is, by **Vacuum Fluctuations(VF).** This theory, "Quantum Electrodynamics"(**QED**), predicts the existence of "fluctuating zero-point" or "vacuum" field even in absence of sources.

In the electromagnetic domain VF are confirmed, for instance, in spontaneous emissions,[2] Lorentz force,[2,10] Lamb shift[11] and measurement of the magnetic moment of the electron.[12]Note that the Casimir effect[12-14] can be estimated taking into account the quantized electromagnetic energy of the vacuum, without reference to VF.[15]

In recent paper,[10] we have shown how the motion of a charge submitted to a "fluctuating vacuum" and to a Lorentz force is related with the "Fluctuation-Dissipation Theorem"(**FDT**).[16-18] According to Dirac´s

equation predictions the $2S_{1/2}$ and $2P_{1/2}$ levels of the hydrogen atom would have the same energy.[6] However, measurements performed by Lamb and Retherford in 1947[19] of the hydrogen microwave spectrum have shown that the difference between these levels was about 1040 megacycles. This difference is named "Lamb shift". In **Section 3** will be briefly shown how to estimate this effect taking account two approaches. One heuristic approach [20,21] taking into account the stochastic VF and another using a quantum mechanical perturbation method adopted by Bethe in 1947.[11] In **Section 4** is seen how to calculate the Casimir force between two metallic plates in vacuum.

(3) Lamb Shift.

(3.1) Heuristic estimation.

As seen elsewhere [2,21] the fluctuations in the electric and magnetic fields associated with the QED vacuum (see **Section 2**) perturbs the proton electric potential in the hydrogen atom. This perturbation causes a fluctuation in the position of the electron. So, this create a difference of potential energy ΔV given by,

$$\Delta V = V(\mathbf{r} + \boldsymbol{\delta r}) - V(\mathbf{r}) = \boldsymbol{\delta r}.\boldsymbol{\nabla}(V(\mathbf{r})) + (1/2)\,(\boldsymbol{\delta r}.\boldsymbol{\nabla})^2 V(\mathbf{r}) + \ldots \quad (3.1.1).$$

Since the VF are isotropic,

$$\langle \boldsymbol{\delta r} \rangle_{vac} = 0\ ,$$

$$\langle (\boldsymbol{\delta r}.\boldsymbol{\nabla})^2 \rangle_{vac} = (1/3)\,\langle (\boldsymbol{\delta r})^2 \rangle_{vac}\, \nabla^2(\ldots). \quad (3.1.2)$$

So, one can obtain,

$$\langle \Delta V \rangle = (1/6)\,\langle (\boldsymbol{\delta r})^2 \rangle_{vac}\ \left\{ \langle \nabla^2(-e^2/4\pi\varepsilon_o r) \rangle \right\}_{atom}. \quad (3.1.3).$$

Fluctuations of the electric field $\mathbf{E}_{k\lambda}(\mathbf{r},t) = -(1/c)\,\partial\mathbf{A}_{k\lambda}(\mathbf{r},t)/\partial t$, where $\mathbf{A}_{k\lambda}(\mathbf{r},t)$ is given by Eq.(2.1), will produce electron displacements that will be indicated, for simplicity, only by $(\boldsymbol{\delta r})_{\mathbf{k}}$. Let us assume that the electron displacement is induced by a single mode **k** and frequency ν and, in addition, that the fluctuations obey the classical equation

$$m\,d^2[(\delta r)_{\mathbf{k}}]/dt^2 = -e\,E_k \quad (3.1.4),$$

where m is the electron mass.

This equation is valid only when the frequency ν is greater than ν_o in the Bohr orbit, that is, $\nu > \nu_o = \pi c/a_o$. The electron is unable to respond to the fluctuating field if $\nu < \nu_o$. Thus, for this oscillating frequency we can put

$$\delta\mathbf{r}(t) \approx \delta\mathbf{r}(0)e^{-ivt} + c.c. \qquad (3.1.5),$$

where,

$$(\delta\mathbf{r})_\mathbf{k} \approx (e/\mu c^2k^2)\, E_\mathbf{k} = (e/mc^2k^2)\, C_\mathbf{k} \{a_\mathbf{k} \exp(-i\nu_k t + \mathbf{k.r}) + h.c.\} \quad (3.1.6),$$

$C_\mathbf{k} = (\hbar ck/2\varepsilon_o V_{ol})$ and V_{ol} is a large normalization volume of a "box" containing the hydrogen atom. Summing the expected values $< 0|(\delta\mathbf{r})^2_\mathbf{k}|0 >$, where $|0 >$ is the vacuum state, over all $\mathbf{k}$ contributions we get

$$<(\delta\mathbf{r})^2>_{vac} = \sum_\mathbf{k} (e/mc^2k^2)^2 < 0|\, (\mathbf{E}_\mathbf{k})^2\, |0 >$$

$$= \sum_\mathbf{k} (e/mc^2k^2)^2\, (\hbar ck/2\varepsilon_o V_{ol}) \qquad (3.1.7),$$

since only the combination $a_{\mathbf{k}\lambda}a^+_{\mathbf{k}\lambda}$ will give a non vanishing contribution. As, for very large volume V_{ol} we can put $\sum_\mathbf{k} \rightarrow 2[V_{ol}(2\pi)^3]\int_o^\infty d^3\mathbf{k}$, Eq.(3.1.7) becomes written as

$$<(\delta\mathbf{r})^2>_{vac} = (2V_{ol}/\pi^2)\int_o^\infty k^2dk(e/mc^2k^2)^2\, (\hbar ck/2\varepsilon_o V_{ol})$$

$$= (e^2/2\pi^2\varepsilon_o\hbar c)(\hbar/mc)^2 \int_o^\infty dk/k \qquad (3.1.8).$$

This result diverges when no limits are imposed on the integral, at large and small frequencies. As mentioned above, the frequencies must obey the condition $\nu > \pi c/a_o$, that is, $k > \pi/a_o$. On the other hand, $k = 2\pi\nu/c$ must be taken only for wavelengths longer than the Compton wavelength = $\hbar/mc$, that is, for $k < mc/\hbar$. With these extreme limits $k_{min} = \pi/a_o$ and $k_{max} = mc/\hbar$ in Eq.(3.1.8) and putting $a_o = 4\pi\varepsilon_o\hbar^2/me^2$ we obtain,

$$<(\delta\mathbf{r})^2>_{vac} \approx (e^2/2\pi^2\varepsilon_o\hbar c)(\hbar/mc)^2 \ln(4\varepsilon_o\hbar c/\, e^2) \qquad (3.1.9).$$

For the atomic orbital and Coulomb potential, from Eq.(3.1.3),

$$\{< \nabla^2(-e^2/4\pi\varepsilon_o r) >\}_{at} = -(e^2/4\pi\varepsilon_o) \int d\mathbf{r}\, \psi^*(\mathbf{r})\, \nabla^2(1/r)\, \psi(\mathbf{r}) = (e^2/\varepsilon_o)\, |\psi(0)|^2$$

remembering that $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$.

The **Lambshift**[11,21]. is given by the energy difference between the two hydrogen levels $^2S_{1/2}$ and $^2P_{1/2}$. As last one vanishes at the origin, only $^2S_{1/2}$ contributes to the shift. So, as $\psi_{2s}(0) = \sqrt{1/8\pi a_o^3}$ we get:

$$\{< \nabla^2(-e^2/4\pi\varepsilon_o r) >\}_{at} = (e^2/\varepsilon_o)\,|\psi_{2s}(0)|^2 = (e^2/8\pi\varepsilon_o\, a_o^3) \qquad (3.1.10).$$

Finally, using (3.1.3), (3.1.9) and (3.1.10), the resulting energy shift $\Delta E_{2s} = < \Delta V >_{2s}$ is given by

$$\Delta E_{2s} = (\alpha^5 mc^2/6\pi)\ln(1/\pi\alpha) \qquad (3.1.11),$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant. According to (3.1.11) the estimated Lamb shift would be $\Delta E_{2s} \approx 1000$ MHz, in fair agreement with the observed value $\{\Delta E_{2s}\}_{exp} \approx 1058$ MHz.[22]

(3.2) Bethe calculation.

A good estimation of the "Lamb Shift" was obtained by Bethe[11] using a noncovariant quantum theory[8,20,21] of the radiation assuming that the electron interaction $W_I(t)$[3] with the VF is given by, neglecting the second order term $e^2\mathbf{A}^2(r,t)$,

$$W_I(t) = -(e/mc)(\mathbf{A}^T.\,\mathbf{p}) \qquad (3.2.1),$$

where e and m are the charge and mass of the electron, respectively and c the light velocity. In Eq.(3.2.1) $\mathbf{A}^T(r,t)$ is the vector potential operator given by Eq.(2.1) where only transverse components are considered and **p** is momentum operator defined by $\mathbf{p} = -i\hbar\,\nabla(\ )$.

In what follows the atomic electron states are indicated by m and n. Taking into account Eq.(3.2.1) and following the usual perturbation theory procedure (see details in reference 20), since only $a_{\mathbf{k}\lambda}\, a_{\mathbf{k}\lambda}^{+}$ will give a nonvanishing contribution, we get the self-energy ΔE^*_α of the electron in a quantum atomic state m,[11,20.21]

$$\Delta E^*_\alpha = -(2e^2/3\pi\hbar c^3)\int_o^{\varepsilon^*} \varepsilon\, d\varepsilon \sum_{\alpha\beta} |v_{\alpha\beta}|^2/(E_\beta - E_\alpha + \varepsilon) \qquad (3.2.2),$$

where P means the "principal value" of the integral, $\varepsilon = \hbar\omega$ is the energy of the quantum, $\varepsilon^* \approx mc^2$ is a "natural cut-off" for the photon energies ε the and $\mathbf{v} = \mathbf{p}/m = (\hbar/im)\nabla$. For the ε extreme limits, relativistic corrections would be necessary but they have not been carried out.[20,21]

For a free electron the interaction with the VF gives origin to a self-energy ΔE_o representing the change of the kinetic energy of the electron for

a fixed momentum, adding the electromagnetic mass to the mass of electron. So, ΔE_o is estimated by

$$\Delta E_o = -(2e^2/3\pi\hbar c^3) \int_o^{\varepsilon^*} \varepsilon \, d\varepsilon \, \mathbf{v}^2/\varepsilon \qquad (3.2.3).$$

For a bound electron, $\mathbf{v}^2$ should be replaced by the expectation value, $(\mathbf{v}^2)_{\alpha\alpha}$, remembering that the matrix elements of $\mathbf{v}$ satisfy the sum rule $\sum_\beta |\mathbf{v}_{\alpha\beta}|^2 = (\mathbf{v}^2)_{\alpha\alpha}$. As this electromagnetic contribution is already contained in the experimental electron mass m considered in Eq.(3.2.2) it must be disregarded. Therefore, the relevant part of the electron self-energy in the m state ΔE_m will be given by

$$\Delta E_\alpha = \Delta E^*_\alpha - \Delta E_o = (2e^2/3\pi\hbar c^3) \int_o^{\varepsilon^*} d\varepsilon \sum_\beta |v_{\alpha\beta}|^2 (E_\beta - E_\alpha)/(E_\beta - E_\alpha + \varepsilon) \quad (3.2.4)$$

It is convenient to integrate Eq.(3.2.4) first over ε. Assuming that ε* to be large compared with all energy differences $E_n - E_m$ in the atom,

$$\Delta E_\alpha = (2e^2/3\pi\hbar c^3) \sum_\beta |v_{\alpha\beta}|^2 (E_\beta - E_\alpha) \ln\{\varepsilon^*/|E_\beta - E_\alpha|\} \qquad (3.2.5).$$

If $E_\beta - E_\alpha$ is negative the Principal Value of integral must be taken. As $\varepsilon^* \approx mc^2$ the argument in the logarithm is very large and it can be taken independent of β and substituted by $\ln\{\varepsilon^*/|E_\beta - E_\alpha|_{Av}\}$ obtaining, [20,21]

$$A = \sum_\beta A_{\beta\alpha} = \sum_\beta |p_{\alpha\beta}|^2 (E_\beta - E_\alpha) = 2\pi\hbar^2 e^2 Z \psi_\alpha^2(0) \qquad (3.2.6),$$

where Z is the nuclear charge and $\psi_\alpha^2(0) = \psi_{n\ell}^2(0)$. For any electron with angular momentum $\ell \neq 0$ the wavefunction $\psi_{n\ell}^2(0) = 0$; therefore the sum A = 0. For a state with $\ell = 0$, however,

$$\psi_{n0}^2(0) = (Z/na_o)^3/\pi \qquad (3.2.7),$$

where n is the principal quantum number and a_o is the Bohr radius.

The **Lamb shift = LS** $= \Delta E_{2s} - \Delta E_{2p}$ is given by the energy difference between the two hydrogen levels: $\psi_{2s}(\mathbf{r})$ and $\psi_{2p}(\mathbf{r})$. As last one vanishes at the origin, only $\psi_{2s}(\mathbf{r})$ contributes to the shift. As $\psi_{2s}(0) = 0$, only the level 2s will contribute to the shift, that is, we have only the energy shift ΔE_{2s}. So, according to Eq.(3.2.7) we get

$$\mathbf{LS} = \Delta E_{2s} = (e^4/12\pi^2 m^2) |\psi_{2S}(0)|^2 . \ln\{mc^2/|E_\beta - E_\alpha|_{Av}\}$$

$$\approx (e^4/12\pi^3 m^2 a_o^3) \ln(\hbar^2 c^3/24e^4)/ n^3 \qquad (3.2.8),$$

taking into account that $|E_\beta - E_\alpha|_{Av} \approx 24m(e^2/\hbar c)^2$ for the 2S state

So, with Eq.(3.2.8) we find, putting n = 2, **LS** = $\Delta E_{2s} \approx 1047$ MHz which agrees rather well with the experimental value 1057.77 ± 0.10. Mhz.

Note that this good agreement between theory and experiment occurs because relativistic effects are very small.[20,21]

(4) Casimir Force.

Note that the Casimir force,[13,23,24] as will be seen now, can be estimated taking into account only the quantized electromagnetic energy of the vacuum, without reference to VF.[13,14]

Let us consider a pair of metal conducting plates at a distance L apart in vacuum with very large dimensions along the x and y-axes. In this case the vacuum energy < E > of the standing waves (**Appendix A**) is written as

$$< E > = \sum_n (1/2)\hbar\omega_n = (\hbar/2) \sum_n \omega_n = (\hbar/2) \{\sum\nolimits^{*} \omega_n \} \qquad (4.1),$$

where $\sum^*$ means a summation over all possible excitation modes with n = 1,2,...,∞.

$$\omega_n = c(K_x^2 + K_y^2 + n^2\pi^2/L^2)^{1/2} \quad ,$$

K_x and K_y are the wave numbers in directions parallel to the plates and $K = n\pi/L$ is the wave number perpendicular to the plates.

Since the area A of the plates is very large, we may sum integrating over two of the dimensions (x and y) of the K-space. The assumption of the periodic boundary conditions yields,

$$< E > = (\hbar/2)2(A/2\pi^2)\iint dK_x\, dK_y \sum_n \omega_n \qquad (n = 1,2,...,\infty) \quad (4.2),$$

where A is the area of the plates and the factor 2 is due to two possible polarizations of the wave. Switching to polar coordinates, Eq.(4.2) becomes, putting $dK_x\, dK_y = 2\pi q^2 dq$,

$$< E > = A(\hbar c/4\pi^2) \sum_n \int_0^\infty 2\pi\, q dq\, [q^2 + n^2\pi^2/L^2]^{1/2} \qquad (4.3).$$

From Eqs.(4.2) or (4.3) it seems that $< E > \to \infty$. However, from the physical point of view we must expect that < E > is finite. In "old school" derivation of < E >, to overcome this difficult, faithful cutoff functions, based in sound physical intuition were used.[23] In modern derivations of

< E > we learned how to overcome using mathematical *regularization transformations*, shown in **Appendix B.** According to this method Eqs.(4.2) and (4.3) are written as, where α is a complex number:

$$\lim_{\alpha\to 0} < E(\alpha) > = (A\hbar/4\pi^2)\lim_{\alpha\to 0} \{\iint dK_x\, dK_y \sum_n \omega_n |\omega_n|^{-2\alpha}\}$$

$$= (\hbar cA/2\pi)\lim_{\alpha\to 0} \{\int_0^{\infty} y(y^2+1)^{1/2-\alpha} \sum_n (n\pi/L)^{3-2\alpha}\} dy \quad (4.4),$$

switching to polar coordinates and putting $y = Lq/n\pi$.

Taking into account that the *improper integral* in y is equal to 1/3, as shown in **Appendix B**, Eq.(4.4) becomes written as

$$< E(s) > = (\hbar c\pi^2 A/6)(1/L^3)\, \lim_{\alpha\to 0}\{\sum_n 1/n^{2\alpha-3}\}$$
$$= (\hbar c\pi^2 A/6)(1/L^3)\, \lim_{\alpha\to 0} \zeta(s) \quad (4.5),$$

where $s \equiv 2\alpha-3$ and $\zeta(s)$ is the *Riemann zeta function*[26] (see **Appendix B).** In this way, Eq.(4.5) becomes,

$$< E(s) > = (\hbar c\pi^2 A/6)(1/L^3)\{\lim_{\alpha\to 0} \xi(2\alpha -3)\} = - (\hbar c\pi^2 A/6L^3)\, \xi(-3) \quad (4.6)$$

Since $\xi(-3) = 1/120$, the final energy is given by

$$< E > = (\hbar c\pi^2 A/720)(1/L^3) \quad (4.7),$$

which is exactly the same result derived by Casimir.[13,23,24].

From Eq.(4.7) we see that there is an **attractive** "Casimir force", per unit area F_c/A between two perfectly conducting idealized plates, in vacuum, given by

$$F_c/A = d(< E >/A)/dL = - \hbar c\pi^2/(240L^4) \quad (4.8),$$

which is very small, since it is proportional to ħ, that measured in the CGS system is given by,

$$F_c/A = -0.013/L_\mu{}^4 \text{ dyne/cm}^2 \quad (4.9),$$

where L_μ is the distance in microns between the plates. For instance, the force F acting on a 1,0 x1,0 cm plates separated by 1μm apart is F = -0.013 dyne. This is comparable to the Coulomb force on the electron in hydrogen atom, the gravitational force between 1/2 kg weights separated by 1,0 cm, or about 1/1000 the weight of a housefly.[13] Since this force is very small, only in 2001 it was precisely measured with success by G. Bressi et al.[24] between two plates.

APPENDIX A. Electromagnetic Vector Potential A(r,t).

We mention here only few properties of the vector potential that will useful to our brief review on Quantum Field Theory. So, since the field **A**(**r**,t) in a free space inside a metallic cubic box of volume $V = L^3$ obeys the Helmholtz equation $(lapl + \mathbf{K}^2)\mathbf{A}(\mathbf{r},t) = 0$ it can be written as

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda} [2\pi\hbar c^2/\omega_{\mathbf{k}} V]^{1/2}\{ a_{\mathbf{k}\lambda}(0)\exp[i(\mathbf{K}.\mathbf{r} - \omega_{\mathbf{k}}t -)] + a^{+}_{\mathbf{k}\lambda}(0)\exp[-i(\mathbf{K}.\mathbf{r}- \omega_{\mathbf{k}}t)\}\ e_{\mathbf{k}\lambda} \quad (A.1),$$

where $e_{\mathbf{k}\lambda}$ is the polarization vector with $\lambda = 1,2$, $\omega_{\mathbf{k}} = Kc$ and $a_{\mathbf{k}\lambda}$,$a^{+}_{\mathbf{k}\lambda}$ are the photon annihilation and creation operators for the mode with the wave vector **k** and polarization λ. In addition,

$$(K_x, K_y, K_z) = (\pi/L)(n_x, n_y, n_z) \qquad (A.2),$$

where n = 0,1,2,....and, for the Coulomb gauge, the transverse gauge condition $\mathbf{K}.\mathbf{e}_{\mathbf{k}} = 0$ is obeyed.

The field Hamiltonian is

$$H = \sum_{\mathbf{k}\lambda} \{\hbar\omega_k (a^{+}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + 1/2)\} \qquad (A.3)$$

and that $\sum_{\mathbf{k}\lambda} (1/2)\hbar\omega_k \rightarrow \infty$ gives the zero-point energy of the vacuum.

Sometimes the exponential functions of Eq.(A.1) are written as $2\pi(kx-\nu t) = (2\pi kx-\omega t) = (Kx - \omega t)$ where, $\mathbf{K} = 2\pi\mathbf{k}$ is the wavenumber. In the quantum mechanical approach we have $(\mathbf{p}.\mathbf{r} - Et)/\hbar = (\mathbf{K}.\mathbf{r} - \omega t)$, where $\mathbf{p} = \hbar\mathbf{k}$ and $E = \hbar\omega_k$. Note that $K = \omega/c$ and $k = \nu/c$, that is, $K\lambda = \omega T$ e $k\lambda = \nu T$. As $\lambda = cT$ we have $K = 2\pi/\lambda$ and $k = 1/\lambda$ and, consequently, $\mathbf{p} = h\mathbf{k}$.

According to Eq.(A.2) $K_n = n\pi/L$ and taking $K = 2\pi/\lambda$ we verify that $\lambda_n = 2L/n$, that is, the wavelengths λ_n of the components decreases as 1/n. In addition, as $E_K = \hbar\omega_K = \hbar Kc$, in a volume V = a.b.c we have

$$E_K = \hbar\omega_K = \hbar|K|c = (\hbar c)(K_x^2 + K_y^2 + K_z^2)^{1/2}$$

$$= (\hbar c)(\pi^2 n^2_x/a^2 + \pi^2 n^2_y/b^2 + \pi^2 n^2_z/c^2)^{1/2} \qquad (A.4).$$

APPENDIX B. Regularization Techniques.

The word **regularize** means to make things regular or acceptable. Regularizations techniques are used to show that some integrals and sums of series, that at a first sight seems to be divergent, are really convergent. We will analyze here only two regularization techniques: (1)*Improper Integrals* and (2)*Riemann Zeta Function* where summation of series is performed in a complex plane.

(B.1) Improper Integrals.[25]

Improper integrals are definite integrals that cover an unbounded area. Not all of them have infinite value. Those that have finite value are called **convergent**, and when it doesn´t are called **divergent.** One type of improper integral are those where at least one of the endpoints is extended to infinity, that is, with **unbounded endpoints**. For example,

$$\int_1^{\infty} dx/x^2 = \lim_{b\to\infty}\{ \int_1^{b} dx/x^2 \} = \lim_{b\to\infty}\{1-1/b \} = 1.$$

Another cases are those with **unbounded function**s

$$\int_o^{1} dx/\sqrt{x} = \lim_{a\to o^+}\{ \int_a^{1} dx/\sqrt{x} \} = \lim_{a\to o^+}\{2-2\sqrt{a} \} = 2.$$

In **Section 4** we have, putting $y^2+1= w$, the improper integral

$$\lim_{s\to o} \{\int_o^{\infty} y(y^2+1)^{1/(2-s)}\, dy \} = \lim_{s\to o} \{\int_1^{\infty} w^{1/(2-s)}\, dw/2 \} =$$
$$= \lim_{s\to o}\{-1/(3-2s)\} = -1/3,$$

(B.2) Riemann Zeta Function.[26]

The Riemann zeta function or Euler-Riemann $\zeta(s)$ is a function of complex variable $s = \sigma + it$. This function can be expressed by

$$\zeta(s) = [1/\Gamma(s)] \int_o^{\infty} x^{s-1}\, dx/(e^x -1) = \sum_1^{\infty} n^{-s} \qquad \text{(B.2.1)},$$

where $\Gamma(s) = \int_o^{\infty} x^{s-1}e^{-x}\, dx$ is a gamma function. It is a function that analytically continues the sum of the Dirichlet series seen in (B.2.1).

Detailed calculations and numerical values of $\zeta(s)$ for some s values are shown in reference [26]. In particular, for nonpositive integers, one has $\zeta(-n) = (-1)^n B_{n+1}/(n+1)$, where B_n are Bernouilli numbers. In the case of the Casimir force, seen in Section 4,we have $\zeta(-3) = 1/120$.

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