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TWO-DIMENSIONAL MASSIVE QUANTUM ELECTRODYNAMICS IN THE UNITARY GAUGE AS A RENORMALIZABLE THEORY

by

M.Gomes. R.Koberle and I.Frenkel
Instituto de Física - Universidade de São Paulo

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ABSTRACT

We discuss two dimensional massive quantum electrodynamics both as a superrenormalizable and as a renormalizable theory, showing their equivalence up to a renormalization. The Green functions are explicitly constructed in zero fermion mass limit.

RESUMO

Discutimos eletrodinâmica quântica massiva em duas dimen - sões tanto como uma teoria superrenormalizâvel como uma teoria renormalizâvel. Mostramos sua equivalência a menos de uma renormalização. As funções de Green são construídas explicitamente no limite em que a massa do fermion tende a zero.

I. INTRODUCTION

The quantum theory of gauge fields has recently received much attention in connection with the unification of electromagnetic and weak interactions. There are also many attempts to incorporate strong interactions in this scheme, the concept of "asymptotic freedom" having played a central role in their endeavour.[1] It is therefore convenient to have a theoretical laboratory at ones disposal in order to study problems connected with gauge invariance. With this idea in mind we discuss 2-dimensional electrodynamics (QED)[2),3)] both as a superrenormalizable and as a renormalizable theory. Although this is only anabelian model, we think it worthwhile to discuss mainly for pedagogical reasons.

One of the peculiar features of 2-dimensional QED is that, due to the fact that the phase space d^2k increases only as k^2 for large k, the theory is renormalizable in the so-called unitary gauge and superrenormalizable in the gauge, which in the fourdimensional world is called renormalizable. The equivalence of these two formulations can be explicity studied. Another advantage is of course the theorie's exact solubility in the zero-fermion-mass limit.

We introduce the usual paraphernalia of Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) perturbation theory[4),5)] in the above mentioned two gauges in sects. II and III. They include the discussion of Ward identities, equations of motion and the zero mass limit. In sect. IV we show the equivalence of the unitary and renornalizable gauge and in sect. V we make contact with the soluble zero mass limit. The conclusions are contained in sect. VI.

II. THE UNITARY GAUGE

Let us consider the 2 dimensional theory specified by the effective Lagrange density

$$\mathcal{L}_{eff} = \frac{i}{2}\bar\psi \overleftrightarrow{\partial\!\!\!/}\psi - M\bar\psi\psi - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}m^2 A'^2 + e'\bar\psi A\!\!\!/'\psi + \frac{g}{2}(\bar\psi\gamma^\mu\psi)^2$$
$$= \mathcal{L}_0 + \mathcal{L}_I \; ; \; \mathcal{L}_I = e'\bar\psi A\!\!\!/'\psi + \frac{g}{2}(\bar\psi\gamma^\mu\psi)^2 \; ; \; F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu \qquad (\mathrm{II}.1)$$

which up to the four-fermion interaction corresponds to massive QED in the so called unitary gauge. The free meson propagator is given by

$$D_{\mu\nu} = \frac{-i}{k^2-m^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right) \qquad (\mathrm{II}.2)$$

Due to the bad asymptotic behaviour of $D_{\mu\nu}$, (II.1) describes in four dimensions a non-renormalizable theory. In two dimensions however $(\bar\psi\gamma_\mu\psi)A'^\mu$ is a super-renormalizable interaction (it has dimension $d=1<2$) and the power counting for a proper graph γ constructed from (II.1) and (II.2) gives

$$d(\gamma) = 2 - \frac{F}{2} - B \; , \quad \begin{array}{l} F: \text{n° of external fermion lines of } \gamma \\ B: \text{n° of external boson lines of } \gamma \end{array} \qquad (\mathrm{II}.3)$$

for the degree function $d(\gamma)$, which measures the superficial divergence of γ. This is the reason for having included the Thirring interaction $(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi)$ in (II.1)[6]; it is necessary in order to have a renormalizable theory. If not present in zeroth order, this coupling would be induced in order e'^2. Thus the theory turns out to be renormalizable, the divergencies of our graphs being either zero or one.

The renormalization scheme we will adopt is a soft version[7),8)] of the BPHZ subtraction procedure. Since it involves changes in the mass parameter m it will be convenient to use the following variables[9)]

$$A_\mu = m A'_\mu$$
$$e = m^{-1} e' \qquad (\mathrm{II}.4)$$

With the definition (II.4) we can rewrite (II.1) and (II.2) as

$$\mathcal{L}_{eff} = \frac{i}{2}\bar\psi\overleftrightarrow{\partial\!\!\!/}\psi - M\bar\psi\psi - \frac{1}{4m^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A^2 +$$
$$+ e\bar\psi A\!\!\!/\psi + \frac{g}{2}(\bar\psi\gamma^\mu\psi)^2 \qquad (\mathrm{II}.5)$$

$$\frac{-i}{k^2-m^2}\left(m^2 g_{\mu\nu} - k_\mu k_\nu\right) \qquad (\mathrm{II}.6)$$

The Green functions of the theory are calculated as a finite part of the Gell-Mann Low formula:

$$G^{(2N,L)}(x_1,\dots x_N; y_1,\dots y_N; z_1,\dots,z_L) = \langle T \prod_{i=1}^{N}\psi(x_i)\prod_{j=1}^{N}\bar\psi(y_j)\prod_{k=1}^{L}A_{\mu_k}(z_k)\rangle =$$
$$= \text{finite part of } {}^{(0)}\langle 0| T\prod_{i=1}^{N}\psi^{(0)}(x_i)\prod_{j=1}^{N}\bar\psi^{(0)}(y_j)\prod_{k=1}^{L}A^{(0)}_{\mu_k}(z_k)\exp i\int d^2x\,\mathcal{L}_I^{(0)}(x)|0\rangle^{(0)} \qquad (\mathrm{II}.7)$$

where the superscript (o) indicates the free fields as specified by $\mathcal{L}_0$. The finite part prescription consists in the application of Zimmermann's forest formula with two generalized Taylor operators[8)] τ^0 and τ^1:

$$\tau^{(0)}F(p,m,M) = F(0,\mu,\mu) \quad \text{for logarithmically divergent graphs}$$
$$p = (p_1, p_2, \dots p_n)$$
$$\tau^{(1)}F(p,m,M) = F(0,0,0) + \qquad (\mathrm{II}.8)$$
$$+ p_i^\mu\left(\frac{\partial F}{\partial p_i^\mu}\right)_{\substack{p=0\\ m=M=\mu}} + M\left(\frac{\partial F}{\partial M}\right)_{\substack{p=0\\ m=M=\mu}} \quad \text{for linearly divergent graphs}$$

The scheme above is adequate for the derivation of homogeneous parametric differential equations and has the advantage that the M and m dependence of the subtraction terms is trivial and zero mass limits can be most easily taken. Since we are interested in the soluble $M \to 0$ limit, this subtraction scheme is very convenient.

Due to our subtraction scheme (II.8), the vertex functions $\Gamma^{(2N,L)}(p_i;q_j;m^2,M,\mu)$ of this model, where p_i and q_j stand for the fermion and meson momenta respectively, satisfy the following normalization conditions

$$\Gamma^{(2,0)}(0;0;0,0,\mu) = 0 \qquad (\text{II}.9)$$

$$\frac{\partial}{\partial M}\Gamma^{(2,0)}(0;0;\mu^2,M,\mu)\Big|_{M=\mu} = -i \qquad (\text{II}.10)$$

$$\frac{\partial}{\partial p^\mu}\Gamma^{(2,0)}(p_\mu;0;\mu^2,\mu,\mu)\Big|_{p=0} = i\gamma_\mu \qquad (\text{II}.11)$$

$$\Gamma^{(4,0)}(0;0;\mu^2,\mu,\mu)\,\delta_{\alpha_1\alpha_2\alpha_3\alpha_4} = +ig \qquad (\text{II}.12)$$

$$\Gamma^{(0,2)}(0;0;\mu^2,\mu,\mu) = 0 \qquad (\text{II}.13)$$

where

$$\delta_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{1}{16}\left(\gamma^\mu_{\alpha_1\alpha_4}\gamma_{\mu\alpha_2\alpha_3} - \gamma^\mu_{\alpha_1\alpha_3}\gamma_{\mu\alpha_2\alpha_4}\right)$$

Observe that the parameters m and M are not the vector meson and fermion physical masses. The fermion physical mass however goes to zero as $M \to 0$.

As we see from (II.3), the two point function of the meson field is only logarithmically divergent. The meson wave function renormalization is therefore finite and accordingly we have not include a counter term of the type

$F_{\mu\nu}F^{\mu\nu}$ in (II.1).

Normal products up to degree $\delta = 2$ are defined as usual.[10] If θ is any combination of the basic fields and its derivatives

of canonical dimension less or equal to two, then the normal product $N_\delta[\theta]$ is defined by

$$\langle T N_\delta[\theta]\,\underline{X}\rangle = \text{finite part of } {}^{(0)}\langle 0|T:\theta^{(0)}:\underline{X}^{(0)}\exp i\int d^2x\,\mathcal{L}^{(0)}_I(x)|0\rangle^{(0)}$$

$$\underline{X} = \prod_{i=1}^{N}\psi(x_i)\prod_{j=1}^{N}\bar\psi(y_j)\prod_{k=1}^{L}A_{\mu_k}(z_k) \qquad (\text{II}.14)$$

With a degree function

$$\delta(\gamma) = \delta - \frac{F}{2} - B \qquad (\text{II}.15)$$

for proper subgraphs containing the special vertex $N_\delta[\theta]$. As we make our subtractions at zero momenta, these normal products satisfy the differentiation formula

$$\partial_\mu\langle T N_\delta[\theta](x)\,\underline{X}\rangle = \langle T N_{\delta+1}[\partial_\mu\theta](x)\rangle \qquad (\text{II}.16)$$

II.1) Equations of Motion and Ward Identities

Equations of motion for the fermion and meson fields and Ward identities can be derived in the standard way.[5] One finds for example

$$\partial^\mu\langle A_\mu(x)\,\underline{X}\rangle = \sum_{i=1}^{L}\partial_{\nu_i}\delta(x-z_i)\langle T\,\underline{X}_{\hat\nu_i}\rangle + e\sum_{j=1}^{N}\left(\delta(x-x_j)-\delta(x-y_j)\right)\langle T\,\underline{X}\rangle \qquad (\text{II}.17)$$

where $\underline{X}_{\hat\nu_\ell} = \prod_{i=1}^{N}\psi(x_i)\prod_{j=1}^{N}\bar\psi(y_j)\,A_{\nu_1}(z_1)\cdots A_{\nu_{\ell-1}}(z_{\ell-1})A_{\nu_{\ell+1}}(z_{\ell+1})\cdots A_{\nu_L}(z_L)$

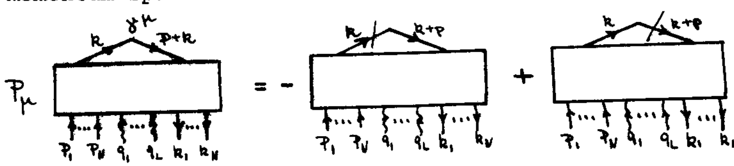
Equation (II.17) can be derived by noting that the line corresponding to the A_μ field can be linked either directly to another meson field (1st term) or to a current vertex (2nd term). In the latter case one uses current conservation expressed by

$$\partial^\mu \langle T N_2(\bar\psi\gamma_\mu\psi)(x)\underline{X}\rangle = \sum_{i=1}^{N} [\delta(x-x_i)-\delta(x-y_i)]\langle T\underline{X}\rangle \qquad (II.18)$$

Equation (II.17) is represented graphically in fig. 1. We sketch derivation of (II.18). First, because of (II.16) we have

$$\partial_\mu\langle T N_1(\bar\psi\gamma^\mu\psi)(x)\underline{X}\rangle = \langle T N_2(\partial_\mu(\bar\psi\gamma^\mu\psi))(x)\underline{X}\rangle \qquad (II.19)$$

Now using the graphical representation for (II.19) in momentum space we have

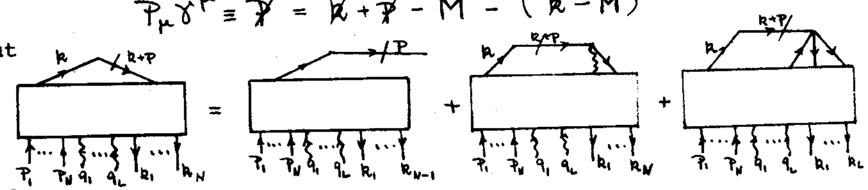


(II.20)

where we used

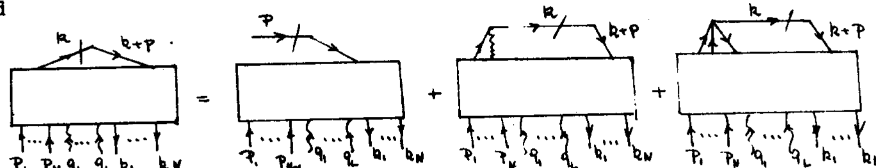
$$p_\mu\gamma^\mu \equiv \not p = \not k + \not p - M - (\not k - M)$$

But



(II.21)

and



(II.22)

and (II.18) follows.

Besides (II.18) we will need the Ward-Takahashi identity for the axial current $N_1(\bar\psi\gamma^\mu\gamma^5\psi)$:

$$\partial^\mu\langle T N_1[\bar\psi\gamma_\mu\gamma^5\psi](x)\underline{X}\rangle = 2i\langle T N_2[M(\bar\psi\gamma^5\psi)](x)\underline{X}\rangle - \sum_{j=1}^{N}[\delta(x-x_j)\gamma^5_{x_j}+\delta(x-y_j)\gamma^{5T}_{y_j}]\langle 0|T\underline{X}|0\rangle \qquad (II.23)$$

which can be shown to be true following the same steps that led to equ. (II.18). (This time however one uses $\not p\gamma^5 = (\not k+\not p-M)\gamma^5+\gamma^5(\not k-M)+2M\gamma^5$.)

We consider now Zimmermann's identity

$$M N_1(\bar\psi\gamma^5\psi) = t\, N_2(M\bar\psi\gamma^5\psi) + r\,\tilde\partial^\mu A_\mu + s\, N_2[\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi)] \qquad (II.24)$$

where $\tilde\partial^\mu = \epsilon^{\mu\nu}\partial_\nu$ and

$$t = 1 - \frac{\mu}{2}\,\mathrm{Tr}\,\gamma_5\left(\frac{\partial}{\partial M}\langle 0|T N_1[\bar\psi\gamma^5\psi](0)\,\tilde\psi(0)\,\tilde{\bar\psi}(0)\rangle^{Prop}\right)\Big|_{M=m=\mu}$$

$$r = i\mu\frac{\partial}{\partial k_\rho}\langle 0|T N_1[\bar\psi\gamma^5\psi](0)\,A_\rho(k)|0\rangle^{Prop}\Big|_{\substack{k=0\\ M=m=\mu}} \qquad (II.25)$$

$$s\,\gamma_\mu\gamma^5 = -i\mu\frac{\partial}{\partial q^\mu}\langle 0|T N_1[\bar\psi\gamma^5\psi](0)\,\tilde\psi(\tfrac{q}{2})\,\tilde{\bar\psi}(\tfrac{q}{2})|0\rangle^{Prop}\Big|_{\substack{q=0\\ M=m=\mu}}$$

This equation can be derived by noting that the difference among vertex functions containing $MN_1(\bar\psi\gamma^5\psi)$ and $N_2(M\bar\psi\gamma^5\psi)$ comes from subtractions for proper graphs that contain these special vertices. For example, graphs with two external fermion lines will require either the application of τ^0 or τ^1, according to wether they contain the degree one or the degree two normal product. This produces an expression of the type

$$M\left(\frac{\partial F(0,\mu,M)}{\partial M}\Big|_{M=\mu}\right) + \not p\left(\frac{\partial F(p,\mu,\mu)}{\partial p}\Big|_{p=0}\right)$$

times the amplitude for the reduced diagram. Since the reduced diagram will have a special vertex with two fermion fields, this will give a contribution to the 1st and the 3rd term in the r.h.s. of (II.24) (Charge conjugation properties have already been applied in order to exclude the vertex $\bar\psi\gamma_\mu\gamma^5\overleftrightarrow{\partial}^\mu\psi$ from (II.24)). The second term in the r.h.s. of (II.24) can be explained by a similar reasoning.

Observe the absence of a four fermion vertex in the r.h.s. of (II.24); as is well known this results from Fermi statistics and specific properties of the two dimensional Dirac matrices. With the information (II.24) and (II.23) we rewrite the axial vector Ward identity as

$$(1-h)\,\partial_\mu\langle T N_1[\bar\psi\gamma^\mu\gamma^5\psi](x)\underline{X}\rangle = \frac{2Mi}{t}\langle T N_1[\bar\psi\gamma^5\psi](x)\underline{X}\rangle + R\langle T\tilde\partial^\mu A_\mu(x)\underline{X}\rangle - \sum_{j=1}^{N}[\delta(x-x_j)\gamma^5_{x_j}+\delta(x-y_j)\gamma^{5T}_{y_j}]\langle T\underline{X}\rangle \qquad (\mathrm{II}.26)$$

with

$$h = \frac{2s}{it}, \qquad R = \frac{2r}{it} \qquad (\mathrm{II}.27)$$

Note that both h and R are mass independent due to (II.24) and (II.25).

II.2) Homogeneous Parametric Equations

The derivation of homogeneous parametric equations is greatly simplified by the introduction of the following differential vertex operations (D.V.O.)[11]

$$\Delta_1 = \frac{i}{2}\int d^2x\, N_2[A_\mu A^\mu](x) \qquad \Delta_2 = \frac{-i}{4m^2}\int d^2x\, N_2[F_{\mu\nu}F^{\mu\nu}](x)$$

$$\Delta_3 = i\int d^2x\, N_2[M\bar\psi\psi](x) \qquad \Delta_4 = \frac{1}{2}\int d^2x\, N_2[\bar\psi\overleftrightarrow{\not\partial}\psi](x)$$

$$\Delta_5 = i\int d^2x\, N_2[\bar\psi\not A\psi](x) \qquad \Delta_6 = \frac{i}{2}\int d^2x\, N_2[(\bar\psi\gamma_\mu\psi)^2](x) \qquad (\mathrm{II}.28)$$

with this notation the Lagrangian (II.5) can be rewritten as

$$i\mathcal{L}_{eff} = \Delta_4 - \Delta_3 + \Delta_2 + \Delta_1 + e\,\Delta_5 + g\,\Delta_6 \qquad (\mathrm{II}.29)$$

Notice that $F_{\mu\nu}F^{\mu\nu}$ is a soft operator[12] since it cancels the longitudinal part of the vector meson propagator (II.6). We have therefore two soft insertions

$$\Delta_0 = -i\int d^2x\, N_1[\bar\psi\psi](x)$$

$$\Delta_0' = \frac{-i}{4m^2}\int d^2x\, N_0[F_{\mu\nu}F^{\mu\nu}](x) \qquad (\mathrm{II}.30)$$

Due to our subtraction scheme (II.9) it is easy to derive the following relations for the vertex functions $\Gamma^{(2N,L)}$

$$m^2\frac{\partial\Gamma^{(2N,L)}}{\partial m^2} = -\,\Delta_0'\,\Gamma^{(2N,L)} \qquad (\mathrm{II}.31)$$

$$\frac{\partial\Gamma^{(2N,L)}}{\partial M} = -\,\Delta_0\,\Gamma^{(2N,L)} \qquad (\mathrm{II}.32)$$

$$M\Delta_0\Gamma^{(2N,L)} = \sum_{i=1}^{6} s_i\,\Delta_i\Gamma^{(2N,L)}\ ,\quad s_2=0 \qquad (\text{II}.33)$$

$$\Delta_0'\,\Gamma^{(2N,L)} = \sum_{i=1}^{6} t_i\,\Delta_i\Gamma^{(2N,L)}\ ,\quad t_2=1 \qquad (\text{II}.34)$$

The peculiar form of (II.31) is a direct consequence of our change of variables (II.4). The μ-dependence of $\Gamma^{(2N,L)}$ is given by

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(2N,L)} = \sum_{i=1}^{7}\alpha_i\,\Delta_i\Gamma^{(2N,L)} \qquad (\text{II}.35)$$

where the coefficients α_i are mass-independent. They can be determined directly by observing that μ enters only via the subtraction terms. For example

$$\begin{aligned}\alpha_4 &= \frac{\mu}{4i}\,\mathrm{Tr}\,\frac{\partial}{\partial\mu}\left\{\gamma^\alpha\frac{\partial}{\partial p^\alpha}\Gamma^{(2,0)}(p,-p)\Big|_{\substack{p=0\\ m=M=\mu}}\right\}\\ \alpha_5 &= \frac{\mu}{4i}\,\mathrm{Tr}\,\frac{\partial}{\partial\mu}\left\{\gamma^\alpha\Gamma^{(2,1)}_\alpha(0,0,0)\Big|_{m=M=\mu}\right\}\end{aligned} \qquad (\text{II}.36)$$

The counting identities

$$\begin{aligned}N\Gamma^{(2N,L)} &= (-2\Delta_3+2\Delta_4+2e\Delta_5+4g\,\Delta_6)\,\Gamma^{(2N,L)}\\ L\,\Gamma^{(2N,L)} &= (2\Delta_1+2\Delta_2+e\,\Delta_5)\,\Gamma^{(2N,L)}\end{aligned} \qquad (\text{II}.37)$$

can be derived by integrating the equations of motion

$$\begin{aligned}\langle TN_2[\bar\Psi(i\partial\!\!\!/-M)\psi](x)\,\underline{X}\rangle &= -\langle TN_2[e\bar\psi A\!\!\!/\psi+g(\bar\psi\gamma_\mu\psi)^2](x)\,\underline{X}\rangle+\\ &\quad+\sum_{k=1}^{N}\delta(x-y_k)\langle T\underline{X}\rangle \end{aligned}\qquad (\text{II}.38)$$

$$\langle TN_2[A_\nu\frac{\partial^\mu\partial^\nu}{m^2}A_\mu-\frac{1}{m^2}A_\nu\partial^2A^\nu-A^2](x)\,\underline{X}\rangle = -i\sum_{k=1}^{L}\delta(x-z_k)\langle T\underline{X}\rangle$$

Making use of eq. (II.31-37) and of

$$\begin{aligned}\frac{\partial\Gamma^{(2N,L)}}{\partial g} &= \Delta_6\Gamma^{(2N,L)}\\ \frac{\partial\Gamma^{(2N,L)}}{\partial e} &= \Delta_5\Gamma^{(2N,L)}\end{aligned} \qquad (\text{II}.39)$$

one can establish a homogeneous parametric differential equation of the Weinberg[13] type

$$\begin{aligned}\Big\{\mu\frac{\partial}{\partial\mu}+\rho_1 m^2\frac{\partial}{\partial m^2}+\rho_2 M\frac{\partial}{\partial M}+\beta_1\frac{\partial}{\partial g}+\\ +\beta_2\frac{\partial}{\partial e}-2N\gamma_1-L\gamma_2\Big\}\Gamma^{(2N,L)} = 0\end{aligned} \qquad (\text{II}.40)$$

The proof of (II.40) is standard.[5] One substitutes the above equations into (II.35) and equates to zero the coefficient of each D.V.O. $\Delta_i,\ i=1,2,\dots,7$. This gives the following system of equations for the ρ's, γ's and β's

$$\alpha_1-\rho_1 t_1-\rho_2 s_1-2\gamma_2 = 0 \qquad (\text{II}.41)$$

$$\rho_1 \qquad\qquad +2\gamma_2 = 0 \qquad (\text{II}.42)$$

$$\alpha_3-\rho_1 t_3-\rho_2 s_3-2\gamma_1 = 0 \qquad (\text{II}.43)$$

$$\alpha_4+\rho_1 t_4-\rho_2 s_4-2\gamma_1 = 0 \qquad (\text{II}.44)$$

$$\alpha_5-\rho_1 t_5-\rho_2 s_5+\beta_2-2\gamma_1 e-\gamma_2 e = 0 \qquad (\text{II}.45)$$

$$\alpha_6-\rho_1 t_6-\rho_2 s_6+\beta_1-4g\gamma_1 = 0 \qquad (\text{II}.46)$$

This system always has a solution in perturbation theory, since its determinant is non vanishing in zero order.

From the equations above we have

$$\beta_2 = e\,\gamma_2 \qquad (\text{II}.47)$$

To see that, one uses the Ward identity

$$\Gamma_\mu^{(2,L)}(p,-p;0) = e\frac{\partial}{\partial p^\mu}\Gamma^{(2,0)}(p,-p) \qquad (\text{II}.48)$$

which follows directly from (II.18). Equation (II.48) implies that $\alpha_5 = e\alpha_4$, $s_5 = es_4$, $t_5 = e\,t_4$ and thus, using (II.44) and (II.45), we obtain (II.47).

We can now show that several parameters occuring in (II.40) are zero, namely

$$\beta_1 = \rho_1 = \beta_2 = \gamma_2 = 0 \qquad (\text{II}.49)$$

In order to show that $\gamma_2 = 0$, we use

$$[\mathcal{D} - 2N\gamma_1 - L\gamma_2 + 2\gamma_2]\,\Delta_0'\,\Gamma^{(2N,L)} = 0 \qquad (\text{II}.50)$$

where $$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \rho_1 m^2\frac{\partial}{\partial m^2} + \rho_2 M\frac{\partial}{\partial M} + \beta_1\frac{\partial}{\partial g} + \beta_2\frac{\partial}{\partial e} \qquad (\text{II}.51)$$

which is easily derived since Δ_0' is an integrated zero order normal product . Now the derivative of (II.40) with respect to m^2 gives

$$[\mathcal{D} - 2N\gamma_1 - L\gamma_2]\,m^2\frac{\partial\Gamma^{(2N,L)}}{\partial m^2} = 0 \qquad (\text{II}.52)$$

Thus comparing (II.51) with (II.52) it follows $\gamma_2 = 0$. From (II.42) and (II.47) we have then $\rho_1 = \beta_2 = 0$.

To show that $\beta_1 = 0$ we follow the recipe of ref.14. Let us use the following notation for proper functions containing only one normal product vertex

Normal product	Notation	
$N_1[\bar\psi\gamma_\mu\psi](x)$	Γ_μ	
$N_2[\bar\psi\gamma^5\psi](x)$	Γ_5	(II.53)
$N_1[\bar\psi\gamma_\mu\gamma^5\psi](x)$	$\Gamma_{\mu 5}$	

Then following the same steps as above we can derive

$$[\mathcal{D} - 2N\gamma_1 - L\gamma_2 + 2\gamma_1]\,\Gamma_\mu^{(2N,L)} = 0 \qquad (\text{II}.54)$$

$$[\mathcal{D} - 2N\gamma_1 - L\gamma_2 + 2\gamma_1 + u]\,\Gamma_5^{(2N,L)} = 0 \qquad (\text{II}.55)$$

Note the additional term in equation (II.55). If the β's are zero, it is related to the so called binding dimension, which is a contribution to the anomalous dimension of the $N_1[\bar\psi\gamma^5\psi]$ field produced in the process of joining $\bar\psi(x)\gamma^5$ and $\psi(y)$ to form the composite object . Because of current conservation the corresponding term is absent from (II.54). Now we apply the operator D to the equations (II.18) and (II.23) and use (II.54) and (II.55) together with the relation $\gamma^\mu\gamma^5 = \varepsilon^{\mu\rho}\gamma_\nu$ to obtain

$$\mathcal{D}h = 0 \qquad (\text{II}.56)$$

$$\mathcal{D}R = 0 \qquad (\text{II}.57)$$

$$\mathcal{D}\left(\frac{M}{t}\right) = u\frac{M}{t} \qquad (\text{II}.58)$$

As we have seen h doesn't depend on the masses. Thus from (II.57) we have

$$\beta_1 \frac{\partial h}{\partial g} = 0 \qquad (\text{II}.59)$$

But $\frac{\partial h}{\partial g} \neq 0$ as a simple calculation shows. Hence

$$\beta_1 = 0 \qquad (\text{II}.60)$$

We can understand these results, perhaps more easily by using the infinite counter term approach [15]. In that language the ρ's, β's and γ's are associated with infinite mass, coupling and wave function renormalizations, respectively. γ_2 for example, is zero because the meson two point function is only logarithmically divergent, implying the absence of infinite wave-function renormalization for A^μ.

In computing this logarithimic divergence of the vector-meson propagator one can set M=0, since terms proportional to M are already finite. But because of the property of the two dimensional Dirac algebra

$$\gamma_\alpha \gamma^\mu \gamma^\alpha = 0 \qquad (\text{II}.61)$$

and symmetric integration, the vector-meson mass renormalization is finite, implying the vanishing of ρ_1. Since in gauge theories $\beta_2 = e\gamma_2$ it follows that $\beta_2 = 0$. $\beta_1 = 0$ finally is a consequence of the fact that the interaction, as $M \to 0$, is of the form $:j^\mu j_\mu:$ with j_μ divergenceless and a combination of free fields as will be shown later (sect. V).

III. The Superrenormalizable Gauge

In four dimensions the non-renormalizability of the model of the previous section is solved by a gauge principle: instead of (II.5) one considers a new Lagrangian

$$\mathcal{L}' = \mathcal{L}_{eff} + \frac{1}{2m_0^2}(\partial_\mu A^\mu)^2 \qquad (\text{III}.1)$$

where the addition of the term $(\partial_\mu A^\mu)^2$ has the effect of improving the ultraviolet behaviour of the vector meson propagator. We have

$$D'_{\mu\nu} = \frac{-i}{k^2-m^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)m^2 + \frac{-i}{k^2-m_0^2}\frac{k_\mu k_\nu}{k^2} m_0^2 \qquad (\text{III}.2)$$

With m_0 finite, (III.2) is a meson propagator in an indefinite metric Hilbert space. As in four dimensions only gauge invariant (i.e. m_0 independent) objects can have physical relevance.

The power counting adequate for (III.1) gives

$$\delta(\gamma) = 2 - \tfrac{1}{2}F - \# v_e \qquad (\text{III}.3)$$

where $\# v_e$ is the number of vertices of the type $\bar\psi A\!\!\!/ \psi$ in γ. Observe from (III.3) that the vertices $A_\mu A^\mu$ and $(\partial_\mu A^\mu)^2$ are trivial from the renormalization point of view: either they belong to a 1PR (one particle reducible) graph or to a finite graph. Thus in the Lagrangian (III.1) these vertices are well defined as ordinary products. If one uses the renormalization scheme (II.8), then the vertex functions of this model will satisfy normalization conditions of the type (II. 9) - (II.13) with the additional requirement that e=0 in these formulas.

The derivation of Ward identities and homogeneous parametric equations can be done similarly to the section II. The gauge criteria for this model, however, deserve some comment.

We have

$$\partial_\mu \langle T A^\mu(x) \underline{X} \rangle = - \sum_{\ell=1}^{L} m_0^2 \partial_{\nu_\ell} \Delta_F(x - z_\ell, m_0^2) \langle T \underline{X}_{\nu_\ell} \rangle + \\ + ie\, m_0^2 \sum_{i=1}^{N} [\Delta_F(x-x_i, m_0^2) - \Delta_F(x-y_i, m_0^2)] \langle T \underline{X} \rangle \qquad (\text{III}.4)$$

which shows that $\partial_\mu A^\mu$ is a free field of mass m_0. Furthermore, because of the superrenormalizability of the interaction $\bar{\psi} \not{A} \psi$, the discussion of m_0 independence of physical quantities is greatly simplified. We have

$$m_0^2 \frac{\partial \Gamma^{(2N,L)}}{\partial m_0^2} = \Delta_7 \Gamma^{(2N,L)} , \quad \Delta_7 = \frac{i}{2 m_0^2} \int dx\, N_0 [(\partial_\mu A^\mu)^2](x) \qquad (\text{III}.5)$$

By using the equations of motion it is a simple matter to verify that vertex functions with only transversal meson and on shell fermion fields are m_0 independent. There will be no anisotropic normal product[5] in the discussion, since graphs with one internal meson line are already convergent. By extension composite objects having degree less or equal to two will be gauge invariant, if they satisfy both the equations (III.4) and (III.5).

IV. An Equivalence Theorem

In the previous sections we have seen two formulations of the theory of a massive vector boson interacting with a massive spinor field in two dimensions. The possibility of a formulation directly without ghost fields is a peculiarity of the two dimensional world and in this section we want to investigate the equivalence of theories that differ by the presence or absence of the ghost field. We will show that for gauge invariant quantities the theories of sections II and III are equivalent up to a renormalization. To this end we consider the class of theories specified by a parameter $0 \le \lambda \le 1$

$$\mathcal{L}_\lambda = \lambda i N [\bar{\psi}(\tfrac{\overleftrightarrow{\not{\partial}}}{2} - ie \not{A})\psi] + (1-\lambda) i \bar{N} [\bar{\psi}(\tfrac{\overleftrightarrow{\not{\partial}}}{2} - ie\not{A})\psi] - \\ - \frac{1}{4m^2}(1-b) N_2[F_{\mu\nu}F^{\mu\nu}] + \tfrac{1}{2}(g+f) N_2[(\bar{\psi}\gamma_\mu\psi)^2] + \\ + \tfrac{1}{2} N_2[A^2] - \tfrac{1}{2}\frac{1}{m_0^2} N_2[(\partial_\mu A^\mu)^2] + (1-c) N_2[M\bar{\psi}\psi] + \\ + i\alpha N[\tfrac{1}{2}\bar{\psi}\overleftrightarrow{\not{\partial}}\psi - ie\bar{\psi}\not{A}\psi] \qquad (\text{IV}.1)$$

The degree function which determines the number of subtractions to be made for proper subgraphs is given by

$$\delta(\gamma) = 2 - \frac{F}{2} - \sum (2 - \delta_a) \qquad (\text{IV}.2)$$

where, with the exception of the vertex $\bar{\psi}\not{A}\psi$, the degree δ_a for the normal products of the Lagrangian (IV.1) is 2. In the case of the vertex $\bar{\psi}\not{A}\psi$ we define

$$\delta_a = 1 \quad \text{for} \quad N[\bar{\psi}\not{A}\psi]$$

whereas for $\overline{N}[\overline{\psi} A\!\!\!/ \psi]$

$$\delta_a = \begin{cases} 1, \text{ if } v_a \text{ is an external vertex, i.e. it has an external } A_\mu \text{ attached to } \overline{N}[\overline{\psi} A\!\!\!/ \psi] \\ 2 \quad \text{otherwise} \end{cases}$$

Thus $$\delta(\gamma) = 2 - \tfrac{1}{2} F_\gamma - \overline{B} - \vartheta \qquad (\underline{IV}.3)$$

where

ϑ : nº of vertices $N[\overline{\psi} A\!\!\!/ \psi]$

$\overline{B}$: nº of external A_μ fields attached to $\overline{N}[\overline{\psi} A\!\!\!/ \psi]$

Up to renormalizations (Two theories are equal up to renormalizations, if they differ only by the values of their counter-terms) we see that the case $\lambda = 1$ corresponds to the superrenormalizable theory of section III and the case $\lambda = 0$ corresponds in the $m_o \to \infty$ to the theory described in section II.

In order to obtain a gauge invariant S-matrix the Green functions will have to satisfy [16]

$$\frac{\partial G^{(2N,L)}}{\partial m_o^2} = \Delta_o G^{(2N,L)} \qquad (\underline{IV}.4)$$

with Δ_o some D.V.O. normalized on mass shell.

This can be established by adjusting conveniently the counter terms in (IV.1) as we will show now. Firstly we have

$$\frac{\partial}{\partial m_o^2} = \frac{1}{m_o^4}\Delta_6 - \frac{\partial c}{\partial m_o^2}\Delta_3 + \frac{\partial d}{\partial m_o^2}\Delta_4 + \frac{\partial f}{\partial m_o^2}\Delta_5 - \frac{\partial b}{\partial m_o^2}\Delta_2 \qquad (\underline{IV}.5)$$

where we are employing the notation

$$\Delta_1 = \frac{i}{2}\int d^2x\, N_2[A^2](x) \qquad \Delta_2 = \frac{-i}{4m^2}\int d^2x\, N_2[F_{\mu\nu}F^{\mu\nu}](x)$$

$$\Delta_3 = i\int d^2x\, N_2[M\overline{\psi}\psi](x)$$

$$\Delta_4 = -\int d^2x\, N[\overline{\psi}(\tfrac{1}{2}\overleftrightarrow{\partial\!\!\!/} - ie A\!\!\!/)\psi](x), \quad \overline{\Delta}_4 = -\int d^2x\, \overline{N}[\overline{\psi}(\tfrac{1}{2}\overleftrightarrow{\partial\!\!\!/} - ie A\!\!\!/)\psi](x)$$

$$\Delta_5 = \frac{i}{2}\int d^2x\, N_2[(\overline{\psi}\gamma^\mu\psi)^2](x), \quad \Delta_6 = \frac{i}{2}\int d^2x\, N_2[(\partial_\mu A^\mu)^2](x) \qquad (\underline{IV}.6)$$

Now we want to prove the identity

$$\Delta_6 = \Delta_o + \sum_{i=2}^{5} \sigma_i \Delta_i + \overline{\sigma}_4 \overline{\Delta}_4 \qquad (\underline{IV}.7)$$

where

$$\begin{aligned} \Delta_o G^{(2N,L)} = i\int d^2x \Big\{ &\sum_{i,j=1}^{L} \partial_{\nu_i}\Delta_F(x-z_i,m^2)\,\Delta_F(x-z_j,m_o^2)\,\langle T\underline{X}_{\nu_i\nu_j}\rangle - \\ &- ie\sum_{i,j=1}^{L} \partial_{\nu_i}\Delta_F(x-z_i,m_o^2)\,[\Delta_F(x-x_j,m_o^2) - \Delta_F(x-y_j,m_o^2)]\,\langle T\underline{X}_{\nu_i}\rangle - \\ &- ie^2\sum_{i\neq j}^{N} [\Delta_F(x-x_i,m_o^2)\,\Delta_F(x-x_j,m_o^2) + \Delta_F(x-y_i,m_o^2)\,\Delta_F(x-y_j,m_o^2)]\,\langle T\underline{X}\rangle \\ &+ e^2\sum_{i,j}^{N} \Delta_F(x-x_i,m_o^2)\,\Delta_F(x-y_j,m_o^2)\,\langle T\underline{X}\rangle \end{aligned} \qquad (\underline{IV}.8)$$

The term $\sigma_1\Delta_1$ is absent from the r.h.s. of equ. (IV.7), because σ_1 is given by $\Delta_o\Gamma^{(0,2)}(0,0)$. But because of current conservation, $\Delta_o\Gamma^{(0,2)}(k_1,k_2)$ is transverse in its external meson lines and thus vanishes at $k_1 = k_2 = 0$.

(IV.7) is proved by iterating the Ward identity

$$\begin{aligned} \langle T\partial_\mu A^\mu(x)\,\underline{X}\rangle = &-\sum m_o^2\,\partial_{\nu_i}\Delta_F(x-z_i,m_o^2)\,\langle T\underline{X}_{\nu_i}\rangle + \\ &+ ie\,m_o^2\sum_{i=1}^{N}[\Delta_F(x-x_i,m_o^2) - \Delta_F(x-y_i,m_o^2)]\,\langle T\underline{X}\rangle \end{aligned} \qquad (\underline{IV}.9)$$

and taking into account the additional terms coming from anisotropies in subtractions for the graphs shown in fig.2.

Observe that these graphs must contain at least one vertex $\overline{N}$.

The $\overline{\Delta}$ insertion can be eliminated from (IV.7), if one uses

$$\overline{\Delta}_4 G^{(2N,L)} - \Delta_4 G^{(2N,L)} = \left[\xi_3\Delta_3 + \xi_4\Delta_4 + \xi_5\Delta_5\right] G^{(2N,L)} \qquad (\mathrm{IV}.10)$$

where the coefficients $\xi_i(g,e,\lambda,\mu)$, i=1,2,3,4 are associated with subtractions present in graphs containing $\overline{\Delta}$, but absent in those containing Δ. Note that the vertex $N_2[A^2]$ is absent from the r.h.s. of (IV.10), by the same reason as in equ. (IV.7). Using (IV.10) the equation (IV.7) can be rewritten as

$$\Delta_6 = \Delta_0 + \sum_{i=2}^{5} \eta_i \Delta_i \qquad (\mathrm{IV}.11)$$

From (IV.5) and (IV.11) we see that in order to satisfy (IV.4) the counter terms must be chosen as

$$\begin{aligned}
\frac{\partial b}{\partial m_0^2} &= \eta_2 , & b &= b_0 - \int_{\mu^2}^{m_0^2} \eta_2 \, d\overline{m}_0^2 \\
\frac{\partial c}{\partial m_0^2} &= \eta_3 , & c &= c_0 + \int_{\mu^2}^{m_0^2} \eta_3 \, d\overline{m}_0^2 \\
\frac{\partial d}{\partial m_0^2} &= -\eta_4 , & d &= d_0 - \int_{m_0}^{m_0^2} \eta_4 \, d\overline{m}_0^2 \\
\frac{\partial f}{\partial m_0^2} &= -\eta_5 & f &= f_0 - \int_{\mu^2}^{m_0^2} \eta_5 \, d\overline{m}_0^2
\end{aligned} \qquad (\mathrm{IV}.12)$$

Thus we still have at our disposal the m_0 independent constants b_0, c_0, d_0 and f_0. These will be fixed by imposing the λ-independence of the S-matrix. We now have

$$\frac{\partial G^{(2N,L)}}{\partial\lambda} = \left[\Delta_4 - \overline{\Delta}_4 - \frac{\partial b}{\partial\lambda}\Delta_2 - \frac{\partial c}{\partial\lambda}\Delta_3 + \frac{\partial d}{\partial\lambda}\Delta_4 + \frac{\partial f}{\partial\lambda}\Delta_5\right] G^{(2N,L)} \qquad (\mathrm{IV}.13)$$

Using (IV.10), (IV.13) becomes

$$\frac{\partial G^{(2N,L)}}{\partial\lambda} = \left[-\left(\frac{\partial c}{\partial\lambda} + \xi_3\right)\Delta_3 - \frac{\partial b}{\partial\lambda}\Delta_2 + \left(\frac{\partial d}{\partial\lambda} - \xi_4\right)\Delta_4 + \left(\frac{\partial f}{\partial\lambda} - \xi_5\right)\Delta_5\right] G^{(2N,L)} \qquad (\mathrm{IV}.14)$$

The remaining step is to rewrite (IV.14) in terms of gauge invariant normal products $\widetilde{N}_2[\mathcal{O}]$. These are linear combinations of the $N_2[\mathcal{O}]$ normal products

$$\widetilde{\Delta}_i = \sum_j v_{ij}\Delta_j \quad , \quad i,j = 2,3,4,5 \qquad (\mathrm{IV}.15)$$

satisfying

$$\frac{\partial}{\partial m_0^2}\widetilde{\Delta}_i G^{(2N,L)} = \Delta_0 \widetilde{\Delta}_i G^{(2N,L)} \qquad (\mathrm{IV}.16)$$

Observe that only formally gauge invariant products $\mathcal{O}_i$ can appear in (IV.15). The matrix $[v]_{ij}$ certainly has an inverse $[w]_{ij}$ in perturbation theory and therefore (IV.14) can be expressed in terms of the $\widetilde{\Delta}_i$ as

$$\frac{\partial G^{(2N,L)}}{\partial\lambda} = \sum_{j=2}^{5}\left[-\left(\frac{\partial c}{\partial\lambda} + \xi_3\right)w_{3j}\widetilde{\Delta}_j - \frac{\partial b}{\partial\lambda}w_{2j}\widetilde{\Delta}_j + \left(\frac{\partial d}{\partial\lambda} - \xi_4\right)w_{4j}\widetilde{\Delta}_j + \left(\frac{\partial f}{\partial\lambda} - \xi_5\right)w_{5j}\widetilde{\Delta}_j\right] G^{(2N,L)} \qquad (\mathrm{IV}.17)$$

The coefficients in (IV.17) must be m_0-independent, since on the fermion mass-shell both $G^{(2N,L)}$ and $\widetilde{\Delta}_i$ are; they can be evaluated by choosing $m_0 = \mu$. Thus imposing λ independence of $G^{(2N,L)}$ will result in the following system of equations

$$-\left(\frac{\partial c_0}{\partial\lambda}+\xi_3\right)w_{32}-\frac{\partial b_0}{\partial\lambda}w_{22}+\left(\frac{\partial d_0}{\partial\lambda}-\xi_4\right)w_{42}+\left(\frac{\partial f_0}{\partial\lambda}-\xi_5\right)w_{52}=0$$

$$-\left(\frac{\partial c_0}{\partial\lambda}+\xi_3\right)w_{33}-\frac{\partial b_0}{\partial\lambda}w_{23}+\left(\frac{\partial d_0}{\partial\lambda}-\xi_4\right)w_{43}+\left(\frac{\partial f_0}{\partial\lambda}-\xi_5\right)w_{53}=0$$

$$-\left(\frac{\partial c_0}{\partial\lambda}+\xi_3\right)w_{34}-\frac{\partial b_0}{\partial\lambda}w_{24}+\left(\frac{\partial d_0}{\partial\lambda}-\xi_4\right)w_{44}+\left(\frac{\partial f_0}{\partial\lambda}-\xi_5\right)w_{54}=0$$

$$-\left(\frac{\partial c_0}{\partial\lambda}+\xi_3\right)w_{35}-\frac{\partial b_0}{\partial\lambda}w_{25}+\left(\frac{\partial d_0}{\partial\lambda}-\xi_4\right)w_{45}+\left(\frac{\partial f_0}{\partial\lambda}-\xi_5\right)w_{55}=0$$

(IV.18)

which can be solved perturbatively for $\frac{\partial b_0}{\partial\lambda}$, $\frac{\partial c_0}{\partial\lambda}$, $\frac{\partial d_0}{\partial\lambda}$ and $\frac{\partial f_0}{\partial\lambda}$.

This concludes the proof of

$$\frac{\partial G^{(2N,L)}}{\partial\lambda}=0 \qquad \text{(IV.19)}$$

Let us now discuss the relation of the theories constructed in this section, to the ones of section II and III. Due to (IV.19) we get the same Green functions for any value of λ. For example for $\lambda=1$, which corresponds, up to renormalizations, to the superrenormalizable case, the Lagrangian (IV.1) contains no D.V.O. of the type $\overline{\Delta}_4$. Thus the anisotropies are absent and the counter terms b,c,d and f are m_0-independent. Since for $\lambda=1$ the number of subtractions is the same as those of the superrenormalizable case the limit $m_0\to\infty$ will not exist, except for gauge-invariant quantities on the mass shell, which are already m_0-independent. When we talk about equivalence up to renormalizations, we allways exclude these gauge invariant objects.

Since our Green functions are λ-independent the $m_0\to\infty$ limit cannot exist either for $\lambda=0$ for gauge-dependent objects. But in this case we did make the same number of subtractions as

in the renormalizable unitary gauge. Thus now the m_0 dependent counter terms diverge in the $m_0\to\infty$ limit. We conclude that in this limit, in which the equivalence up to renormalizations obviously continues to hold, one needs an infinite renormalization to go from the theories of this section to the unitary gauge.

V. THE SOLUBLE ZERO MASS LIMIT

Two dimensional QED is known to be soluble, if the mass of the fermion is zero, even if the vector field has a bare mass different from zero. Actually this model is an example of a dynamical generation of mass in which the vector field gets a mass through the interaction. We want to consider here the limit $M \to 0$ of the model of section III. Due to the presence of vertices of the super-normalizable type in (III.1) some remarks are needed.

i) Due to the renormalization condition $\underbrace{(\text{II.9})}_{\text{with } e=0}$, reduced graphs with vertices with two fermion lines will have a momentum factor, which improves the infrared convergence of the integral in the loop momenta of these lines (see fig. 3) This is necessary if one wants to avoid infrared divergencies arising from the fact, that, we have two legs with zero mass in the unsubtracted integrand.[17]

ii) Increasing the number of vertices of the type $\bar\psi\gamma_\mu\psi A^\mu$ in a graph does not introduce infrared problems if the mass of the vector boson is mantained different from zero. This won't be true in general if m'=0. Even in the Landau gauge ($m_o=0$) there will be divergencies associated with graphs of the type of fig.4 and the perturbation series in e' won't exist. However because of generation of mass an exact solution will exist. To obtain this solution one should first take the limit $M \to 0$ maintaining m_o and m different from zero, then sum the perturbative series to get the exact solution and then discuss the other zero mass limits for gauge invariant quantities.

Let us begin discussing the $M \to 0$ limit. From (III.4) the vector meson propagator satisfies

$$\partial_\mu \langle T A^\mu(x) A_\nu(y)\rangle = -\frac{m_o^2}{m'^2}\,\partial_\nu \Delta_F(x-y;m_o^2) \qquad (\text{V.1})$$

whereas for the rotational of A_μ we have

$$\tilde\partial_\mu \langle T A^\mu(x) A_\nu(y)\rangle = -\tilde\partial_\nu \Delta_F(x-y;m_o^2) \; +$$
$$+\, e\int d^2x'\, \Delta_F(x-x';m'^2)\,\tilde\partial_\lambda \langle T j^\lambda(x')\, A_\nu(y)\rangle \qquad (\text{V.2})$$
$$= -\tilde\partial_\nu \Delta_F(x-y;m_o^2) - \alpha\int d^2x'\,\Delta_F(x-x';m'^2)\,\tilde\partial_\lambda \langle T A_\lambda(x')\, A_\nu(y)\rangle$$

with

$$\Delta_F(x-x';m'^2) = \int e^{-i(x-x')\cdot k}\,\frac{1}{k^2-m'^2}\,\frac{d^2k}{(2\pi)^2}$$

$$j_\mu(x) = N_2[\bar\psi\gamma_\mu\psi](x)$$

In obtaining (V.2) we used the axial vector current conservation

$$\langle T\tilde\partial^\mu j_\mu(x)\,\underline{X}\rangle = -\frac{\alpha}{e}\langle T\partial_\lambda A^\lambda(x)\,\underline{X}\rangle + \beta\sum_{j=1}^{N}\left[\delta(x-x_j)\gamma^5_{x_j} + \delta(x-y_j)\gamma^{5T}_{y_j}\right]\langle T\,\underline{X}\rangle$$

where α and β are known functions of the masses and coupling constants.

The equation (V.2) can be easily integrated

$$\tilde\partial_\mu \langle T A^\mu(x) A_\nu(y)\rangle = -\tilde\partial_\nu \Delta_F(x-y;m'^2+\alpha) \qquad (\text{V.3})$$

which shows explicitly that $\tilde\partial_\mu A^\mu$ is a free field of mass $m'^2+\alpha$. The generation of mass is, as we see, a direct consequence of the anomaly in the axial vector Ward identity. Using the identity

$$a^\mu = -\partial^\mu\int d^2y\, D(x-y)\,\partial^\nu a_\nu(y) + \tilde\partial^\mu\int d^2y\, D(x-y)\,\tilde\partial^\nu a_\nu(y)$$

with

$$\Box D(x) = -\delta(x) \qquad (\text{V.4})$$

which expresses the vector a^μ in terms of its divergence and rotational,[18] we obtain

$$\langle T A_\mu(x) A_\nu(y)\rangle = -\frac{\partial_\mu\partial_\nu}{m'^2}\left[D(x-y) - \Delta_F(x-y; m_0^2)\right] - \frac{\tilde\partial_\mu\tilde\partial_\nu}{m'^2+\alpha}\left[D(x-y) - \Delta_F(x-y; m'^2+\alpha)\right] \qquad (V.5)$$

Other Green functions with at least one vector meson can be calculated in a similar way. If $\underline{Y} = \prod_{i=1}^{N}\psi(x_i)\prod_{j=1}^{N}\bar\psi(y_j)$ then we have for example

$$\begin{aligned}\langle T A_\mu(x)\underline{Y}\rangle = {} & \frac{e'}{m'^2}\sum_{i=1}^{N}\partial^\mu\left[D(x-x_i) - D(x-y_i) + \Delta_F(x-y_i; m_0^2) - \Delta_F(x-x_i; m_0^2)\right]\langle T\underline{Y}\rangle + \\ & - \frac{e}{m'^2+\alpha}\sum_{i=1}^{N}\tilde\partial^\mu\left[\left(D(x-x_i) - \Delta_F(x-x_i; m'^2+\alpha)\right)\gamma^5_{x_i} - \right. \\ & \left. + \left(D(x-y_i) - \Delta_F(x-y_i; m'^2+\alpha)\right)\gamma^{5T}_{y_i}\right]\langle T\underline{Y}\rangle\end{aligned} \qquad (V.6)$$

The above formulae indicate that A_μ can be written as

$$A_\mu = \partial_\mu\varphi_1 + \tilde\partial_\mu\varphi_2 \qquad (V.7)$$

with

$$\varphi_1 = \varphi_{10} + \varphi_{11}$$

$$\varphi_2 = \varphi_{20} + \varphi_{21}$$

where φ_{10} and φ_{20} are zero mass scalar fields and φ_{11} and φ_{21} are scalar fields of $(\text{mass})^2$ m_0^2 and $m'^2+\alpha$ respectively.

We can now integrate the vector current and axial vector current Ward identities to obtain

$$\begin{aligned}\langle T j^\mu(x)\underline{Y}\rangle = {} & -\partial^\mu\sum_{i=1}^{N}\left(D(x-x_i) - D(x-y_i)\right)\langle T\underline{Y}\rangle + \\ & + \frac{\alpha}{m'^2+\alpha}\tilde\partial^\mu\sum_{i=1}^{N}\left[\left(D(x-x_i) - \Delta_F(x-x_i; m'^2+\alpha)\right)\gamma^5_{x_i} + \right. \\ & \left. + \left[D(x-y_i) - \Delta_F(x-y_i; m'^2+\alpha)\right)\gamma^{5T}_{y_i}\right]\langle T\underline{Y}\rangle + \\ & + \beta\sum_{i=1}^{N}\tilde\partial^\mu\left[D(x-x_j)\gamma^5_{x_j} + D(x-y_j)\gamma^{5T}_{y_j}\right]\langle T\underline{Y}\rangle\end{aligned} \qquad (V.8)$$

Green functions containing only fermion fields need a little bit more of discussion. We start from the Dirac equation

$$\begin{aligned} i\partial\!\!\!/\langle T\psi(x)\underline{Y}\rangle = {} & i\sum_{k=1}^{N}\delta(x-y_k)\langle T\underline{Y}_{\hat{y}_k}\rangle(-1)^{N+k} \\ & - e\langle T(A\!\!\!/\psi)(x)\underline{Y}\rangle - g\langle T N_{3/2}[(\bar\psi\gamma_\mu\psi)\gamma^\mu\psi](x)\underline{Y}\rangle\end{aligned} \qquad (V.9)$$

and use the Wilson[13] identity

$$\begin{aligned}\langle T{:}N(\bar\psi\gamma_\mu\psi)(x+\varepsilon)\gamma^\mu\psi(x){:}\underline{X}\rangle = {} & \\ = a_1\langle T N_{3/2}[(\bar\psi\gamma_\mu\psi)\gamma^\mu\psi](x)\underline{X}\rangle & + a_2\partial\!\!\!/\langle T\psi(x)\underline{X}\rangle + \\ + a_3\langle T\psi(x)\underline{X}\rangle & + a_4\langle T A\!\!\!/(x)\psi(x)\underline{X}\rangle\end{aligned} \qquad (V.10)$$

Note that a_1, a_2 and a_3 are independent of e', M and m', while a_4 is linear in e'. Moreover $a_3=0$ because in the zero mass limit it is given by

$$\langle T{:}N_1(\bar\psi\gamma^\mu\psi)(0)\gamma^\mu\psi(0){:}\tilde{\bar\psi}(0)\rangle^{prop}\Big|_{\substack{M=0\\ e'=0}} \qquad (V.11)$$

since it results from the first subtraction term for linearly divergent graphs. But using the normalization condition(II.9) and

$$\langle T N_{3/2}[\mathcal{O}(0)]\tilde\varphi(p)\rangle^{prop}\Big|_{\substack{p=0\\ M=0\\ e'=0}} = \left\{\begin{array}{l}\text{contribution of the}\\ \text{trivial graph}\end{array}\right\}$$

in eq. (V.10) we obtain the result that (V.11) is equal to zero. Substituting (V.10) into (V.9) we obtain

$$Z_1(\epsilon)\,\not\partial\,\langle T\psi(x)\,\bar{Y}\rangle = i\sum(-1)^{N+k}\delta(x-y_k)\langle T\,\bar{Y}_{\hat{y}_k}\rangle -$$

$$- e' Z_2(\epsilon)\langle T(\not{A}\psi)(x)\,\bar{Y}\rangle - g\,Z_3(\epsilon)\langle :N_1(\bar\psi\gamma_\mu\psi)(x)\gamma^\mu\psi(x):\bar{Y}\rangle \qquad (\text{V}.12)$$

Applying $\mu\,\partial/\partial\mu$ to (V.11) and using (II.48) we obtain

$$\mu\frac{\partial}{\partial\mu}\left(\frac{Z_2}{Z_1}\right) = 0\ , \quad \mu\frac{\partial}{\partial\mu}\left(\frac{Z_3}{Z_1}\right) = 0$$

and

$$\mu\frac{\partial}{\partial\mu}\left(\frac{1}{Z_1}\right) = 2\gamma_2\,\frac{1}{Z_1}$$

which shows that as $\epsilon\to 0$, Z_2/Z_1 and Z_3/Z_1 are finite constants, but $Z_1 = c_1(\mu^2\epsilon^2)^{\gamma_2}$ with c_1 a finite constant.

Using these results we can rewrite (V.12) as

$$i\not\partial\langle T\psi(x)\,\bar{Y}\rangle = i\sum_{k=1}^{N}(-1)^{N+k}\delta(x-y_k)\langle T\,\bar{Y}_{\hat{y}_k}\rangle -$$

$$- \bar{e}\langle T(\not{A}\psi)(x)\,\bar{Y}\rangle - \bar{g}\,\langle T:(j^\mu\gamma_\mu\psi)(x):\bar{Y}\rangle \qquad (\text{V}.13)$$

where the Z_1 factor has been absorbed in ψ and $\bar{e} = Z_2/Z_1 e'$, $\bar{g} = Z_3/Z_1\, g$.
From (V.6) and (V.8) we have[20]

$$\langle T(A_\mu\psi)(x)\,\bar\psi(y)\rangle = \frac{e'}{m'^2}\left[\partial_\mu\left(\Delta_F(x-y;m_0^2) - D(x-y)\right)\right]\langle T\psi(x)\bar\psi(y)\rangle -$$

$$- \frac{e'}{m'^2+\alpha}\,\tilde\partial_\mu\left[D(x-y) - \Delta_F(x-y;m'^2+\alpha)\right]\gamma_y^{5T}\langle T\psi(x)\bar\psi(y)\rangle$$

$$\langle T:(j_\mu\psi)(x):\bar\psi(y)\rangle = \partial_\mu D(x-y) +$$

$$+ \frac{\alpha}{m'^2+\alpha}\,\tilde\partial_\mu\left[D(x-y) - \Delta_F(x-y;m'^2+\alpha)\right]\gamma_y^{5T}\langle T\psi(x)\bar\psi(y)\rangle$$

$$+ \beta\,\tilde\partial_\mu D(x-y)\,\gamma^{5T}\langle T\psi(x)\bar\psi \ldots\rangle \qquad (\text{V}.14)$$

Thus the fermion two point function is

$$\langle T\psi(x)\bar\psi(y)\rangle = e^{-iF(x,y)}\,{}^{(0)}\langle T\psi^{(0)}(x)\,\bar\psi^{(0)}(y)\rangle^{(0)} \qquad (\text{V}.15)$$

where

$$F(x,y) = \left(\frac{e'\bar{e} - \alpha\bar{g}}{m'^2+\alpha}\right)\left(D(x-y) - \Delta_F(x-y;m'^2+\alpha)\right) -$$

$$+ \frac{e'\bar{e}}{m'^2}\left(D(x-y) - \Delta_F(x-y;m_0^2)\right) \qquad (\text{V}.16)$$

$$- \bar{g}(1+\beta)\,D(x-y)$$

Green functions with more than two fermi fields can be constructed similarly.

From (V.5) and (V.16) we can verify equ. (III.5). Furthermore we can see explicitly that the m_0-dependence can be gauged away.

VI. CONCLUSION

We have shown how to construct Green functions in gauges, which differ in the high energy behavior of the photon propagator. Yet they all lead to the same S-matrix due to the presence of suitable counterterms, which in the $m_o \to \infty$ limit become infinite in order to absorb the difference between a superrenormalizable and a renormalizable theory. Observables are of course m_o-independent.

The considerations of this paper can be extend to four dimensions[21], where one has an infinite number of counter terms, whose job is to ensure that the renormalizable and the non-renormalizable theory produce both the same S-matrix.

FOOTNOTES AND REFERENCES

1) See for example S. Coleman, "Secret Symmetry: An introduction to spontaneous symmetry breakdown and gauge fields"; Lectures given at the 1973 International School of Physics Ettore Majorana.

2) J. Schwinger, Phys. Rev. 128, 2425 (1962); Theoretical Physics, Trieste Lectures, 1962, pg. 89; I.A.E.A., Vienna, 1963.

3) J.H.Lowenstein and J.A.Swieca, Ann. Phys., 68, 172 (1971)

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5) J.H.Lowenstein, "Normal Product Methods in Renormalized Perturbation Theory", Univ. of Maryland Technical Report Nº 73-068 (1972); "BPHZ Renormalization", Lectures given at the International School of Math. Phys., Erice, Sicily, 1975.

6) W.Thirring, Ann. Phys. 3, 91 (1958).

7) M.Gomes, J.H.Lowenstein and W.Zimmermann, Commun. Math. Phys. 39, 81 (1974).

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9) The introduction of the variables e and m is only matter of convenience and everything can be done without them.

10) We will not use normal products with degree higher than two; if needed they can be defined as usual.

11) J.H.Lowenstein, Commun. Math. Phys. 24, 1 (1971).

12) By soft we mean an operator with very good short distance

behaviour; in the case at hand it means that :
is already well defined.

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15) See e.g. N.N. Bogoliubov and D.W.Shirkow, "Introduction to the Theory of Quantized Fields", New York, Interscience Publ. 1959.

16) J.H.Lowenstein and B.Schroer, Phys. Rev. D6, 1553 (1972).

17) J.H.Lowenstein, "Convergence Theorems for Renormalized Feynman Integrals with Zero-Mass Propagators", to be published in Commun. Math. Phys.
Note also that graphs, which are already finite due to the vertex $\bar{\psi} A \cdot \gamma \psi$, do not create infrared divergencies due to the conservation of the axial-vector current, which excludes the generation of a fermion mass.

18) K.Johnson, N.Cimento 20, 773 (1961).

19) K.Wilson, Phys. Rev. 179, 1499 (1969).

20) Here we have used
$\lim_{x \to x_i} \Delta_F (x - x_i ; m_0^2) = D(x - x_i)$ for $m_0^2 < \infty$.

21) M. Gomes and R.Köberle, to be published.

FIGURE CAPTIONS

Fig. 1 The $\partial_\mu A^\mu$ line can be attached only to the longitudinal part of the meson line or to a entering or leaving fermion line.

Fig. 2 Graphs contributing to anisotropie $\dot{s}$.

Fig. 3 The reduced vertex has a momentum factor which improves the infrared behaviour of this graph.

Fig. 4 Graphs which diverg if $m = M = 0$.

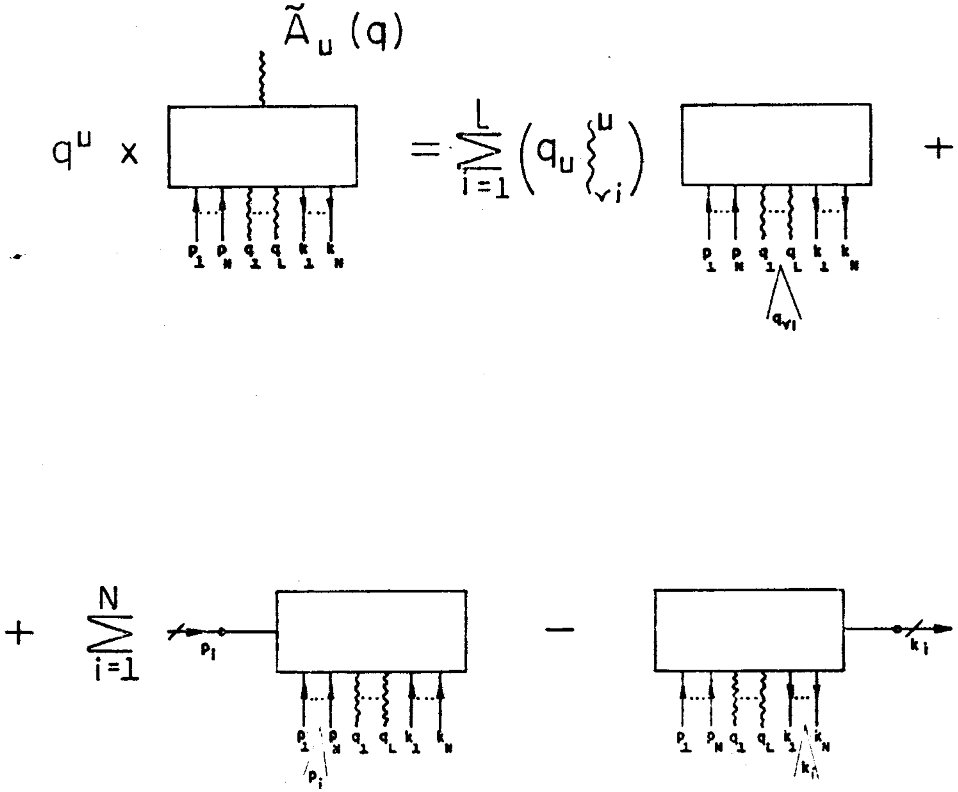

Fig. 1

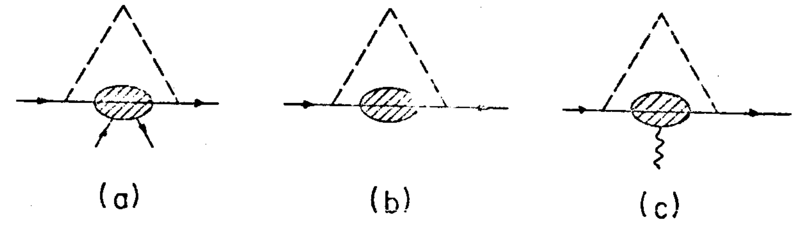

Fig. 2

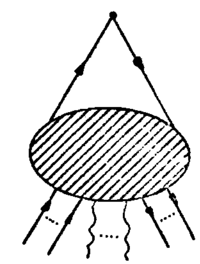

Fig. 3

Fig. 4