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HOW TO COMPUTE EFFECTIVE POTENTIALS

by

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Abstract

A simple way of computation is used to get the effective potential of the Coleman-Weinberg lagrangian in a one-parameter class of gauges involving ghosts.

Resumo

Uma técnica de computação muito simples permite a obtenção do potencial efetivo do Lagrangeano de Coleman-Weinberg em uma classe de "gauges" de um parâmetro que contém os chamados "ghosts".

I. The problem of computing effective potentials has only been solved by displays of virtuosism. Coleman and E. Weinberg's[3] way and Jackiw's[5] are examples of that: for anything but the very simplest models, it is hard to get the one-loop contribution and almost impossible to go further.

In 1975, S.Y. Lee and A.M. Sciaccaluga[7] introduced a very clever way of computing the effective potential of the $\lambda\varphi^4$ theory, including two-loop contributions. Apparently they were not much interested in applying it to gauge theories, where the problems really are.

In this paper we do precisely that. After solving some complications with the gauge-fixing terms, one has a very efficient technique that allows the computation of the effective potential up to (and including) two-loop contributions in a one-parameter class of gauges involving ghosts.

Our version of the justification for the method is given in Section 2. This justification, required by some delicate points related to the gauge invariance, is, to our knowledge, the only complete one. In Section 3 we apply it to the Coleman-Weinberg lagrangian and compute the one-loop contribution. A discussion about the choice of gauge is also presented. The analysis of the two-loop contribution is rather long and will be deferred to another publication.

2. Consider the lagrangian

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = -\frac{1}{2}\, \partial_\mu \varphi\, \partial_\mu \varphi - U(\varphi) \tag{1}$$

where $U(\varphi)$ is a polynomial in the scalar field φ. It is easy to see that in the classical field theory described by this lagrangian $U(\varphi)$ is the energy per unit volume for that state in which the field takes the constant value φ. The generalization of this quantity to a quantum field theory is called the effective potential. Is is defined in the following way[1]: let the functional generator of the IPI Green functions, $\Gamma[\varphi]$, be developed in a power expansion of the derivatives of the field,

$$(2) \qquad -\Gamma[\varphi] = \int d^4x \left[V(\varphi) + \frac{1}{2} Z \, (\partial_\mu \varphi)^2 + \cdots \right]$$

where V, Z,... are ordinary functions of φ $V(\varphi)$ is the effective potential.

The usual way to write $\Gamma[\varphi]$ is of course,

$$(3) \qquad \Gamma[\varphi] = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \; \Gamma^{(n)}(x_1, \ldots x_n) \, \varphi(x_1) \cdots \varphi(x_n)$$

where $\Gamma^{(n)}(x_1, \ldots x_n)$ is the n-point IPI Green function. Taking the Fourier transform and using the translation invariance[2] one gets

$$(4) \qquad \Gamma[\varphi] = \sum_n \frac{1}{n!} \int d^4x \, [\varphi(o)]^n \, \tilde{\Gamma}^{(n)}(0, \cdots 0)$$

the Fourier components being calculated at zero momenta. Comparing to (2) one gets, for constant φ,

$$(5) \qquad V(\varphi) = - \sum_n \frac{1}{n!} \tilde{\Gamma}^{(n)}(0, \ldots 0) \, \varphi^n$$

which allows the computation of the effective potential in terms of an infinite series whose coefficients are the IPI Green functions of the theory. Though this is hardly a convenient method of computation, Coleman and Weinberg[3] used it to obtain, with great ingenuity, the one-loop approximations in some cases. Symanzik[4] has shown that $V(\varphi)$ is the expectation value of the energy per unit volume in a state for which the field has the constant value φ. Clearly, in the tree approximation, $V(\varphi)$ coincides with $U(\varphi)$ of equation (1). For a theory in which the vacuum expectation value of the field operator does not vanish, but has the value v, one has, instead of (5),

$$(6) \qquad V(\varphi) = -\sum_n \frac{1}{n!} \tilde{\Gamma}^{(n)}(0, \dots 0)\,(\varphi - v)^n .$$

From this equation it follows that

$$(7) \qquad \left.\frac{dV(\varphi)}{d\varphi}\right|_{\varphi = v} = 0$$

which allows the computation of the vacuum expectation value as the solution of a minimum problem. This is, perhaps, the most useful property of the effective potential: its connection with the spontaneous breakdown of symmetries.

Computing $V(\varphi)$ has been a hard task. The summation of an infinite number of Green functions was performed, as mentioned above, in ref. (3); functional integration techniques were used by Jackim[5]; Steven Weinberg[6] investigated the connection of $V(\varphi)$ with tadpole diagrams. This last technique inspired a very clever procedure invented by Lee and Sciaccaluga[7] which is, to our knowledge, the only simple and efficient way of computing $V(\varphi)$. The original presentation treated only very simple cases, however, and the power of the method was, apparently, not recognized. We adapt it here to gauge theories.

Consider, for simplicity, a theory with just one scalar field, given by a lagrangian $\mathcal{L}(\varphi, \partial_\mu \varphi)$. Introduce a new field

$$\varphi'(x) = \varphi(x) - v$$

where v is an arbitrary constant. In terms of φ' the lagrangian becomes $\mathcal{L}'(\varphi', \partial_\mu \varphi')$, with some new vertices which depend on v. We can rewrite eq. (4) as

$$(8) \quad \Gamma[\varphi'+v] = \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, \Gamma^{(n)}_{(x_1, \dots x_n)} [\varphi'(x_1)+v] \cdots [\varphi'(x_n)+v] \; .$$

Defining

$$\Gamma'[\varphi'] = \Gamma[\varphi'+v]$$

and resuming eq. (8), one gets

$$(9) \quad \Gamma'[\varphi'] = \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, \bar{\Gamma}^{(n)}(x_1, \dots x_n)\, \varphi'(x_1) \cdots \varphi'(x_n) \; .$$

In the tree approximation $\Gamma'[\varphi']$ coincides with the lagrangian $\mathcal{L}'(\varphi', \partial_\mu \varphi')$. It is then clear that $\bar{\Gamma}^{(n)}(x_1, \dots x_n)$ are proper vertices computed with the lagrangian $\mathcal{L}'(\varphi', \partial_\mu \varphi')$. Fourier transforming eq. (9) one gets

$$(10) \quad \Gamma'[\varphi'] = \sum_n \frac{1}{n!} \int d^4x \, [\varphi'(x)]^n \, \tilde{\bar{\Gamma}}^{(n)}(0, \dots 0)$$

where $\tilde{\bar{\Gamma}}^{(n)}(0, \dots 0)$ is the n-point proper vertex computed, in momentum space, with the Feynman rules of the lagrangian $\mathcal{L}'$.

Going on, one has

$$V'(\varphi') = V(\varphi' + v) = -\sum_n \frac{1}{n!} \tilde{\tilde{\Gamma}}^{(n)}_{(0,\dots 0)} \varphi'^{\,n} \tag{11}$$

for constant φ'. It follows that

$$\left.\frac{dV(\varphi'+v)}{d\varphi'}\right|_{\varphi'=0} = -\tilde{\tilde{\Gamma}}^{(1)}_{(0)} \;. \tag{12}$$

As

$$\left.\frac{dV(\varphi'+v)}{d\varphi'}\right|_{\varphi'=0} = \left.\frac{dV(\varphi'+v)}{d(\varphi'+v)}\right|_{\varphi'+v=v} = \left.\frac{dV(\varphi)}{d\varphi}\right|_{\varphi=v} \tag{13}$$

equation (12) becomes

$$\left.\frac{dV}{d\varphi}\right|_{\varphi=v} = \frac{dV}{dv} = -\tilde{\tilde{\Gamma}}^{(1)}_{(0)} \;. \tag{14}$$

That is, one gets a simple differential equation involving the tadpole term computed according to the rules of lagrangian $\mathcal{L}'$. Putting $v=\varphi$ at the end, one gets the effective potential $V(\varphi)$.

3. We apply this method now to the Coleman-Weinberg lagrangian: the electrodynamics of zero-mass scalar bosons.

(15) $$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}(\partial_\mu \varphi_1 + e A_\mu \varphi_2)^2 - \frac{1}{2}(\partial_\mu \varphi_2 - e A_\mu \varphi_1)^2 - \\ - \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2 \; .$$

Our goal is to find the effective potential as a function of φ_1 and φ_2, that is, for $A_\mu = 0$. To our avail the U(1) global gauge symmetry wil. be used: it says that $V(\varphi_1, \varphi_2)$ is a function only of $\varphi_1^2 + \varphi_2^2$, restricting the problem to that of determining the dependence with, say, φ_1.

Let us first define the theory by choosing the following class of gauges:

(16) $$\mathcal{L}_G = -\frac{1}{2}\xi\left(\partial_\mu A_\mu - \frac{e}{\xi}\varphi_1\varphi_2\right)^2$$

with the corresponding Faddeev-Popov[9] term

(17) $$\mathcal{L}_{FP} = \bar{c}\left\{\partial^2 - \frac{e^2}{\xi}(\varphi_1^2 - \varphi_2^2)\right\}c \; .$$

By introducing a new, shifted field $\varphi_1' = \varphi_1 - v$ we pass to the transformed lagrangian

$$
\begin{aligned}
(18)\qquad \mathcal{L} = {} & \frac{1}{2}\partial_\mu A_\nu \partial_\nu A_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{e^2 v^2}{2} A_\mu A_\mu - \frac{1}{2}(\partial_\mu \varphi_1)(\partial_\mu \varphi_1) - \\
& - \frac{\lambda v^2}{4}\varphi_1^2 - \frac{1}{2}(\partial_\mu \varphi_2)(\partial_\mu \varphi_2) - \frac{\lambda v^2}{12}\varphi_2^2 - \frac{\lambda v^3}{3!}\varphi_1 - e(\partial_\mu \varphi_1) A_\mu \varphi_2 + \\
& + \frac{e^2}{2}(\partial_\mu \varphi_2)\frac{1}{e} A_\mu \varphi_1 + \frac{e}{2}(\partial_\mu \varphi_2) A_\mu \varphi_1 - \frac{e^2}{2} A_\mu A_\mu \varphi_1^2 - \\
& - \frac{e^2}{2} A_\mu A_\mu \varphi_2^2 + e v (\partial_\mu \varphi_2) A_\mu - e^2 v A_\mu A_\mu \varphi_1 - \\
& - \frac{\lambda v}{3!}\varphi_1^3 - \frac{\lambda v}{3!}\varphi_1 \varphi_2^2 - \frac{\lambda}{4!}\varphi_1^4 - \frac{\lambda}{4!}\varphi_2^4 - \frac{2\lambda}{4!}\varphi_1^2 \varphi_2^2 - \\
& - \frac{1}{2}\xi \left\{ \partial_\mu A_\mu - \frac{e}{\xi}(\varphi_1 + v)\varphi_2 \right\}^2 + \bar{c}\left\{ \partial^2 \cdot \frac{e^2}{\xi}\left[(\varphi_1 + v)^2 - \varphi_2^2\right]\right\} c
\end{aligned}
$$

where, to simplify the notation, we omitted the prime of the field operator.

The Feynman rules corresponding to this lagrangian are given in Fig. 1.

Fig. 1

The one-loop tadpole diagrams are given in Fig. 2,

Fig. 2

and contribute

a) $\dfrac{-\lambda v}{32\pi^2}\,\dfrac{\lambda v^2}{2}\ln\dfrac{\lambda v^2}{2}$

b) $-\dfrac{v\left(\frac{\lambda}{3} + \frac{2e^2}{\xi}\right)}{32\pi^2}\left(\lambda v^2 + \dfrac{e^2 v^2}{\xi}\right)\ln\left(\dfrac{\lambda v^2}{6} + \dfrac{e^2 v^2}{\xi}\right)$

c) $\dfrac{e^4 v^3}{16\pi^2}\,\dfrac{1}{\xi^2}\ln\dfrac{e^2 v^2}{\xi}$

d) $\dfrac{e^4 v^3}{16\pi^2}\left\{ \dfrac{1}{2\xi^2} - \dfrac{1}{\xi^2}\ln\dfrac{1}{\xi} - \left(3+\dfrac{1}{\xi^2}\right)\ln e^2 v^2 - \dfrac{1}{2}\right\}$.

The renormalization performed here was the 't Hooft Veltman's[8] "pole subtraction". To be able to compare our results to some ~~existents~~ ones it will be found convenient to introduce below other renormalization prescriptions.

The basic equation that gives the one-loop contribution to the effective potential is

$$\frac{dV}{dv} = -\frac{e^4 v^3}{16\pi^2}\left\{\frac{1}{2\xi^2} - 3\ln(e^2v^2) - \frac{1}{2}\right\} + \frac{\lambda^2 v^3}{64\pi^2}\ln\frac{\lambda v^2}{2} + \frac{\left(\frac{\lambda}{3}+\frac{2e^2}{\xi}\right)\left(\frac{\lambda}{6}+\frac{e^2}{\xi}\right)}{32\pi^2} v^3 \ln\left[v^2\left(\frac{\lambda}{6}+\frac{e^2}{\xi}\right)\right] \tag{19}$$

which gives

$$V(v) = -\left\{\frac{e^4}{16\pi^2}\left(\frac{1}{2\xi^2}-\frac{1}{2}\right) - \frac{\lambda^2}{64\pi^2}\ln\frac{\lambda}{2} - \frac{3e^4}{16\pi^2}\ln e^2 - \frac{\left(\frac{\lambda}{3}+\frac{2e^2}{\xi}\right)\left(\frac{\lambda}{6}+\frac{e^2}{\xi}\right)}{32\pi^2}\ln\left(\frac{\lambda}{6}+\frac{e^2}{\xi}\right)\right\}\frac{v^4}{4} + \left\{\frac{3e^4}{16\pi^2}+\frac{\lambda^2}{64\pi^2}+\frac{\left(\frac{\lambda}{3}+\frac{2e^2}{\xi}\right)\left(\frac{\lambda}{6}+\frac{e^2}{\xi}\right)}{32\pi^2}\right\}\frac{v^4}{4}\left(\ln v^2 - \frac{1}{2}\right) \tag{20}$$

Putting $v = (\varphi_1^2 + \varphi_2^2)^2$ and adding the zero-loop term $\frac{\lambda}{4!}(\varphi_1^2+\varphi_2^2)^2$, we have the effective potential up to one-loop contributions. If, following Coleman and Weinberg, we use the renormalization prescriptions

$$\left.\frac{\partial^2 V}{\partial \varphi_1^2}\right|_{\varphi_1=0} = 0 \tag{21}$$

(22) $$\left.\frac{\partial^4 V}{\partial \varphi_1^4}\right|_{\varphi_1 = M} = \lambda_R$$

the result will be

(23) $$V(\varphi_1, \varphi_2) = \frac{\lambda_R}{4!}(\varphi_1^2 + \varphi_2^2)^2 +$$

$$+\left\{\frac{3e^4}{64\pi^2} + \frac{5\lambda^2}{1152\pi^2} + \frac{\lambda e^2}{192\pi^2\xi} + \frac{e^4}{64\pi^2\xi^2}\right\}(\varphi_1^2+\varphi_2^2)^2\left\{\ln\frac{(\varphi_1^2+\varphi_2^2)}{M^2} - \frac{25}{6}\right\}$$

For $\xi \to \infty$ (Landau gauge) one gets exactly the results of ref.(3). Analogue renormalization prescriptions for φ_2 were omitted and made unnecessary by the use of the U(1) symmetry. Note that the gauge-fixing condition is compatible with $A_\mu = 0$, the point at which the effective potential was calculated.

Some more comments about the gauge-fixing terms are in order. What we did was to define the theory according to Faddeev-Popov[9] through the gauge-fixing and ghost terms given by eqs. (16) and (17) and then shifting the theory, including $\mathcal{L}_G$ and $\mathcal{L}_{FP}$, by putting $\varphi_1' = \varphi_1 - v$. This is the cleanest way to proceed, but is not the only one. One could think of acting in the following way: first, shift the field φ_1 in the lagrangian at eq. (15), then add gauge-fixing and ghost terms. In the shifted theory, one can then compute the tadpole diagrams. By solving eq. (14) one has the effective potential of the theory without shift. But, in which gauge? The natural answer is: in the gauge determined by the limit of the gauge-fixing condition $\mathcal{L}_G$ as $v \to 0$. We illustrate this with an example.

Consider the lagrangian (18), omit the last terms, replace them by

$$\mathcal{L}_G = -\frac{1}{2}\xi\left(\partial_\mu A_\mu - \frac{ev}{\xi}\varphi_2\right)^2$$

$$\mathcal{L}_{FP} = \bar{c}\left\{\partial^2 - \frac{e^2 v}{\xi}(\varphi_1 + v)\right\} c \quad .$$

What we obtain is the Coleman-Weinberg lagrangian in the well-known R_ξ gauge(10). By proceeding in an analogous way one has

$$(24) \qquad V = \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2 + \left\{\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2} + \frac{\lambda e^2}{384\pi^2\xi} + \frac{e^4}{128\pi^2\xi^2}\right\}(\varphi_1^2+\varphi_2^2)^2\left\{\ln\frac{\varphi_1^2+\varphi_2^2}{M^2} - \frac{25}{6}\right\} .$$

This should be, if this procedure is correct, the effective potential of the unshifted theory in the gauge

$$\mathcal{L}_G = -\frac{1}{2}\xi(\partial_\mu A_\mu)^2 .$$

Now, in this gauge, Jackim(5) has the result (one-loop):

$$(25) \qquad V = \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2 + \left\{\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2} - \frac{e^2\lambda}{192\pi^2\xi}\right\}(\varphi_1^2+\varphi_2^2)^2\left\{\ln\frac{\varphi_1^2+\varphi_2^2}{M^2} - \frac{25}{6}\right\}$$

which does not agree with equation (24). Hence, the second approach is not trustworthy.

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References

(1) J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. 127, 965 (1962); G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

(2) Good reviews that explain these matters are S. Coleman, Secret Symmetry (Lectures given at the 1973 International Summer School of Physics Ettore Majorana). J. Iliopoulos, C. Itzykson, A. Martin, Revs. Mod. Phys. 47, 165 (1975).

(3) S. Coleman, E. Weinberg, Phys. Rev. D7, 1888 (1973).

(4) K. Symanzik, Comm. Math. Phys. 16, 48 (1970.

(5) R. Jackiw, Phys. Rev. D9, 1886 (1974).

(6) S. Weinberg, Phys. Rev. D7, 2887 (1973).

(7) S.Y. Lee, A. M. Sciaccaluga, Nucl. Phys. B96, 435 (1975).

(8) G. 't Hooft, M. Veltman, Nucl. Phys. B44, 189 (1972).

(9) L. Faddeev, V.N. Popov, Phys. Lett. 25B, 29 (1967).

(10) K. Fujikawa, B. Lee, A. Sanda, Phys. Rev. D6, 2923 (1972).

Figure captions

Fig. 1 - The Feynman rules for the graphs a) ... p) are given by the following expressions:

a) $\frac{1}{i} \frac{1}{k^2 + e^2 v^2} \left\{ \delta_{\alpha\beta} + \frac{k_\alpha k_\beta (1-\xi)}{\xi k^2 + e^2 v^2} \right\}$

b) $\frac{1}{i} \frac{1}{k^2 + \frac{\lambda v^2}{2}}$

c) $\frac{1}{i} \frac{1}{k^2 + \frac{\lambda v^2}{6} + \frac{e^2 v^2}{\xi}}$

d) $\frac{1}{i} \frac{1}{k^2 + \frac{e^2 v^2}{\xi}}$

e) $\frac{1}{i} \lambda v$

f) $\frac{1}{i} v \left(\frac{\lambda}{3} + \frac{2e^2}{\xi} \right)$

g) $\frac{1}{i} 2e^2 v \, \delta_{\alpha\beta}$

h) $\frac{1}{i} \frac{2e^2}{\xi} v$

i) $-2e \, (k_\mu + q_\mu)$

j) $\frac{1}{i} \left(\frac{\lambda}{3} + \frac{2e^2}{\xi} \right)$

k) $\frac{1}{i} \lambda$

l) $\frac{1}{i} \lambda$

m) $\frac{1}{i} 2e^2 \delta_{\alpha\beta}$

n) $\frac{1}{i} 2e^2 \delta_{\alpha\beta}$

o) $\frac{1}{i} \frac{2e^2}{\xi}$

p) $i \, \frac{2e^2}{\xi}$.

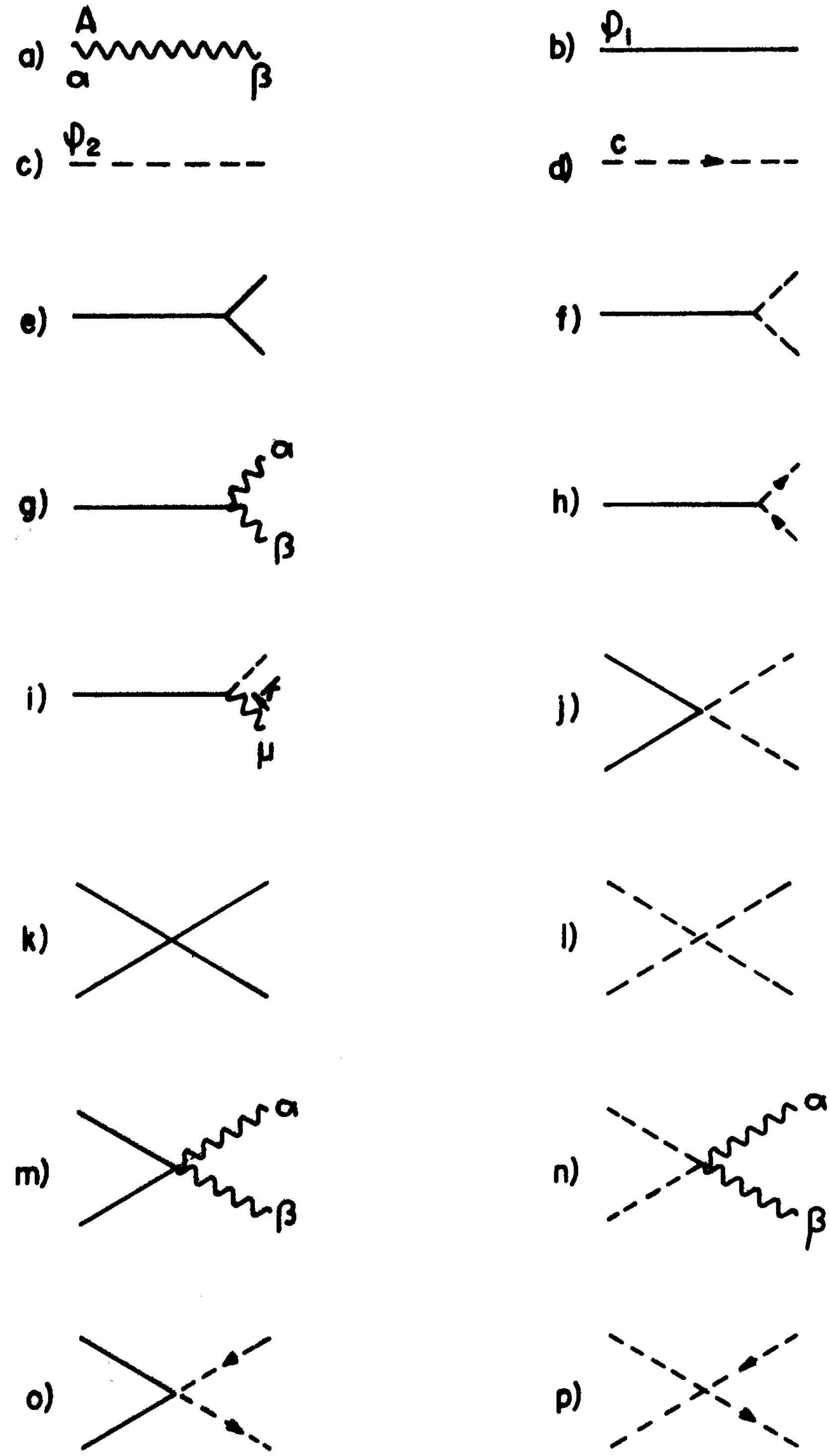

Fig. 1

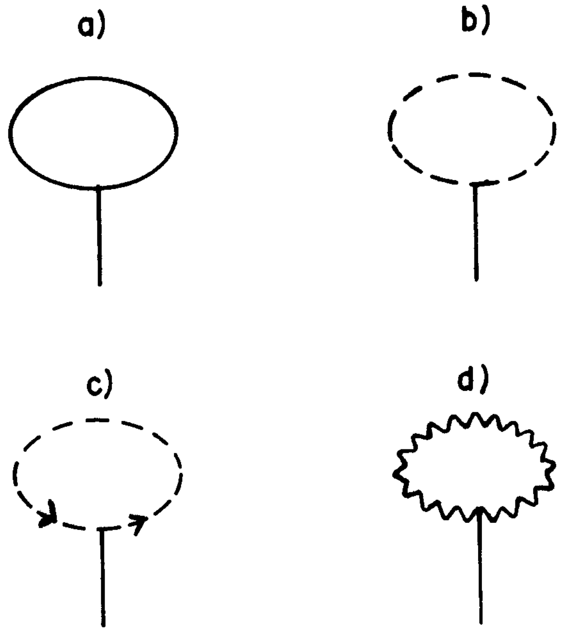

Fig. 2