IFUSP/P-113

# ASPECTS OF ALGEBRAIC QUANTUM THEORY



by

IVAN F. WILDE*,

Instituto de Física, Universidade de São Paulo

Brazil



*Supported by BNDE.

Present Adress
Department of Mathematics
Bedford College, London

## PREFACE

These notes constituted a postgraduate course given in the physics department of the University of São Paulo during the winter semester (March - June) of 1974.

The intention was to give a self-contained introduction to the methods and results of algebraic quantum theory - assuming a working knowledge of quantum field theory (i.e. essentially the free field). Some effort was made to develop the mathematical background of operator algebras, at the cost of neglecting the Hilbert space aspects, although these are perhaps more familiar. Further mathematical background can be found, for example, in Dunford and Schwartz (1966), Kadison (1967), Kato (1966), Lanford (1972), Naimark (1964) and Reed and Simon (1972).

In chapters 1 and 2, some basic results concerning C*-algebras and Von Neumann algebras are presented. In chapter 3, canonical commutation relations are discussed, in both the Heisenberg and Weyl form, and Von Neumann's uniqueness theorem is proved. It is also shown that it does not hold for infinitely many degrees of freedom.

In chapter 4 and 5, the axiomatic schemes of Segal and Haag and Kastler are presented, and some of their consequences are given.

The theory of the free charged bose field is developed in detail in chapter 6, which leads naturally to chapter 7, the theory of superselection sectors of Doplicher, Haag and Roberts.

Finally, in chapter 8, a two-dimensional model is presented, which exhibits explicitly the philosophy developed in chapter 7.

In is a pleausre to thank J.Fernando Perez, Ely Joana Belotto Silva, Cezar Augusto Bonato, and Ricardo Azevedo Neves da Silva for many comments during the course and J.Fernando Perez for many discussions.

Furthermore, I would like to thank Henrique Fleming, Roland Koberle and J.Fernando Perez for the hospitality of the Physics Institute, and BNDE for financial support.

Finally, I would like to thank Ivonne for her excellent typing of the manuscript, and Ricardo Neves da Silva for reading the manuscript and writing in the symbols.

Ivan F.Wilde.

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# 1. $C^*$ - Algebras

We shall develop here some of the basic theory of $C^*$ - algebras. The standart references are Dixmier (1969a), Naimark (1964), Rickart (1960), and Sakai (1971). (See also Lanford (1972) and Kadison (1967)).

The study of properties of general $C^*$ - algebras can often be reduced to that of a commutative algebra, which, in turn, reduces to the study of continuous functions on a compact Hausdorff space. We shall begin, then, with Gelfand's theory of commutative $C^*$ - algebras. This seems to be the natural setting for the spectral theory of self-adjoint operators on a Hilbert space.

After this, we consider representations of $C^*$ - algebras (which is continued in 2.6), states, their associated representations, and some of their properties.

## 1.1) Banach Algebras

These first two sections are based on the lectures of Simon (1972).

1.1.1) Definition - A Banach algebra (with identity), $\mathfrak{A}$, is a complex Banach space together with:

(i) an associative, distributive multiplication with identity, $\mathbb{1}$ ;

(ii) $\|ab\| \leq \|a\| \|b\|$ for all $a,b \in \mathfrak{A}$ ;

(iii) $\|\mathbb{1}\| = 1$.

(Unless stated otherwise, we will assume that our algebras have an identity).

1.1.2) Proposition - Let $\mathfrak{A}$ be a Banach algebra. Then the following hold.

a) The set, $\mathcal{I}$ , of invertible elements of $\mathfrak{A}$ is open, and the inverse operation is continuous from $\mathcal{I}$ to $\mathcal{I}$ ;

b) Maximal proper ideals are closed;

c) If $x \in \mathfrak{A}$, the spectrum of $x$,

$$\sigma(x) \equiv \{\lambda \in \mathbb{C} \mid x - \lambda\mathbb{1} \notin \mathfrak{I}\}, \text{ is a compact subset of } \mathbb{C};$$

d) For any $x \in \mathfrak{A}$, $\sigma(x) \neq \emptyset$ ;

e) Let $x \in \mathfrak{A}$. Then $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists, and is equal to $\sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Proof :

a) Let $a \in \mathfrak{I}$, and let $b \in \mathfrak{A}$ with $\|b\| \le \|a^{-1}\|^{-1}$. Then the geometric series $a^{-1} \sum_{n=0}^{\infty} (ba^{-1})^n$ converges in $\mathfrak{A}$, and is an inverse for a+b. Thus, if $c \in \mathfrak{A}$ satisfies $|c-a| \le |a^{-1}|^{-1}$, we conclude that $c = a + (c-a)$ is invertible, i.e. $\mathfrak{I}$ is open.

Now let $x_n \to x$ in $\mathfrak{A}$ as $n \to \infty$, with $x_n, x \in \mathfrak{I}$. Then

$$x_n^{-1} = (x + (x_n - x))^{-1}$$

$$= x^{-1} \Sigma_{n=0}^{\infty} (x^{-1}(x-x_n))^n \quad \text{for large } n. \text{ Therefore}$$

$$x_n^{-1} - x^{-1} = x^{-1} \Sigma_{n=1}^{\infty} (x^{-1}(x-x_n))^n \to 0 \text{ as } n \to \infty.$$

b) Let $\mathfrak{M}$ be a maximal proper ideal of $\mathfrak{A}$. Then $\mathfrak{M}$ cannot contain any invertible elements (otherwise we would have $\mathfrak{M} = \mathfrak{A}$), i.e. $\mathfrak{M} \subset \mathfrak{A} \setminus \mathfrak{I}$. By (a), $\mathfrak{I}$ is open and so $\mathfrak{A} \setminus \mathfrak{I}$ is closed. Hence $\overline{\mathfrak{M}}$, the closure of $\mathfrak{M}$, is contained in $\mathfrak{A} \setminus \mathfrak{I}$. In particular, $\overline{\mathfrak{M}} \neq \mathfrak{A}$. But $\overline{\mathfrak{M}}$ is an ideal containing $\mathfrak{M}$ and must therefore be equal to $\mathfrak{M}$ by the supposed maximality of $\mathfrak{M}$.

c) Let $x \in \mathfrak{A}$, and suppose $|\lambda| > \|x\|$. Then the sum

$$\mathbb{1} + \lambda^{-1}x + (\lambda^{-1}x)^2 + \ldots.$$

converges to $(\mathbb{1} - \lambda^{-1}x)^{-1}$. But $(x-\lambda)^{-1} = (-\lambda^{-1})(\mathbb{1} - \lambda^{-1}x)^{-1}$ and so $\lambda \notin \sigma(x)$: Hence $\sigma(x)$ is a bounded subset of $\mathbb{C}$.

Let $\phi$ be the map $\phi : \lambda \to x - \lambda\mathbb{1}$. Then $\lambda \in \sigma(x)$ iff $(x-\lambda) \in \mathfrak{I}$ iff $\lambda \in \phi^{-1}(\mathfrak{I})$. That is, $\mathbb{C} \setminus \sigma(x) = \phi^{-1}(\mathfrak{I})$. However, $\phi$ is continuous and $\mathfrak{I}$ is open, so $\sigma(x)$ is closed. Therefore $\sigma(x)$ is compact.

d) Let $x \in \mathfrak{A}$, and suppose $\sigma(x)=\emptyset$. The map $\phi:\lambda \to (x-\lambda)$ is an entire function. Since $\sigma(x)=\emptyset$, $\phi(\lambda)$ has an inverse for all $\lambda \in \mathbb{C}$, and so $\phi(\lambda)^{-1}$ is also entire. But, for $|\lambda|$ large, we have

$$\phi(\lambda)^{-1} = (x-\lambda)^{-1} = (-\lambda)^{-1} \Sigma_{n=0}^{\infty} (\lambda^{-1}x)^n$$

$$\to 0 \quad \text{as} \quad |\lambda| \to \infty.$$

By Liouville's theorem, $\phi(\lambda)^{-1} \equiv 0$, which is a contradiction.

e) Let $\ell(m) = \log \|x^m\|$. Then $\ell(m+n) \le \ell(m)+\ell(n)$. Fix $n$ and write $k = \alpha n+\beta$, where $\alpha,\beta$ are integers with $0 \le \beta < n$. Then $\ell(k) \le \alpha\ell(n)+\ell(\beta)$.

Hence

$$\frac{\ell(k)}{k} \le \frac{\alpha n\ell(n)}{nk} + \frac{\ell(\beta)}{k} = \frac{\ell(n)}{n} - \frac{\beta}{kn} + \frac{\ell(\beta)}{k}$$

$$\Rightarrow \quad \overline{\lim}\, \frac{\ell(k)}{k} \le \frac{\ell(n)}{n}$$

$$\Rightarrow \quad \overline{\lim}\, \frac{\ell(k)}{k} \le \inf \frac{\ell(n)}{n} \le \underline{\lim}\, \frac{\ell(n)}{n} .$$

Therefore $\lim \ell(k)/k$ exists, and is equal to $\inf \ell(k)/k$. The result follows by taking exponentials.

Let us denote this limit by $r(x)$.

Then $r(x) \le \inf ||x^n||^{1/n} \le ||x||$, and so $1/||x|| \le 1/r(x)$.

If $|\lambda| < 1/r(x)$, then $(\mathbb{1}-\lambda x)^{-1}$ has a series expansion given by $\Sigma_{n=0}^{\infty}(\lambda x)^n$. The radius of convergence is given precisely by $|\lambda| < 1/r(x) = (\overline{\lim}\, \|x^n\|^{1/n})^{-1}$. In other words, $(\mathbb{1} - \lambda x)^{-1}$ is analytic in $|\lambda| < 1/r(x)$, but not in any bigger disc.

However, $\lambda \to \mathbb{1} - \lambda x$ is entire, and so its inverse is analytic whenever it exists. This is whenever $1 \notin \sigma(\lambda x)$, or $\lambda^{-1} \notin \sigma(x)$. This holds if $\lambda$ is such that $|\lambda| < \{\sup_{\mu \in \sigma(x)} |\mu|\}^{-1}$.

Thus $(\mathbb{1} - \lambda x)^{-1}$ is analytic in the disc $|\lambda| < \{\sup_{\mu\in\sigma(x)} |\mu|\}^{-1}$. Hence, by our previous reasoning, $\{\sup_{\mu\in\sigma(x)} |\mu|\}^{-1} \leq 1/r(x)$, i.e. $r(x) \leq \sup_{\mu\in\sigma(x)} |\mu|$.

But if $|\mu| > r(x)$, we know that $(\mu - x)^{-1}$ has a series expansion, i.e. $\mu \in \mathbb{C} \setminus \sigma(x)$.

Hence $\sup_{\mu\in\sigma(x)} |\mu| \leq r(x)$, and so equality follows.

QED.

$r(x)$ is called the spectral radius of $x$. It is the radius of the smallest disc centred at the origin containing $\sigma(x)$, the spectrum of $x$.

1.1.3) Theorem (Gelfand-Mazur)

Let $\mathfrak{A}$ be a Banach algebra.

a) Suppose $\{0\} \cup \mathcal{G} = \mathfrak{A}$. Then $\mathfrak{A} \simeq \mathbb{C}$.

b) Suppose $\mathfrak{A}$ is commutative. Then there is a canonical bijection between maximal ideals $J \subset \mathfrak{A}$ and continuous non-zero homomorphisms, $\ell$, from $\mathfrak{A}$ to C, given by $J = \ker \ell$.

Proof. a) Let $x \in \mathfrak{A}$. Then $\sigma(x) \neq \phi$, and so there is $\lambda \in \mathbb{C}$ such that $x - \lambda\mathbb{1} \notin \mathcal{G}$. It follows that $x - \lambda\mathbb{1} = 0$, i.e. $x = \lambda\mathbb{1}$, some $\lambda \in \mathbb{C}$.

b) Let $J \subset \mathfrak{A}$ be maximal. By 1.1.2, $J$ is closed. It follows that $\mathfrak{A}/J$ is a Banach algebra with respect to the norm $\|\text{class } x\| \equiv \inf_{j\in J} \|x+j\|$.

Suppose $\mathfrak{A}/J$ contains a non-zero non-invertible element, $\text{cl}a$. Then $(\mathfrak{A}/J)\,\text{cl}a$ is a proper ideal in $\mathfrak{A}/J$ since it does not contain $\text{cl}\mathbb{1}$, and is not $\{0\}$. Hence $\mathfrak{A}a+J$, the pre-image of $(\mathfrak{A}/J)\,\text{cl}a$ under the canonical morphism $\mathfrak{A} \to \mathfrak{A}/J$, is a proper ideal in $\mathfrak{A}$ which strictly contains $J$. This contradicts the maximality of $J$, and so all non-zero elements of $\mathfrak{A}/J$ are invertible. By a), $\mathfrak{A}/J \simeq \mathbb{C}$. If we denote this isomorphism by

$\phi$ , and the canonical morphism $\mathfrak{A} \to \mathfrak{A}/J$ by $\pi$ , we see that $\phi \cdot \pi$ is a homomorphism $: \mathfrak{A} \to \mathbb{C}$. The kernel of $\phi \cdot \pi$ is exactly J. To show that $\phi \cdot \pi$ is continuous, suppose the contrary. Then there is a sequence $a_n \in \mathfrak{A}$ such that $\phi \cdot \pi(a_n) \to -1$ and $a_n \to 0$ as $n \to \infty$. Then $a_n - \phi \cdot \pi(a_n)\mathbb{1} \in J$ and converges to $\mathbb{1} \notin J$, which contradicts the fact that J is closed.

Conversely, let $\ell : \mathfrak{A} \to \mathbb{C}$ be a non-zero continuous homomorphism, and let $J = \ker \ell$. $J \neq \mathfrak{A}$ since $\ell$ is non-zero. Let $a \notin J$. Then any $b \in \mathfrak{A}$ can be written as

$$b = a\frac{\ell(b)}{\ell(a)} + \left(b - a\frac{\ell(b)}{\ell(a)}\right).$$

Since $b - \dfrac{a\ell(b)}{\ell(a)} \in \ker \ell = J$, we see that J is maximal.

We have, then, an association between maximal ideals, J, and continuous homomorphisms, $\ell$, with $\ker \ell = J$. This association is one-one since $\ell$ is uniquely determined by its kernel. Indeed, let $\ell$ and $\ell'$ have the same kernel. Then, for any $a \in \mathfrak{A}$, we have $a - \ell(a)\mathbb{1} \in \ker \ell$, so $a - \ell(a)\mathbb{1} \in \ker \ell'$, i.e. $\ell'(a) = \ell(a)$

QED

1.1.4) <u>Definition</u>

A continuous homomorphism $\mathfrak{A} \to \mathbb{C}$ is called a multiplicative linear functional, or a character.

1.1.3 says that there is a one-one correspondence between maximal ideals and characters of a commutative Banach algebra such that $J \leftrightarrow \ell$ if and only if $J = \ker \ell$.

1.1.5) <u>Definition</u>

The set of characters of a commutative Banach algebra, $\mathfrak{A}$ , is called the spectrum of $\mathfrak{A}$ , and is denoted $\mathrm{Sp}\,\mathfrak{A}$.

1.1.6) <u>Theorem</u> (Gel'fand)

Let $\mathfrak{A}$ be a commutative Banach algebra with spectrum $Sp\,\mathfrak{A}$. Then:

a) $Sp\,\mathfrak{A}$ is a w*-closed subset of the unit ball of $\mathfrak{A}^*$, the dual of $\mathfrak{A}$.

b) $Sp\,\mathfrak{A}$ is a compact Hausdorff space (with the induced w*-topology).

c) Given $x \in \mathfrak{A}$, $\ell \in Sp\,\mathfrak{A}$, define $\hat{x}(\ell) = \ell(x)$. Then the function $\hat{x}$ on $Sp\,\mathfrak{A}$ satisfies ran $\hat{x} = \sigma(x)$.

d) $\hat{}: \mathfrak{A} \to C(Sp\,\mathfrak{A})$ is a homomorphism, and $||\hat{x}||_\infty \leq ||x||$.

Proof -a) Let $\ell \in Sp\,\mathfrak{A}$ and let $J = \ker \ell$. Let $x \in \mathfrak{A}$. Then $x - \ell(x) \in J$ and so $x - \ell(x)$ is not invertible, i.e. $\ell(x) \in \sigma(x)$.

But $\sigma(x) \subset \{\lambda \mid |\lambda| \leq r(x)\}$, and $r(x) \leq ||x||$. Hence $|\ell(x)| \leq ||x||$, and so $\ell$ belongs to the unit ball of $\mathfrak{A}^*$.

Now let $\ell_\alpha \to \ell$ in the w*-topology of $\mathfrak{A}^*$, with each $\ell_\alpha \in Sp\,\mathfrak{A}$. Then, for any $a, b \in \mathfrak{A}$, $\ell_\alpha(ab) = \ell_\alpha(a)\ell_\alpha(b)$ converges to $\ell(a)\ell(b)$ (- this follows because $|\ell_\alpha(a)|$ is bounded uniformly in $\alpha$). On the other hand, $\ell_\alpha(ab) \to \ell(ab)$. Hence $\ell \in Sp\,\mathfrak{A}$.

-b) It is easy to see that the w*-topology of $\mathfrak{A}^*$ is Hausdorff. By the Banach-Alaoglu theorem (Dunford and Schwartz (1966)), the unit ball of $\mathfrak{A}^*$ is w*-compact. Since $Sp\,\mathfrak{A}$ is a w*-closed subset of the unit ball of $\mathfrak{A}^*$, it is w*-compact.

-c) As in a), $\ell(x) \in \sigma(x)$ for $x \in \mathfrak{A}$, $\ell \in Sp\,\mathfrak{A}$. Hence ran $\hat{x} \subset \sigma(x)$.

Let $\lambda \in \sigma(x)$. Then $x-\lambda$ is not invertible, and so belongs to some maximal ideal, J, say. Let $\ell \in Sp\,\mathfrak{A}$ be such that $\ker \ell = J$, as in 1.1.3. Then $x-\lambda \in J$ implies that $\ell(x) = \lambda$. Therefore $\hat{x}(\ell) = \ell(x) = \lambda$, and ran $\hat{x} = \sigma(x)$.

-d) Clearly $\hat{}$ is a homomorphism. We must show that

$\hat{x}(\cdot) \in C(\mathrm{Sp}\,\mathfrak{A})$. Let $\ell_\alpha \to \ell$ in $\mathrm{Sp}\,\mathfrak{A}$ with the $w^*$-topology. Then, by definition of this topology, $\ell_\alpha(x) \to \ell(x)$ for each $x \in \mathfrak{A}$, i.e. $\hat{x}(\ell_\alpha) \to \hat{x}(\ell)$ and so $\hat{x}(\cdot)$ is continuous.

$$\begin{aligned} ||\hat{x}||_\infty &\equiv \sup\{|\hat{x}(\ell)| \;|\ell \in \mathrm{Sp}\,\mathfrak{A}\} \\ &= \sup\{|\lambda| \;|\lambda \in \sigma(x)\} \quad \text{by (c)} \\ &= r(x) \quad \text{by 1.1.2} \\ &\leq ||x|| \end{aligned}$$

QED

a) implies that multiplicative functionals are automatically continuous. The map $x \to \hat{x}$ is called the Gelfand transform.

1.1.7) <u>Theorem</u>

Let $\mathfrak{A}$ be a commutative Banach algebra generated by one element: i.e. there is $a \in \mathfrak{A}$ such that polynomials in $a$ are dense in $\mathfrak{A}$. Then the map $\hat{a}$: $\mathrm{Sp}\,\mathfrak{A} \to \sigma(a) \subset \mathbb{C}$ is a homeomorphism.

Proof - $\hat{a}$ is continuous from $\mathrm{Sp}\,\mathfrak{A}$ onto $\sigma(a)$ by 1.1.6. Moreover, $\mathrm{Sp}\,\mathfrak{A}$ and $\sigma(a)$ are both compact Hausdorff spaces, so we need only prove that $\hat{a}$ is bijective. As just noted, $\hat{a}$ is onto $\sigma(a)$, so we need only show that it is injective.

Suppose that $\hat{a}(\ell_1) = \hat{a}(\ell_2)$. Since $\ell_1$ and $\ell_2$ are multiplicative, and $\hat{a}(\ell) = \ell(a)$, we conclude that

$$\ell_1\left(\sum_{n=0}^{N} c_n a^n\right) = \ell_2\left(\sum_{n=0}^{N} c_n a^n\right).$$

But $\ell_1$ and $\ell_2$ are continuous and $a$ generates $\mathfrak{A}$, so $\ell_1 = \ell_2$.

QED

If $a_1,\ldots,a_k$ generate $\mathfrak{A}$, then in exactly the same way, one shows that $\mathrm{Sp}\,\mathfrak{A}$ is homeomorphic to $\sigma(a_1)\times\ldots\times\sigma(a_k)$ as a subset of $\mathbb{C}^k$ under the map $\hat{a}_1 \otimes \ldots \otimes \hat{a}_k$.

## 1.2) C* - Algebras

### 1.2.1) Definition -

A Banach *-algebra is a Banach algebra together with a map $a \to a^*$ satisfying

i) * is conjugate linear,

ii) $a^{**} = a$ for all $a \in \mathfrak{A}$,

iii) $(ab)^* = b^*a^*$, all $a,b \in \mathfrak{A}$,

iv) $\|a^*\| = \|a\|$

A C*-algebra is a Banach *-algebra which satisfies

v) $\|a^*a\| = \|a\|^2$ for all $a \in \mathfrak{A}$.

An algebra satisfying all of 1.2.1 is also called a B*-algebra or an abstract C*-algebra. Clearly, a norm closed algebra of bounded operators on a complex Hilbert space also closed with respect to taking adjoints is a C*-algebra. Such an algebra is sometimes called a concrete C*-algebra since it is "concretely" given as an algebra of operators. However, we shall see that every abstract C*-algebra is isomorphic to such a concrete C*-algebra, so there is really no difference between them.

### 1.2.2) Theorem (Gelfand-Naimark)

Let $\mathfrak{A}$ be a commutative Banach *-algebra. Then the Gelfand transform $\hat{}: \mathfrak{A} \to C(\mathrm{Sp}\,\mathfrak{A})$ is an isometric *-isomorphism if and only if $\mathfrak{A}$ is a C*-algebra.

Proof - If $\hat{}$ is an isometric *-isomorphism, then the C*-property 1.2.1 v) follows from

$$\|f\|_\infty^2 = \|\,|f|^2\,\|_\infty \quad \text{for} \quad f \in C(\mathrm{Sp}\,\mathfrak{A}).$$ So $\mathfrak{A}$ is a C*-algebra.

Conversely, suppose that $\mathfrak{A}$ is a C*-algebra. Let $h \in \mathfrak{A}$ with $h = h^*$. Then set

$$u_t = e^{ith} \equiv \sum_{n=0}^{\infty} \frac{(it)^n}{n!} h^n \quad , \quad t \in \mathbb{R}.$$

We see that $u_t^* = u_{-t}$, and $u_t^* u_t = u_0 = \mathbb{1}$. Therefore

$$\| u_t^* u_t \| = \| u_t \|^2 = 1 ,$$

and so

$$\| u_t \| = \| u_t^* \| = \| u_{-t} \| = 1.$$

Now let $\ell \in \mathrm{Sp}\,\mathfrak{A}$. Then, since $\ell$ is continuous, we have

$$\ell(u_t) = e^{it\ell(h)}, \text{ and } \ell(u_{-t}) = e^{-it\ell(h)}.$$

By 1.1.6, $|\ell(u_{\pm t})| \leq ||u_{\pm t}|| = 1$. This holds for all $t \in \mathbb{R}$, so $\ell(h) \in \mathbb{R}$. In other words, $\hat{h}(\cdot)$ is real-valued.

If $x \in \mathfrak{A}$, we can write $x = \frac{1}{2}(x+x^*)+i\,\frac{i}{2}(x^*-x)$ and $\frac{1}{2}(x+x^*)$ and $\frac{i}{2}(x^*-x)$ are hermitian (i.e. invariant under*).

Hence

$$(x^*)\hat{}(\ell) = \ell(x^*) = \overline{\ell(x)} = \overline{\hat{x}(\ell)}$$

and so $\hat{}$ is a *-homomorphism.

To show that $\hat{}$ is isometric, consider again $h=h^* \in \mathfrak{A}$. Then $||h||^2 = ||h^2||$, and so $||h||^{2^n} = ||h^{2^n}||$.

Therefore

$$\| \hat{h} \|_\infty = r(h) = \lim_{n\to\infty} \| h^{2^n} \|^{1/2^n} = \| h \| .$$

Now if $x \in \mathfrak{A}$, we have $||\hat{x}||_\infty^2 = ||\bar{\hat{x}}\hat{x}||_\infty$

$= ||(x^*x)\hat{}||_\infty$ since $\hat{}$ is a *-homomorphism

$= ||x^*x||$ since $x^*x$ is hermitian

$= ||x||^2$

Thus $\hat{}$ is isometric, and hence injective. It remains only to show that $\hat{}$ is surjective. By the Stone-Weierstrass theorem, it is enough to show that ran $\hat{}$ is closed in $C(\mathrm{Sp}\,\mathfrak{A})$ and separates points of $\mathrm{Sp}\,\mathfrak{A}$.

Since $\hat{}$ is isometric, it follows that ran $\hat{}$ is complete (since $\mathfrak{A}$ is complete) and therefore closed in $C(\mathrm{Sp}\,\mathfrak{A})$. If $\ell_1 \neq \ell_2$ in $\mathrm{Sp}\,\mathfrak{A}$, then there is $x \in \mathfrak{A}$ such that $\ell_1(x) \neq \ell_2(x)$, i.e. $\hat{x}(\ell_1) \neq \hat{x}(\ell_2)$. So ran $\hat{}$ separates points of $\mathrm{Sp}\,\mathfrak{A}$.

QED

Example: Let $\mathfrak{A}$ be the algebra of functions analytic in $\{z \mid |z|<1\}$ and continuous on the boundary with $||f|| = \sup\limits_{|z|=1} |f(z)|$.

Then the function $g(z)=z$ belongs to $\mathfrak{A}$, and $\sigma_{\mathfrak{A}}(g)=\{\lambda : |\lambda| \leq 1\}$. Let $\mathfrak{B} = C(\{z : |z| = 1\})$ with the supremum norm. Then $\mathfrak{A} \subset \mathfrak{B}$.

However, $\sigma_{\mathfrak{B}}(g)=\{\lambda : |\lambda| = 1\}$. So we see that if we enlarge the algebra, we may shrink the spectrum of an element. This is to be expected since the spectrun is defined in terms of inverses existing, and these may exist if the algebra is enlarged. However, this situation does not occur for C*-algebras.

1.2.3) Theorem

Let $x \in \mathfrak{A} \subset \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are (not necessarily commutative) C*-algebras. Then $\sigma_{\mathfrak{A}}(x) = \sigma_{\mathfrak{B}}(x)$.

Proof: We need only show that whenever $x-\lambda$ has an inverse in $\mathfrak{B}$ this inverse is in the C*-algebra generated by $x$ and $x^*$. Equivalently, we need only show that if $a \in \mathfrak{B}$ is invertible, then $a^{-1}$ is in the C*-algebra generated by $a$ and $a^*$.

Suppose, first, that $h=h^*$ and $h^{-1}$ exists.

Let $\mathcal{A}_1$ denote the C*-algebra generated by $h$, and $\mathcal{A}_2$ that generated by $h$ and $h^{-1}$. Then $\mathcal{A}_2$ is commutative, and $\mathcal{A}_1 \subset \mathcal{A}_2$.

By 1.2.2, $\mathcal{A}_2 \simeq C(\mathrm{Sp}\,\mathcal{A}_2)$. Let $\ell_1, \ell_2 \in \mathrm{Sp}\,\mathcal{A}_2$, and suppose $\ell_1(h) = \ell_2(h)$.

Then $\ell_1(hh^{-1}) = \ell_1(h)\ell_1(h^{-1}) = \ell_1(\mathbb{1}) = 1$.

Hence $\ell_1(h^{-1})=\ell_1(h)^{-1}=\ell_2(h)^{-1}=\ell_2(h^{-1})$. Since $h, h^{-1}$ generate $\mathcal{A}_2$, it follows that $\ell_1=\ell_2$. Hence $\mathcal{A}_1$ considered as a subset of $C(\mathrm{Sp}\,\mathcal{A}_2)$ separates points of $\mathrm{Sp}\,\mathcal{A}_2$. Since $\mathcal{A}_1$ is an algebra closed under the $*$-operation containing constants, we conclude by the Stone-Weierstrass theorem, that $\hat{\mathcal{A}}_1$ is dense in $C(\mathrm{Sp}\,\mathcal{A}_2)$. But $\mathcal{A}_1$ is already norm closed, so we must have $\hat{\mathcal{A}}_1=C(\mathrm{Sp}\,\mathcal{A}_2)$, i.e. $\mathcal{A}_1=\mathcal{A}_2$. Hence $h^{-1}\in\mathcal{A}_1$.

Now let $a\in\mathfrak{A}$ be arbitrary. If $a^{-1}$ exists in $\mathfrak{B}$, so does $(a^*a)^{-1}=a^{-1}(a^{-1})^*$. Since $a^*a$ is hermitian, $(a^*a)^{-1}$ belongs to the C*-algebra generated by $a^*a$, and so belongs to that generated by $a$ and $a^*$. But $a^{-1}=(a^*a)^{-1}a^*$, and so $a^{-1}$ is in this last C*-algebra.

QED.

1.2.4) <u>Proposition</u> - Let $\mathfrak{A}$ be a C*-algebra generated by a single hermitian element $h$. Then $\mathfrak{A}$ is isometrically $*$-isomorphic to the algebra of continuous functions on $\sigma(h)$, in such a way that polynomials in $h$ are isomorphic to the same polynomial on $\sigma(h)$.

<u>Proof</u>: By 1.2.2, $\mathfrak{A}$ is isometrically $*$-isomorphic to $C(\mathrm{Sp}\,\mathfrak{A})$ under the Gelfand transform. By 1.1.7, $\hat{h}:\mathrm{Sp}\,\mathfrak{A}\to\sigma(h)$ is a homeomorphism. Hence, there is a one-one correspondence between $C(\mathrm{Sp}\,\mathfrak{A})$ and $C(\sigma(h))$ given by $f\in C(\mathrm{Sp}\,\mathfrak{A})\to f.\hat{h}^{-1}\in C(\sigma(h))$. Let $\alpha:C(\mathrm{Sp}\,\mathfrak{A})\to C(\sigma(h))$ be this isomorphism. Let $P(h)$ be a polynomial in $h$. Then, under the Gelfand transform, $P(h)$ becomes $P(\hat{h})\in C(\mathrm{Sp}\,\mathfrak{A})$, and

$$\alpha(P(\hat{h}))(\lambda) = P(\hat{h})(\hat{h}^{-1}(\lambda)) = P(\lambda)$$

QED.

It is convenient to state here the following result, without proof.

1.2.5) <u>Proposition</u> - Let $\mathfrak{A}$ be a C*-algebra and let J be a closed two-sided ideal in $\mathfrak{A}$. Then $\mathfrak{A}/J$ with respect to the norm $\|c\ell x\| = \inf_{j\in J} \|x+j\|$ is a C*-algebra.

This can be proved in several ways. For a proof using the notion of approximate identity see Dixmier (1969a). Naimark (1964) uses quasi-inverses, and Sakai (1971) uses bipolars and central projections in $\mathfrak{A}^{**}$.

1.3) <u>Spectral Theory</u> -

There are several forms of the spectral theorem, the most convenient of which depends on the context. We shall give three different formulations.

1.3.1) <u>Theorem</u> - Let A be a bounded self-adjoint operator on a Hilbert space, $\mathfrak{H}$. Then there exists a family $\{\mu_\alpha\}$ of real Baire measures, each on $\sigma(A)$, such that $\mathfrak{H}$ is unitarily equivalent to $\bigoplus_\alpha L^2(\sigma(A), d\mu_\alpha)$ and A is unitarily equivalent to multiplication by $\lambda$, i.e. if $f \in \bigoplus_\alpha L^2(\sigma(A), d\mu_\alpha)$, $(Af)(\lambda) = \lambda f(\lambda)$.

<u>Proof</u> - Let $\mathfrak{A}$ be the C*-algebra generated by the self-adjoint bounded operator A. $\mathfrak{A}$ is a commutative sub-algebra of $\mathfrak{B}(\mathfrak{H})$, the C*-algebra of all bounded operators on $\mathfrak{H}$.

By 1.2.3, $\sigma_{\mathfrak{A}}(A) = \sigma_{\mathfrak{B}(\mathfrak{H})}(A) \equiv \sigma(A)$, and so, by 1.2.4, $\mathfrak{A}$ is isometrically *-isomorphic to $C(\sigma(A))$. Let $\phi : C(\sigma(A)) \to \mathfrak{A}$ be the inverse of this isomorphism.

Suppose there is a vector $\xi \in \mathfrak{H}$ such that $\{\mathfrak{A}\xi\}$ is dense in $\mathfrak{H}$. Then $\mu_\xi$ defined on $C(\sigma(A))$ by

$$\mu_\xi(f) = (\xi, \phi(f)\xi)$$

is a positive linear functional. By the Riesz-Markov theorem there is a measure $d\mu_\xi$ on $\sigma(A)$ with

$$\mu_\xi(f) = \int_{\sigma(A)} f d\mu_\xi$$

Define $U : C(\sigma(A)) \to \mathfrak{H}$ by $Uf = \phi(f)\xi$. Then

$$||Uf||^2 = (\xi, \phi(f)^*\phi(f)\xi) = \int_{\sigma(A)} |f|^2 d\mu_\xi$$

$$= ||f||^2_{L^2(\sigma(A), d\mu_\xi)}$$

U is isometric with a dense range and a dense domain of definition in $L^2(\sigma(A), d\mu_\xi)$. It therefore defines a unitary operator from $L^2(\sigma(A), d\mu_\xi)$ onto $\mathfrak{H}$.

Let $f \in C(\sigma(A))$. Then

$$\begin{aligned}(U^{-1} A\, U\, f)(\lambda) &= (U^{-1} A \phi(f)\xi)(\lambda) \\ &= \phi^{-1}(A\phi(f))(\lambda) \\ &= \phi^{-1}(A)(\lambda) f(\lambda) \\ &= \lambda f(\lambda) \quad \text{by } 1.2.4\end{aligned}$$

It is easy to extend this to any $f \in L^2(\sigma(A), d\mu_\xi)$. If there does not exist a $\xi \in \mathfrak{H}$ as above, then, using Zorn's lemma, we can find orthogonal subspaces $\{\mathfrak{H}_\alpha\}$ in $\mathfrak{H}$, with $\bigoplus_\alpha \mathfrak{H}_\alpha = \mathfrak{H}$, and vectors $\xi_\alpha \in \mathfrak{H}_\alpha$ with $\mathfrak{A}\xi_\alpha$ dense in $\mathfrak{H}_\alpha$. Then, as above, we construct $U_\alpha : L^2(\sigma(A), d\mu_{\xi_\alpha}) \to \mathfrak{H}_\alpha$. $\bigoplus_\alpha U_\alpha$ gives the required equivalence.

The more conventional form of the spectral theorem is the following.

1.3.2) <u>Theorem</u> - Let A be a bounded self-adjoint operator on a Hilbert space, $\mathfrak{H}$. Then there is a family $\{E_\lambda | \lambda \in \mathbb{R}\}$ of projections on $\mathfrak{H}$ satisfying:

(i) $E_\lambda$ is a strong limit of polynomials in A,

(ii) $E_\lambda E_\mu = E_\mu$ if $\mu \leq \lambda$,

(iii) $\underset{\varepsilon \downarrow 0}{\text{s-lim}}\, E_{\lambda+\varepsilon} = E_\lambda$, $\underset{\lambda \to -\infty}{\text{s-lim}}\, E_\lambda = 0$, $\underset{\lambda \to +\infty}{\text{s-lim}}\, E_\lambda = \mathbb{1}$.

(iv) $A = \int_{\mathbb{R}} \lambda dE_\lambda = \lim_{\varepsilon \downarrow 0} \int_{-||A||-\varepsilon}^{||A||} \lambda dE_\lambda$, where the integral is a Stieltjes integral in the norm topology.

The family $\{E_\lambda\}$ is uniquely determined by (ii), (iii) and (iv).

<u>Proof</u> - First we shall construct such a family $\{E_\lambda\}$ of projections on $\mathfrak{H}$. Let $\mathfrak{A}$ be the $C^*$-algebra generated by A.

Then $\mathfrak{A}$ is commutative and so it is isometrically *-isomorphic to $C(K)$, where $K=\mathrm{Sp}\,\mathfrak{A}$, by 1.2.2. $A$ is isomorphic to $\hat{A}(\cdot)$, which is real-valued since $A$ is self-adjoint (1.2.2). Let $\lambda \in \mathbb{R}$. Define a function $P_\lambda(\cdot)$ on $K$ by

$$P_\lambda(\cdot) = \chi_{\{\kappa \in K | \hat{A}(\kappa) \leq \lambda\}}(\cdot)$$

i.e. $P_\lambda(\kappa) = 1$ if $\hat{A}(\kappa) \leq \lambda$, otherwise $P_\lambda(\kappa) = 0$. Set $\Lambda = \{\kappa \in K | \hat{A}(\kappa) \leq \lambda\}$. Since $\hat{A}(\cdot)$ is continuous, $\Lambda$ is closed in $K$. Let $\zeta(\cdot)$ be a non-negative function in $C(K)$ such that $0 \leq \zeta(\kappa) < 1$, $\kappa \notin \Lambda$, and $\zeta(\kappa) = 1$, $\kappa \in \Lambda$. (Such functions exist by Uhrysohn's lemma. See for example Naimark (1964)). Clearly $\zeta^n(\kappa)$ converges to $P_\lambda(\kappa)$ as $n \to \infty$, for each $\kappa \in K$.

Let $\xi \in \mathfrak{H}$. Then the map $\hat{T} \to (\xi, T\xi)$, $T \in \mathfrak{A}$, defines a positive linear functional on $C(K)$. By the Riesz-Markov representation theorem, there is a regular Borel measure $\mu_\xi$ on $K$ such that

$$(\xi, T\xi) = \int_K \hat{T}(\kappa)\, d\mu_\xi(\kappa)$$

By the dominated convergence theorem, we see that $\zeta^n$ converges to $P_\lambda$ in $L^2(K, d\mu_\xi)$. In particular, $\zeta^n$ is $L^2$-Cauchy. Since $\zeta \in C(K)$, there is $T \in \mathfrak{A}$ such that $\zeta = \hat{T}$, $\zeta^n = (T^n)\hat{}$. We have

$$\|\zeta^n - \zeta^m\|^2_{L^2(K, d\mu_\xi)} = (\xi, (T^n - T^m)^*(T^n - T^m)\xi)$$

$$= \|(T^n - T^m)\xi\|^2_{\mathfrak{H}}$$

i.e. $\{T^n \xi\}$ is Cauchy in $\mathfrak{H}$, for each $\xi \in \mathfrak{H}$. It follows that $\operatorname{s-lim}_{n\to\infty} T^n$ exists, and defines a bounded operator, denoted $E_\lambda$. Since $\zeta$ is real, each $T^n$ is self-adjoint, so $E_\lambda = E_\lambda^*$. Furthermore, since $\|T^n\|$ is uniformly bounded (by 1), we have

$$E_\lambda^2 = \operatorname{s-lim}_{n\to\infty} T^n T^n = \operatorname{s-lim}_{n\to\infty} T^{2n} = E_\lambda.$$

Therefore $E_\lambda$ is a projection on $\mathcal{H}$, for each $\lambda \in \mathbb{R}$.

It is clear that $E_\lambda$ is a strong limit of polynomials in A because it is a strong limit of elements of $\mathfrak{A}$, and each element of $\mathfrak{A}$ is a norm limit of polynomials in A. This proves (i).

To prove (ii), we note that if $\mu \leq \lambda$, then $\zeta_\lambda \zeta_\mu = \zeta_\mu$, and so $T_\lambda T_\mu = T_\mu$, which implies that $E_\lambda E_\mu = E_\mu$.

Now, as before, we have, for $\xi \in \mathcal{H}$,

$$\|(E_{\lambda+\varepsilon} - E_\lambda)\xi\|^2 = \int_K |P_{\lambda+\varepsilon} - P_\lambda|^2 d\mu_\xi .$$

But $P_{\lambda+\varepsilon}$ converges pointwise to $P_\lambda$ as $\varepsilon \downarrow 0$, and so by Lebesgue's dominated convergence theorem, $||(E_{\lambda+\varepsilon} - E_\lambda)\xi|| \to 0$ as $\varepsilon \downarrow 0$, for each $\xi \in \mathcal{H}$. That is, $E_\lambda = \operatorname{s-lim}_{\varepsilon\downarrow 0} E_{\lambda+\varepsilon}$.

In exactly the same way, $||(E_\lambda - 0)\xi|| \to 0$ as $\lambda \to -\infty$, and $||(E_\lambda - \mathbb{1})\xi|| \to 0$ as $\lambda \to \infty$. This proves (iii).

We have $|\hat{A}(\cdot)| \leq ||A||$. Let us divide the interval $(-||A||-\varepsilon, ||A||]$ into n equal parts, which we denote by $I_j$, $1 \leq j \leq n$. $I_j$ is the half-open interval $(a_j, a_{j+1}]$. Then

$$\chi_{\{\kappa \in K \mid \hat{A}(\kappa) \in I_j\}}(\cdot) = (P_{a_{j+1}} - P_{a_j})(\cdot)$$

It is easy to see that $\Sigma_{j=1}^{n} a_{j+1} (P_{a_{j+1}} - P_{a_j})(\cdot) \equiv S_n$ converges uniformly to $\hat{A}(\cdot)$ as $n \to \infty$.

Hence, for given $\delta > 0$, there is an N such that for all $n > N$,

$$\int_K |S_n - \hat{A}|^2 d\mu_\xi < \delta \int_K d\mu_\xi = \delta(\xi, \xi) = \delta \|\xi\|^2$$

That is, for $n>N$

$$\left\| \left( \sum_{j=1}^{n} a_{j+1} (E_{a_{j+1}} - E_{a_j}) - A \right) \xi \right\|^2 < \delta \|\xi\|^2$$

and we see that the sum converges in norm to **A**. Thus

$$A = \int_{-||A||-\varepsilon}^{||A||} \lambda dE_\lambda , \quad \varepsilon > 0 \quad \text{arbitrary.}$$

If $\lambda < -||A||$, $E_\lambda = 0$, and if $\lambda > ||A||$, $E_\lambda = \mathbb{1}$ so we can write

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

The proof of (iv) is now complete.

To see that $\{E_\lambda\}$ is unique, we note that since $E_\lambda - E_\mu$ is orthogonal to $E_\alpha - E_\beta$ if $(\lambda,\mu] \cap (\alpha\ \beta] = \emptyset$, we have

$$A^2 = \int \lambda^2 dE_\lambda \quad , \text{ or } \quad A^n = \int \lambda^n dE_\lambda .$$

Let $f(\lambda)$ be the characteristic function of the interval $(-||A||-1, \mu]$. Then, for $\xi \in \mathfrak{H}$,

$$\int f(\lambda)\, d(\xi, E_\lambda \xi) = \lim \sum_{\lambda_i \le \mu} (\xi, (E_{\lambda_i} - E_{\lambda_{i-1}})\xi)$$

$$= \lim (\xi, (E_\mu - E_{\lambda_{n-1}})\xi) + (\xi, (E_{\lambda_{n-1}} - E_{\lambda_{n-2}})\xi)$$

$$+ \dots + (\xi, (E_{\lambda_2} - E_{\lambda_1})\xi)$$

$$= (\xi, E_\mu \xi) - (\xi, E_{\lambda_1} \xi) = (\xi, E_\mu \xi)$$

since $E_\lambda = 0$ if $\lambda < -\|A\|$,

$$= \|E_\mu \xi\|^2 .$$

Let $\wp_n(\lambda)$ be a sequence of polynomials converging pointwise to $f(\lambda)$, $\lambda \in (-||A||-1, ||A||]$, and uniformly bounded on this interval. Then

$$\|E_\mu \xi\|^2 = \int f(\lambda)\, d(\xi, E_\lambda \xi) = \lim_n \int \wp_n(\lambda)\, d(\xi, E_\lambda \xi)$$

$$= \lim_n (\xi, \wp_n(A)\xi) .$$

Since $(\xi, \wp_n(A)\xi)$ is defined independently of $E_\mu$, we see that $E_\mu$ is, indeed, uniquely determined by A.

QED.

The projections $\{E_\lambda\}$ are called the spectral projections of A.

We note that by (i) and (iv), an operator $B \in \mathcal{B}(\mathcal{H})$ commutes with A if and only if B commutes with all spectral projections of A.

i) gives only strong convergence. The $E_\lambda$ may not belong to $\mathfrak{A}$. Indeed, $\mathfrak{A}$ may possess no projections.

1.3.3.) <u>Theorem:</u> Let $\mathfrak{A}$ be a commutative C*-algebra of operators on a Hilbert space $\mathcal{H}$. Then there exists a Hausdorff space K and a Borel measure $\mu$ on K such that $\mathcal{H}$ is unitarily equivalent to $L^2(K,d\mu)$ and each $A \in \mathfrak{A}$ is unitarily equivalent to multiplication by a continuous function on K.

<u>Proof</u> - The proof is just as in 1.3.1, with $K = \mathrm{Sp}\,\mathfrak{A}$ and $\mu = \bigoplus_\alpha \mu_\alpha$ is the measure on K given by the vectors $\xi_\alpha \in \mathcal{H}_\alpha$ such that $\mathfrak{A}\xi_\alpha$ is dense in $\mathcal{H}_\alpha$ and $\bigoplus_\alpha \mathcal{H}_\alpha = \mathcal{H}$. The unitaries, $U_\alpha$, are defined by $U_\alpha^{-1} : \mathfrak{A}\xi_\alpha \to C(K)$,

$$U_\alpha^{-1} A \xi_\alpha = \hat{A}(\cdot).$$

QED.

1.3.4) <u>Definition</u> - Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that A and B commute if and only if the unitary groups $\exp(is\,A)$ and $\exp(it\,B)$ commute for all $s,t \in \mathbb{R}$.

If A and B are bounded, this is equivalent to the usual notion of commutativity.

1.3.5) <u>Corollary</u> - Let $\{A_\alpha\}_{\alpha\in I}$ be a family of commuting self-adjoint operators on a Hilbert space $\mathcal{H}$. Then there is a Hausdorff space K and a Borel measure $\mu$ on K such that $\mathcal{H}$ is unitarily equivalent to $L^2(K,d\mu)$ and such that each $A_\alpha$ is unitarily equivalent to multiplication by some real measurable function.

<u>Proof</u> - The C*-algebra generated by the unitary operators $\{\exp(is A_\alpha) \mid s \in \mathbb{R}, \alpha \in I\}$ is commutative and so by 1.3.3, $\mathcal{A} \simeq L^2(K, d\mu)$, some $K, \mu$, and each $\exp(is A_\alpha)$ is given by multiplication on $L^2(K, d\mu)$ by a continuous function. One can then show that this function is the exponential of a real function on $K$, multiplication by which is unitarily equivalent to $A_\alpha$.

QED.

For further details and a discussion of many related topics we refer to Segal and Kunze (1968).

1.3.3 and 1.3.5 express the simultaneous diagonalizability of commuting operators. Indeed, 1.3.5 is the definition of a self-adjoint operator as given by Segal and Kunze (1968).

## 1.4) POSITIVE ELEMENTS OF A C*-ALGEBRA

1.4.1) <u>Definition:</u> An element $a$ in a C*-algebra $\mathfrak{A}$ is called positive if and only if $a = b^2$ for some hermitian $b \in \mathfrak{A}$. We write $a \geq 0$.

1.4.2) <u>Proposition:</u> Let $\mathfrak{A}$ be a C*-algebra, and let $a \in \mathfrak{A}$ with $a = a^*$. Then $a \geq 0$ if and only if $\sigma(a) \subset [0, \infty)$.

<u>Proof</u> - Suppose $\sigma(a) \subset [0, \infty)$. Then, if $\mathcal{A}$ is the C*-algebra generated by $a$, we have $\mathcal{A} \simeq C(\mathrm{Sp}\,\mathcal{A})$ and ran $\hat{a} = \sigma(a) \subset [0, \infty)$, i.e. $\hat{a}(.) \geq 0$. Let $f \in C(\mathrm{Sp}\,\mathcal{A})$ be the positive square root of $\hat{a}(.)$. Let $b \in \mathcal{A}$ such that $\hat{b} = f$. Then $b = b^*$ and $b^2 = a$, since $f^2 = \hat{a}$. Hence $a \geq 0$.

Conversely, suppose $a \geq 0$. Then $a = b^2$ for some $b = b^* \in \mathfrak{A}$, and so $a = a^*$ and $\sigma(a) \subset \mathbb{R}$. Let $\mu \in \mathbb{R}$. Then

$$(a + \mu^2 \mathbb{1}) = (b^2 + \mu^2 \mathbb{1}) = (b - i\mu)(b + i\mu)$$

However, $b = b^*$ implies that $\sigma(b) \subset \mathbb{R}$ and so $(b \pm i\mu)$ is invertible. Hence $(a + \mu^2 \mathbb{1})$ is invertible for all $\mu \in \mathbb{R}$, i.e. $(a - \lambda \mathbb{1})^{-1}$ exists for all $\lambda < 0$. In other words, $\sigma(a) \subset [0, \infty)$.

1.4.3) <u>Corollary</u> - Let $a \in \mathfrak{A}$, $a \geq 0$. Then there is a unique $s \in \mathfrak{A}$ such that $s \geq 0$ and $s^2 = a$.

<u>Proof</u> - By 1.4.2, there is $b \in \mathcal{A}(a)$, the C*-algebra generated by a, with $b \geq 0$ and $b^2 = a$. We only need to prove uniqueness. Suppose $t \in \mathfrak{A}$, $t \geq 0$ and $t^2 = a$. Since a commutes with t and $b \in \mathcal{A}(a)$, b commutes with t.
Hence $\mathcal{A}(a,t)$ the C*-algebra generated by a and t, is commutative and contains a, b and t. Realizing $\mathcal{A}(a,t)$ as C(K) some K, we have $\hat{a} = \hat{b}^2 = \hat{t}^2$, $\hat{b} \geq 0$, $\hat{t} \geq 0$. It follows that $\hat{b} = \hat{t}$ and so b=t.

QED.

1.4.4) <u>Proposition</u> - Let $\mathfrak{A}$ be a C*-algebra and let $k, h \in \mathfrak{A}$ with $k \geq 0$, $h \geq 0$. Then $h+k \geq 0$.

<u>Proof</u> - Let $a = a^* \in \mathfrak{A}$, with $||a|| \leq 1$. By realizing a as a continuous function on the spectrum of the C*-algebra it generates, we see that $a \geq 0$ if and only if $||\mathbb{1} - a|| \leq 1$. Now

$$\left\| \mathbb{1} - \frac{h+k}{\|h\|+\|k\|} \right\| = \frac{\| \|h\| - h + \|k\| - k \|}{\|h\| + \|k\|}$$

$$\leq \frac{\| \|h\| - h \| + \| \|k\| - k \|}{\|h\| + \|k\|}$$

$$= \frac{\|h\| \, \|(\mathbb{1} - h/\|h\|)\| + \|k\| \, \|(\mathbb{1} - k/\|k\|)\|}{\|h\| + \|k\|}$$

$\leq 1$ since $h/||h|| > 0$ and $k/||k|| \geq 0$.

Thus $(h+k)/\|h\|+\|k\| \geq 0$, i.e. $h+k \geq 0$.

QED

1.4.5) <u>Proposition</u> - Let $\mathfrak{A}$ be a C*-algebra, and $a,b \in \mathfrak{A}$. Then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

<u>Proof</u> - Suppose $\lambda \neq 0$, and $ab-\lambda\mathbb{1}$ has an inverse, $u$. Then

$$\begin{aligned}(ba-\lambda)(bua-\mathbb{1}) &= babua - ba - \lambda bua + \lambda \\ &= b(ab-\lambda)ua + \lambda bua + ba - \lambda bua + \lambda \\ &= ba - ba + \lambda = \lambda\mathbb{1}.\end{aligned}$$

Similarly, $(bua - \mathbb{1})(ba - \lambda) = \lambda\mathbb{1}$. Hence $ba-\lambda$ is invertible. It follows that $\sigma(ab) \cup \{0\} \supset \sigma(ba)$. In the same way, we have $\sigma(ba) \cup \{0\} \supset \sigma(ab)$. The result follows.

QED

1.4.6) <u>Theorem</u> : Let $\mathfrak{A}$ be a C*-algebra, and let $a \in \mathfrak{A}$. Then $a \geq 0$ if and only if $a = x^*x$, some $x \in \mathfrak{A}$.

<u>Proof</u> - If $a \geq 0$, then $a = b^2$, some $b = b^* \in \mathfrak{A}$.

Conversely, suppose $a = x^*x$, some $x \in \mathfrak{A}$. Then $a=a^*$. Suppose $x^*x$ is not positive. Let $\mathcal{A}$ be the commutative C*-algebra generated by a. Then $\mathcal{A}$ is isomorphic to $C(\mathrm{Sp}\mathcal{A})$ under the Gelfand transform. Since, by assumption, a is not positive, neither is $\hat{a}(\cdot)$, and so there is an $\ell_0 \in \mathrm{Sp}\mathcal{A}$ such that $\hat{a}(\ell_0) < 0$. But $\hat{a}$ is continuous, hence there is a neighbourhood N of $\ell_0$ in $\mathrm{Sp}\mathcal{A}$ such that $\hat{a}(\ell) < 0$ for all $\ell \in N$. Let $f \in C(\mathrm{Sp}\mathcal{A})$ be such that f is zero on the closed set $\mathrm{Sp}\mathcal{A} \setminus N$, and strictly positive on N. (Such an f exists, by Uhrysohn's Lemma). Let $b \in \mathcal{A}$ with $\hat{b} = f$. Then since $f\hat{a}f < 0$, we have $bab \leq 0$, (i.e. $-bab \geq 0$). That is, $bx^*xb \leq 0$.

Write $xb = h+ik$ with $h,k \in \mathfrak{A}$, $h=h^*$, $k=k^*$. $(2h = xb + (xb)^*$, $2ik = xb - (xb)^*)$.
Then $(xb)^*(xb) = h^2 + k^2 + ihk - ikh$ and, $(xb)(xb)^* = h^2+k^2+ikh-ihk$.

Hence $(xb)^*(xb) + (xb)(xb)^* = 2(h^2+k^2) \geq 0$, by 1.4.4. But, $-bx^*xb = -(xb)^*(xb) \geq 0$, and so, by 1.4.4,

$$(xb)(xb)^* = -bx^*xb + 2(h^2+k^2) \geq 0.$$

On the other hand, by 1.4.5, $\sigma((xb)(xb)^*)\cup\{0\} = \sigma((xb)^*(xb))\cup\{0\}$ and so $\sigma((xb)(xb)^*) \subset \mathbb{R}^-$ (by 1.4.2, since $bx^*xb \leq 0$). That is, $(xb)(xb)^* \leq 0$. This implies that $(xb)(xb)^* = 0$ which is false ($\hat{f}\hat{a}\hat{f} \neq 0$). We must therefore have that $x^*x \geq 0$.

QED

1.4.7) Propositon - Let $\mathfrak{A}$ be a C*-algebra, and $a \in \mathfrak{A}$. Then a can be written as a linear combination of

i) two hermitian elements of $\mathfrak{A}$,

ii) four positive elements of $\mathfrak{A}$,

iii) four unitary elements of $\mathfrak{A}$.

Proof: i) $a = 1/2\ (a+a^*) + i/2\ ((a-a^*)/i)$.

ii) Let $h=h^* \in \mathfrak{A}$. Let $|h|$ denote the positive square root of $h^2$ (1.4.3). As in the proof of 1.4.2, one sees that $h + |h|$ and $|h| - h$ are both positive. But $h = \frac{1}{2}(h+|h|) - \frac{1}{2}(|h|-h)$. Now use (i).

iii) Let $h = h^* \in \mathfrak{A}$. Then $h^2 \geq 0$. Suppose $\|h\| < 1$. Then $\mathbb{1} - h^2 \geq 0$ and so has a positive square root, $(\mathbb{1} - h^2)^{1/2}$

Let $u = h + i(\mathbb{1} - h^2)^{1/2}$. Then $u^*u = uu^* = \mathbb{1}$, i.e. u is unitary. Moreover, $h = \frac{1}{2}(u + u^*)$.

Now, if $\|h\| \geq 1$, consider $\alpha h$ with $\alpha = (2\|h\|)^{-1}$. As above $\alpha h = \frac{1}{2}(v+v^*)$, with v unitary, i.e. $h = \frac{1}{2\alpha}(v+v^*)$. Now use (i).

QED.

## 1.5) Homomorphisms

1.5.1) Definition - Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras. A homomorphism $\phi: \mathfrak{A} \to \mathfrak{B}$ is a linear map such that:

(i) $\phi(ab) = \phi(a)\phi(b)$, any $a,b \in \mathfrak{A}$,

(ii) $\phi(\mathbb{1}) = \mathbb{1}$,

(iii) $\phi(a^*) = \phi(a)^*$, all $a \in \mathfrak{A}$.

1.5.2) <u>Definition</u> - A map $\phi : \mathfrak{A} \to \mathfrak{B}$ is said to be order preserving if $a \geq 0$ implies $\phi(a) \geq 0$, all $a \in \mathfrak{A}$.

1.5.3) <u>Proposition</u> - Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras, and $\phi : \mathfrak{A} \to \mathfrak{B}$ a homomorphism. Then $\phi$ is order preserving, and is norm decreasing.

<u>Proof</u> - Let $a \in \mathfrak{A}$, $a \geq 0$. Then $a = b^2$, $b = b^* \in \mathfrak{A}$. Thus $\phi(a) = \phi(b^2) = \phi(b)^2$, and $\phi(b)^* = \phi(b^*) = \phi(b)$. Hence $a \geq 0$ implies that $\phi(a) \geq 0$.

Let $a \in \mathfrak{A}$, $a = a^*$. By 1.2.2, $-||a||\mathbb{1} \leq a \leq ||a||\mathbb{1}$. Since $\phi$ is order preserving, we have $-||a||\mathbb{1} \leq \phi(a) \leq ||a||\mathbb{1}$. By 1.2.2, we conclude that $||\phi(a)|| \leq ||a||$.
Now, for arbitrary $a \in \mathfrak{A}$, $||\phi(a)||^2 = ||\phi(a)^*\phi(a)|| = ||\phi(a^*a)||$.

$$\leq ||a^*a|| = ||a||^2$$

QED.

1.5.4. <u>Proposition</u> - Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras, and $\phi$ a one-one homomorphism. Then $\phi^{-1}$ is order preserving, and $\phi$ is norm preserving.

<u>Proof</u> - Let $\phi(a) \geq 0$. We must show that $a \geq 0$. Consider

$$\begin{aligned} \phi\left(\left\{\frac{a+a^*}{2}\right\}^2\right) &= \frac{1}{4}\phi\left(a^2 + a^*a + aa^* + a^{*2}\right) \\ &= \frac{1}{4}\left(\phi(a)^2 + \phi(a)^*\phi(a) + \phi(a)\phi(a)^* + \phi(a)^{*2}\right) \\ &= \phi(a)^2, \text{ since } \phi(a) \geq 0 \text{ implies } \phi(a)^* = \phi(a). \end{aligned}$$

By the uniqueness of the positive square root, we see that

$$\phi\left(\left\{\left(\frac{a+a^*}{2}\right)^2\right\}^{1/2}\right) = \phi(a).$$

But $\phi$ is one-one, so

$$a = \left\{ \left( \frac{a+a^*}{2} \right)^2 \right\}^{1/2} \geq 0$$

as required.

Exactly as in 1.5.3, we obtain for $b=b^* \in \phi(\mathfrak{A})$,

$$\| \phi^{-1}(b) \| \leqslant \| b \|$$

i.e. $\| a \| \leqslant \| \phi(a) \|$ for all $a = a^* \in \mathfrak{A}$, and hence for all $a \in \mathfrak{A}$. Together with 1.5.3, this implies that $\|\phi(a)\| = \|a\|$ for all $a \in \mathfrak{A}$.

QED.

1.5.5) <u>Corollary</u> - The norm in a C*-algebra is unique. Moreover, if $|||\cdot|||$ is a norm on $\mathfrak{A}$ with respect to which the completion of $\mathfrak{A}$ is a C*-algebra, then $|||\cdot||| = \|\cdot\|$.

<u>Proof</u> - Let $i : \mathfrak{A} \to \mathfrak{B}$ be the identification mapping of $\mathfrak{A}$ into $\mathfrak{B}$, the completion of $\mathfrak{A}$ w.r.t. $|||\cdot|||$. Then, by 1.5.4,

$$|||\, i(a) \,||| = \| a \| \qquad \text{for all } a \in \mathfrak{A},$$

i.e. $|||a||| = \|a\|$.

QED

1.5.6) <u>Definition</u> - Let $\mathfrak{A}$ be a C*-algebra. An endomorphism is a homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}$.

$\phi$ is a monomorphism if $\phi$ is a one-one endomorphism, and an automorphism if it is an endomorphism which is both one-one and onto. The family of automorphisms of a C*-algebra $\mathfrak{A}$ is denoted by Aut $\mathfrak{A}$. Evidently Aut $\mathfrak{A}$ is a group under composition.

1.5.7) <u>Propositon</u> - Let $\mathfrak{A}$ be a C*-algebra, and $\phi$ a monomorphism. Them $\phi$ is norm preserving.

<u>Proof</u> - Immediate from 1.5.4.

1.5.8) <u>Definition</u> - An automorphism $\phi$ of a C*-algebra is called inner if there exists $u \in \mathfrak{A}$, u unitary, such that $\phi(a) = uau^*$ for all $a \in$ .

Obviously, any unitary $u \in \mathfrak{A}$ defines an automorphism by $a \longrightarrow uau^*$, but not all automorphisms arise in this way. We also note that if $\phi$ is inner, then the u is not necessarily unique. Indeed, if $v \in Z(\mathfrak{A}) = \{b \in \mathfrak{A} \mid ab = ba \text{ for all } a \in \mathfrak{A}\}$, with $v$ unitary, then $v a v^* = a$. So u and $uv$ give the same automorphism.

1.5.9) <u>Definition</u> - Let $\mathfrak{A}$ be a C*-algebra of operators on a Hilbert space $\mathcal{H}$, and let $\gamma \in \operatorname{Aut} \mathfrak{A}$. $\gamma$ is said to be implementable if there is a unitary operator U on $\mathcal{H}$ such that

$$\gamma(a) = U a U^* \qquad \text{for all } a \in \mathfrak{A}.$$

Clearly, if $\gamma$ is inner it is implementable, but the converse is not true in general. As before, U may not be unique.

1.6) <u>REPRESENTATIONS</u>

1.6.1) <u>Definition</u> - Let $\mathfrak{A}$ be a C*-algebra. A representation of $\mathfrak{A}$ is a pair $(\mathcal{H}, \pi)$ consisting of a Hilbert space $\mathcal{H}$, and a homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$, from $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$, the set of all bounded operators on $\mathcal{H}$. $(\mathcal{H}, \pi)$ is said to be a faithful representation if $\ker \pi = \{0\}$, i.e. $\pi(a) = 0$ implies $a=0$.

1.6.2) <u>Proposition</u> - Let $(\mathcal{H}, \pi)$ be a representation of a $C^*$-algebra, $\mathfrak{A}$. Then $\pi(\mathfrak{A})$ is a $C^*$-algebra in $\mathcal{B}(\mathcal{H})$. <u>Proof</u> - Let $\mathcal{J} = \ker \pi$. Since $\pi$ is norm decreasing, it is continuous, and so $\mathcal{J}$ is closed. Also $\mathcal{J}$ is a two-sided ideal and $a \in \mathcal{J}$ implies that $a^* \in \mathcal{J}$. Hence $\mathfrak{A}/\mathcal{J}$ is a $C^*$-algebra (1.2.5).

Let $p: \mathfrak{A} \to \mathfrak{A}/\mathcal{J}$ be the canonical map. Let $\xi \in \mathfrak{A}/\mathcal{J}$,

$a \in \xi$. Define $\phi(\xi) = \pi(a)$. We easily see that $\phi$ is well-defined on $\mathcal{A}/\mathcal{J}$, and defines a homomorphism: $\mathcal{A}/\mathcal{J} \to \mathcal{B}(\mathcal{H})$. Moreover, we see that $\phi$ is one-one onto $\pi(\mathcal{A})$. Therefore $\phi$ is norm preserving and so $\pi(\mathcal{A})$ is isometrically isomorphic to $\mathcal{A}/\mathcal{J}$. Therefore $\pi(\mathcal{A})$ is a $C^*$-algebra.

QED.

1.6.3) <u>Proposition</u> - Let $(\mathcal{H}, \pi)$ be a representation of a $C^*$-algebra $\mathcal{A}$. Then $\pi$ is norm decreasing. $\pi$ is norm preserving if and only if $(\mathcal{H}, \pi)$ is faithful. <u>Proof</u> - By 1.5.3, $\pi$ is norm decreasing, and by 1.5.4, norm preserving if $(\mathcal{H}, \pi)$ is faithful.

However, if $\pi$ is norm preserving, then $\pi(a) = 0$ implies $0 = \|\pi(a)\| = \|a\|$, i.e. $a = 0$. That is, $(\mathcal{H}, \pi)$ is faithful.

QED.

1.6.4) <u>Definition</u> - Let $(\mathcal{H}, \pi)$ be a representation of a $C^*$-algebra $\mathcal{A}$. $(\mathcal{H}, \pi)$ is said to be irreducible if and only if the only closed subspaces of $\mathcal{H}$ invariant under $\pi(\mathcal{A})$ are $\{0\}$ and $\mathcal{H}$ itself.

1.6.5) <u>Definition</u> - Let $\mathcal{M}$ be a set of operators in $\mathcal{B}(\mathcal{H})$. The commutant $\mathcal{M}'$ of $\mathcal{M}$ is the set

$$\mathcal{M}' = \{ b \in \mathcal{B}(\mathcal{H}) \mid ab = ba \text{ for all } a \in \mathcal{M} \}$$

1.6.6) <u>Theorem</u> - Let $(\mathcal{H}, \pi)$ be a representation of a $C^*$-algebra $\mathcal{A}$. $(\mathcal{H}, \pi)$ is irreducible if and only if $\pi(\mathcal{A})' = \mathbb{C}\mathbb{1}$

<u>Proof</u> - Suppose $\pi(\mathcal{A})' = \mathbb{C}\mathbb{1}$, and suppose that $(\mathcal{H}, \pi)$ is not irreducible. Then there is a proper closed subspace $V$ of $\mathcal{H}$ such that $\pi(\mathcal{A}) V \subset V$.

Let $P$ be the orthogonal projection of $\mathcal{H}$ onto $V$. Then $P \in \pi(\mathcal{A})'$. To see this, let $A \in \pi(\mathcal{A})$. Let $\xi, \eta \in \mathcal{H}$, and write $\xi = P\xi + P^{\perp}\xi$, $P^{\perp} = \mathbb{1} - P$, $\eta = P\eta + P^{\perp}\eta$. We have $PP^{\perp} = 0$.

$$\text{Then } (\xi, PA\eta) = (P\xi + P^{\perp}\xi, PA(P+P^{\perp})\eta)$$
$$= (P\xi, AP\eta) + (A^*P\xi, P^{\perp}\eta)$$
$$= (P\xi, AP\eta) \quad \text{since } A^*: V \to V$$
$$= (\xi, AP\eta) \quad \text{since } A: V \to V$$

i.e. $(\xi, PA\eta) = (\xi, AP\eta)$ for all $\xi, \eta \in \mathcal{H}$. Hence $PA=AP$ and $P \in \pi(\mathfrak{A})'$ as asserted. However, this contradicts $\pi(\mathfrak{A})' = \mathbb{C}\mathbb{1}$, and so $(\mathcal{H}, \pi)$ is irreducible.

To prove the converse, suppose $(\mathcal{H}, \pi)$ is irreducible, but $\pi(\mathfrak{A})' \neq \mathbb{C}\mathbb{1}$.

Since $A \in \pi(\mathfrak{A})$ implies that $A^* \in \pi(\mathfrak{A})$, we easily deduce that $B \in \pi(\mathfrak{A})'$ implies that $B^* \in \pi(\mathfrak{A})'$. Clearly, $\pi(\mathfrak{A})'$ is linear, and so any $B \in \pi(\mathfrak{A})'$ can be writen as a linear combination of two hermitian elements of $\pi(\mathfrak{A})'$. Thus $\pi(\mathfrak{A})' \neq \mathbb{C}\mathbb{1}$ implies that there is $C = C^* \in \pi(\mathfrak{A})'$ with $C \neq \lambda\mathbb{1}$, $\lambda \in \mathbb{C}$.

By the spectral theorem, 1.3.2, we can write

$$C = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda$$

for some family of projections $\{E_\lambda\}$. Since $C \neq \lambda\mathbb{1}$, there exists at least one $E_\mu$ with $E_\mu \neq 0$ and $E_\mu \neq \mathbb{1}$. Moreover, $E_\mu$ is a strong limit of polynomials in $C$, and hence $E_\mu \in \pi(\mathfrak{A})'$.

Let $V = E_\mu \mathcal{H}$. Then, for $\xi \in V$,

$\pi(A)\xi = \pi(A)E_\mu\xi = E_\mu \pi(A)\xi \in V$, for all $A \in \mathfrak{A}$, i.e. $V$ is a proper closed subspace invariant under $\pi(\mathfrak{A})$. This contradicts the assumed irreducibility.

QED.

1.6.7) <u>Definition</u> - Let $(\mathcal{H}, \pi)$ be a representation of a $C^*$-algebra $\mathfrak{A}$. A vector $\xi \in \mathcal{H}$ is called cyclic (for $(\mathcal{H}, \pi)$) if the set $\{\pi(\mathfrak{A})\xi\}$ is dense in $\mathcal{H}$.

1.6.8) <u>Propositon</u> - A representation $(\mathcal{H},\pi)$ of a C*-algebra $\mathfrak{A}$ is irreducible if and only if every non-zero vector in $\mathcal{H}$ is cyclic.

<u>Proof</u> - Suppose $(\mathcal{H},\pi)$ is irreducible, and let $\xi \in \mathcal{H}$, $\xi \neq 0$. Then the closure of $\pi(\mathfrak{A})\xi$ is a non-zero invariant closed subspace of $\mathcal{H}$, which must be equal to $\mathcal{H}$ by the irreducibility of $(\mathcal{H},\pi)$. In other words, $\xi$ is cyclic.

Conversely, suppose every non-zero $\xi \in \mathcal{H}$ is cyclic.

If V is a non-zero, proper closed subspace in $\mathcal{H}$, with $\pi(\mathfrak{A})V \subset V$, then no vector in V can be cyclic. Indeed, $\xi \in V$ implies that $\{\pi(\mathfrak{A})\xi\} \subset V$. Therefore $(\mathcal{H},\pi)$ is irreducible.

QED.

It does not follow that $(\mathcal{H},\pi)$ is irreducible under the assumption that $\mathcal{H}$ contains only a dense set of cyclic vectors.

1.7) <u>States on a C*-algebra</u>

1.7.1) <u>Definition</u> - A state on a C*-algebra $\mathfrak{A}$ is a positive linear functional $\omega$, with $\omega(\mathbb{1}) = 1$. That is, $\omega : \mathfrak{A} \to \mathbb{C}$ such that,

(i) $\omega$ is linear,

(ii) $a \in \mathfrak{A}$, $a \geq 0$ implies $\omega(a) \geq 0$

(iii) $\omega(\mathbb{1}) = 1$

1.7.2) <u>Proposition</u> - Let $\omega$ be a state on $\mathfrak{A}$. Then $\omega$ satisfies a Schwarz inequality:

$$|\omega(a^*b)| \leq \omega(a^*a)^{1/2}\,\omega(b^*b)^{1/2}$$

for all $a,b \in \mathfrak{A}$.

<u>Proof</u> - The form $\langle a,b\rangle = \omega(a^*b)$ is a sesquilinear form with $\langle a,a\rangle \geq 0$.

QED

1.7.3) <u>Proposition</u> - Let $\omega$ be a state on a C*-algebra $\mathfrak{A}$.

Then $\omega$ is continnous, and $\|\omega\| = \sup\{|\omega(a)| : \|a\| \leq 1\}$ is equal to one.

<u>Proof</u> - We need only show that $|\omega(a)| \leqslant \|a\|$ for any $a \in \mathfrak{A}$. Let $h \in \mathfrak{A}$, $h = h^*$. Then

$$-\|h\| \mathbb{1} \leqslant h \leqslant \|h\| \mathbb{1}$$

and therefore $-\|h\| \leqslant \omega(h) \leqslant \|h\|$,
i.e. $|\omega(h)| \leqslant \|h\|$.
For any $a \in \mathfrak{A}$, we have

$$|\omega(a)| = |\omega(\mathbb{1}\, a)|$$

$$\leqslant \omega(a^*a)^{1/2} \quad \text{by 1.7.2}$$

$$\leqslant \|a^*a\|^{1/2} \quad \text{since } a^*a \text{ is hermitian}$$

$$= \|a\| ,$$

i.e. $|\omega(a)| \leqslant \|a\|$.

QED.

1.7.4) <u>Definition</u> - Let E be a set in a linear space X. E is called convex if $x, y \in E$ implies that $\alpha x + (1-\alpha)y \in E$ for all $0 \leqslant \alpha \leqslant 1$.

A point $z \in E$, a convex set, is an extreme point (with respect to E) if $z = \alpha x + (1-\alpha)y$, with $0<\alpha<1$, $x, y \in E$ has only the solution $x = y = z$,
i.e. z is not a convex combination of two distinct points of E.

1.7.5) <u>Definition</u> - Let E be the set of states on a C*-algebra $\mathfrak{A}$. Then E is a convex set in $\mathfrak{A}^*$, the dual of $\mathfrak{A}$. The extreme points of E are called pure states. If a state $\omega$ is not pure, then it is called a mixture.

It is easy to see that $\omega$ is a mixture if and only if there are states $\omega_1$ and $\omega_2$ different from $\omega$ such that

$$\omega = \tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2 .$$

1.7.6) <u>Theorem</u> - Let $\mathfrak{A}$ be a commutative C*-algebra. Then the set of pure states on $\mathfrak{A}$ is exactty Sp $\mathfrak{A}$.
<u>Proof</u>. Let $\omega \in \mathrm{Sp}\,\mathfrak{A}$, and suppose $\omega = \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$.

Let $a \in \mathfrak{A}$ with $a = a^*$. Then

$$\omega(a^2) = \tfrac{1}{2}(\omega_1(a^2) + \omega_2(a^2)) = \omega(a)^2, \text{ since } \omega \in \mathrm{Sp}\,\mathfrak{A},$$
$$= \tfrac{1}{4}(\omega_1(a) + \omega_2(a))^2.$$

Hence
$$0 = (\omega_1(a^2) - \omega_1(a)^2) + (\omega_2(a^2) - \omega_2(a)^2)$$
$$+\omega_1(a^2) - 2\omega_1(a)\omega_2(a) + \omega_2(a^2)$$
$$\geqslant (\omega_1(a) - \omega_2(a))^2$$
because $\omega_1(a^2) \geqslant \omega_1(a)^2$ and $\omega_2(a^2) \geqslant \omega_2(a)^2$

by Schwarz' inequality.

It follows that $\omega_1(a) = \omega_2(a)$ and so $\omega_1(a) = \omega_2(a) = \omega(a)$, for each $a = a^* \in \mathfrak{A}$. By 1.4.7 (i), we conclude that $\omega_1 = \omega_2 = \omega$, and so $\omega$ is pure.

For the converse, suppose $\omega$ is a pure state on $\mathfrak{A}$.
Suppose $a = a^* \in \mathfrak{A}$, $0 \leq a \leq \mathbb{1}$, $0 \neq \omega(a) \neq 1$.
Define, for $b \in \mathfrak{A}$,
$$\omega_1(b) = \omega(ab)/\omega(a), \quad \omega_2(b) = \omega((\mathbb{1}-a)b)/\omega(\mathbb{1}-a).$$
Then $\omega_1$ and $\omega_2$ are states on $\mathfrak{A}$, and we see that

$$\omega(a)\omega_1(b) + \omega(\mathbb{1}-a)\,\omega_2(b) = \omega(b),$$
i.e.
$$\omega = \omega(a)\,\omega_1 + (1 - \omega(a))\,\omega_2.$$
Since $\omega$ was assumed to be pure, we have $\omega = \omega_1 = \omega_2$,
i.e. $\omega(b) = \omega_1(b) = \omega(ab)/\omega(a)$,
i.e. $\omega(ab) = \omega(a)\omega(b)$, for all $b \in \mathfrak{A}$, and a as above.

Now suppose $a \geqslant 0$ and $\omega(a) = 0$. Then
$$|\omega(ab)| \leqslant |\,\omega(a^{1/2}\,a^{1/2}\,b)\,|$$

$$\leq \omega(a)^{1/2}\, \omega(b^*ab)^{1/2}$$

$$= 0$$

Hence, using $\omega(a) = 0$, we have

$$\omega(ab) = 0 = \omega(a)\omega(b), \text{ for all } b \in \mathfrak{A}.$$

If $a \leq \mathbb{1}$ and $\omega(a) = 1$, then $\omega(\mathbb{1} - a) = 0$ and $\mathbb{1} - a \geq 0$, so that, as above,

$$\omega((\mathbb{1} - a)\, b) = \omega(\mathbb{1} - a)\, \omega(b) \quad , \text{ for all } b \in \mathfrak{A},$$

i.e. $\omega(ab) = \omega(a)\,\omega(b)$.

We have shown that for any $a \in \mathfrak{A}$, with $0 \leq a \leq \mathbb{1}$, and for any $b \in \mathfrak{A}$, we have

$$\omega(ab) = \omega(a)\ \omega(b).$$

By linearity, this holds for all $0 \leq a$ and then for all $a \leq 0$.
By 1.4.7, it holds for all $a \in \mathfrak{A}$.
This means that $\omega \in \mathrm{Sp}\, \mathfrak{A}$.

QED.

1.7.7) <u>Theorem</u> - Let $\mathfrak{A}$ be a C*-algebra, and let $\omega$ be a bounded linear functional with $\|\omega\| = \omega(\mathbb{1}) = 1$. Then $\omega$ is a state on $\mathfrak{A}$.

<u>Proof</u>. We only have to show that $\omega$ is positive.
Let $h \in \mathfrak{A}$, $h = h^*$. We claim that $\omega(h) \in \mathbb{R}$.

Suppose $\omega(h) = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$. Then $\omega(h + i\lambda\mathbb{1}) =$
$= \alpha + i(\lambda + \beta)$ , for all $\lambda \in \mathbb{R}$.

Hence

$$|\omega(h + i\lambda\mathbb{1})| \geq |\beta + \lambda|.$$

On the other hand,

$$|\omega(h + i\lambda\mathbb{1})| \leq \|\omega\|\ \|h + i\lambda\mathbb{1}\|$$

$$= ( \| h \|^2 + \lambda^2 )^{1/2}$$

(using Gelfand's theorem to realize h as a function).

Therefore $|\beta + \lambda|^2 \leq \| h \|^2 + \lambda^2$ for all real $\lambda$.

This is impossible unless $\beta = 0$, and so $\omega(h)$ is real as asserted.

Now suppose $h \geq 0$, and $\| h \| \leq 1$. By the above, $\omega(h) \in \mathbb{R}$.

Suppose $\omega(h) < 0$.

Then $\omega( \mathbb{1} - h ) = 1 - \omega(h) > 1$.

However, $| \omega( \mathbb{1} - h )| \leq \| \mathbb{1} - h \| \leq 1$.

This is a contradiction. Therefore $\omega(h) \geq 0$.

The result follows.

QED

1.7.8) <u>Theorem</u> - Let $\mathfrak{A} \subset \mathfrak{B}$ be C*-algebras, and let $\omega$ be a state on $\mathfrak{A}$. Then there exists a state $\rho$ on $\mathfrak{B}$ such that $\rho \upharpoonright \mathfrak{A} = \omega$. In other words, a state on a C*-algebra can always be extended to a state on a larger C*-algebra.

<u>Proof</u> - Since $\omega$ is a state on $\mathfrak{A}$, we have $\| \omega \| = \omega( \mathbb{1} ) = 1$.

By the Hahn - Banach theorem (Dunford and Schwartz (1966)), there exists a linear functional, $\rho$ say, on $\mathfrak{B}$, such that $\rho$ is bounded, $\| \rho \| = \| \omega \|$ and $\rho \upharpoonright \mathfrak{A} = \omega$. Since $\rho \upharpoonright \mathfrak{A} = \omega$, we have

$$\rho( \mathbb{1} ) = \omega( \mathbb{1} ) = 1 \quad ( \mathbb{1} \in \mathfrak{A} \subset \mathfrak{B} ).$$

But then

$$\| \rho \| = \| \omega \| = \omega( \mathbb{1} ) = \rho( \mathbb{1} ) = 1.$$

By 1.7.7, $\rho$ is a state on $\mathfrak{B}$.

QED.

In order to extend this result to pure states, we shall need a result on the existence of extreme points in a convex set. This is the Krein-Milman theorem, which, when applied to the set of states on a C*-algebra, implies that any $w^*$- closed convex set F of states contains extreme points (with respect to F). A precise statement of the Krein-Milman theorem and further details may be found in Dunford and Schwartz (1966).

1.7.9) Theorem - Let $\mathfrak{A} \subset \mathfrak{B}$ be C*-algebras, and suppose that $\omega$ is a pure state on $\mathfrak{A}$. Then $\omega$ has an extension to a pure state on $\mathfrak{B}$.

Proof. - By 1.7.8, we know that $\omega$ has extensions to states on $\mathfrak{B}$. Let $F = \{\rho$ state on $\mathfrak{B} \mid \rho \restriction \mathfrak{A} = \omega\}$. Then $F \neq \emptyset$. Moreover, it is obvions that F is convex and W*-closed. Therefore F has extreme points. Let $\rho$ be such an extreme point. We claim that $\rho$ is pure on $\mathfrak{B}$. Suppose not ; i.e. there are states $\rho_1, \rho_2$ on $\mathfrak{B}$, and $0 < \alpha < 1$ such that $\rho = \alpha \rho_1 + (1 - \alpha) \rho_2$.

Now $\rho \in F$ implies that $\omega = \alpha \rho_1 \restriction \mathfrak{A} + (1 - \alpha) \rho_2 \restriction \mathfrak{A}$. But $\rho_1 \restriction \mathfrak{A}$ and $\rho_2 \restriction \mathfrak{A}$ are both states on $\mathfrak{A}$, and $\omega$ is pure. Hence $\rho_1 \restriction \mathfrak{A} = \rho_2 \restriction \mathfrak{A} = \omega$, i.e. $\rho_1$ and $\rho_2$ belong to F. This contradicts the fact that $\rho$ is an extreme point of F.

We conclude that $\rho$ is pure on $\mathfrak{B}$.

QED.

1.7.10) Corollary - Let $\mathfrak{A}$ be a C*-algebra; then the pure states separate points of $\mathfrak{A}$.

Proof - Let $a \in \mathfrak{A}$, and let $a = h + ik$ with $h, k, \in \mathfrak{A}$, hermitian. By 1.4.7, $\omega(h)$ and $\omega(k)$ are real for any state $\omega$. Hence $\omega(a) = 0$ implies that $\omega(h) = \omega(k) = 0$. It is enough, then, to show that if $h = h^* \in \mathfrak{A}$ and $\omega(h) = 0$ for all pure states $\omega$ on $\mathfrak{A}$, then $h = 0$.

Suppose, then, $h = h^* \in \mathfrak{A}$, $h \neq 0$.
Let $\mathcal{A}$ be the commutative C*-algebra generated by h. Since $h \neq 0$, there is $\ell \in \mathrm{Sp}\mathcal{A}$ such that $\hat{h}(\ell) \neq 0$, i.e. $\ell(h) \neq 0$. By 1.7.6, $\ell$ is a pure state on $\mathcal{A}$, which therefore has an extension to a pure state, $\omega$ say, on $\mathfrak{A}$ (by 1.7.9).
But then $\omega(h) = \ell(h) \neq 0$. The result follows.

QED.

1.7.11) <u>Corollary</u> The involution $a \to a^*$ in a C*-algebra is unique.

<u>Proof</u>. Let $a \to a'$ be another involution. By 1.7.3 and 1.7.7, $\omega$ is a state with respect to $(\mathfrak{A}, *)$ if and only if $\omega$ is a state with respect to $(\mathfrak{A}, ')$.
By 1.4.7, if $\omega$ is a state, we have, for $a \in \mathfrak{A}$, $\omega(a^*) = \overline{\omega(a)}$.
Similarly, $\omega(a') = \overline{\omega(a)}$.
Hence $\omega(a^*) = \omega(a')$ for all states, and all $a \in \mathfrak{A}$.
By 1.7.10, $a^* = a'$, for all $a \in \mathfrak{A}$.

QED.

## 1.8. <u>The Gelfand, Naimark, Segal Construction</u>.

We shall discuss, in this section, a certain connection between states on a C*-algebra, and representations, and related results.

1.8.1.) <u>Theorem</u> (Gelfand, Naimark, Segal) Let $\mathfrak{A}$ be a C*-algebra, and $\omega$ a state on $\mathfrak{A}$. Then there is a representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ with a cyclic vector $\Omega \in \mathcal{H}$ such that $\omega(a) = (\Omega, \pi(a)\Omega)$ for all $a \in \mathfrak{A}$. Moreover, the triple $(\mathcal{H}, \pi, \Omega)$ is unique up to unitary equivalence: i.e. if $(\mathcal{H}', \pi', \Omega')$ is another such triple, then there is a unitary operator $U: \mathcal{H}' \to \mathcal{H}$ such that $U\Omega' = \Omega$ and $U\pi'(a)U^* = \pi(a)$ for all $a \in \mathfrak{A}$.

<u>Proof</u>. Let $N = \{x \in \mathfrak{A} \mid \omega(x^*x) = 0\}$. Using the Schwarz inequality, it is easy to see that N is a left ideal in $\mathfrak{A}$.

Let K be the linear space $\mathfrak{A}/N$. Let $\xi, \eta \in K$, and $x \in \xi$, $y \in \eta$. We define

$$\langle \xi, \eta \rangle = \omega(x^*y).$$

By Schwarz' inequality, we see that this is a well defined sesquilinear form on K. Using $\omega(a^*) = \overline{\omega(a)}$ (which follows from 1.4.7 (ii)), we obtain $\langle \xi, \eta \rangle = \overline{\langle \eta, \xi \rangle}$. So $\langle\ ,\ \rangle$ is an inner product on K. Moreover, $\|\xi\|_\omega^2 \equiv \langle \xi, \xi \rangle = \omega(x^*x)$, $x \in \xi$ defines a norm on K because $\omega(x^*x) = 0$ implies $x \in N$, and so $\xi = 0$.

For $a \in \mathfrak{A}$, and $\xi \in K$, we define

$$L_a \xi = a\xi,$$

where $a\xi$ is the class containing $ax$, $x \in \xi$. This is well-defined because N is a left ideal. To see that $L_a$ is bounded, we compute

$$\|L_a \xi\|_\omega^2 = \langle L_a \xi, L_a \xi \rangle = \omega((ax)^* ax),\ x \in \xi,$$
$$= \omega(x^*a^*ax).$$

Define $\rho(b) = \omega(x^*bx)$, $b \in \mathfrak{A}$. Then $\rho$ is a positive linear functional on $\mathfrak{A}$. Hence

$$|\rho(b)| \leq \rho(\mathbb{1})\,\|b\|, \text{ for all } b \in \mathfrak{A}.$$

That is,

$$|\omega(x^*bx)| \leq \omega(x^*x)\,\|b\|.$$

Setting $b = a^*a$, we have

$$|\omega(x^*a^*ax)| \leq \omega(x^*x)\,\|a^*a\|$$
$$= \omega(x^*x)\,\|a\|^2$$
$$= \langle \xi, \xi \rangle\,\|a\|^2,$$

i.e. $\|L_a \xi\|_\omega \leq \|a\|\,\|\xi\|_\omega$, and so $L_a$ is bounded on K.

It is easy to see that $L_{a+b} = L_a + L_b$, $L_{ab} = L_a L_b$, $L_{\mathbb{1}} = \mathbb{1}_K$, and $\langle \xi, L_{a^*}\eta \rangle = \langle L_a \xi, \eta \rangle$ for all $\xi, \eta \in K$, $a, b \in \mathfrak{A}$.

Let $\Omega \in K$ be the class containing $\mathbb{1}$. Then any $\xi \in K$ can be written as $\xi = L_x \Omega$, where $x \in \xi$.

Let $\mathcal{H}$ be the completion of K w.r.t. the norm $\|\cdot\|_\omega$. Then $\mathcal{H}$ is a Hilbert space and contains K as a dense subset. Since $L_{(\cdot)}$ is bounded, it has a unique bounded extension, say $\pi(\cdot)$, to $\mathcal{H}$, which evidently is a representation of $\mathfrak{A}$ in $\mathcal{H}$. It is also clear that $\Omega$ is ciclic for $(\mathcal{H}, \pi)$, and the construction of $(\mathcal{H}, \pi, \Omega)$ is complete.

To prove uniqueness, let $(\mathcal{H}', \pi', \Omega')$ be another triple. Define $U : \mathcal{H}' \to \mathcal{H}$ by $U\pi'(a)\Omega' = \pi(a)\Omega$.

Then

$$\| U \pi'(a)\Omega' \|^2_{\mathcal{H}} = \| \pi(a)\Omega \|^2_{\mathcal{H}} = \omega(a^*a)$$
$$= \| \pi'(a)\Omega' \|^2_{\mathcal{H}'}$$

So U is isometric from a dense set in $\mathcal{H}'$ to a dense set in $\mathcal{H}$ . It therefore extends to a unitary from $\mathcal{H}'$ to $\mathcal{H}$, with the required properties.

QED.

Given $\mathfrak{A}$ and $\omega$, $(\mathcal{H}, \pi, \Omega)$ is called the GNS triple associated to $\mathfrak{A}$ and $\omega$ . The notation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is also used to emphasise the dependence on $\omega$ .

It is worth remarking here that, in general, $\mathcal{H}_\omega$ is <u>not</u> a separable Hilbert space.

1.8.2) <u>Definition</u> - Let $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha \in I}$ be representations of a C*-algebra $\mathfrak{A}$ . The direct sum of the representations $(\mathcal{H}_\alpha, \pi_\alpha)$ is the representation $(\mathcal{H}, \pi)$ with $\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha$, and $\pi(a) = \bigoplus_{\alpha \in I} \pi_\alpha(a)$ , $a \in \mathfrak{A}$ .

It is denoted $(\bigoplus_\alpha \mathcal{H}_\alpha, \bigoplus_\alpha \pi_\alpha)$.

1.8.3) <u>Corollary</u> - Any C*-algebra $\mathfrak{A}$ is isometrically isomorphic to a C*-algebra of operators on a Hilbert space.

<u>Proof</u> - Let S be a family of states which separates points of $\mathfrak{A}$ (i.e. $\omega(a) = 0$ for all $\omega \in S$ implies $a = 0$ ). Let $(\mathcal{H}_\omega, \pi_\omega)$ be the GNS representation associated with $\omega \in S$ (1.8.1) Let $(\mathcal{H}, \pi)$ be the direct sum $\mathcal{H} = \bigoplus_{\omega \in S} \mathcal{H}_\omega$ , $\pi = \bigoplus_{\omega \in S} \pi_\omega$ .

Suppose $\pi(a) = \pi(b)$, $a, b \in \mathfrak{A}$. Then $\pi_\omega(a) = \pi_\omega(b)$ for all $\omega \in S$.

Hence $\pi_\omega(a - b) = 0$ for all $\omega \in S$. This implies that

$(\Omega_\omega, \pi_\omega(a-b)\Omega_\omega) = 0$ for all $\omega \in S$, i.e. $\omega(a-b) = 0$ for all $\omega \in S$. Since $S$ separates points of $\mathfrak{A}$, we have $a = b$.

Thus $\pi$ is 1 - 1, and so $\mathfrak{A}$ is isometrically isomorphic to $\pi(\mathfrak{A})$.

QED.

1.8.4) <u>Theorem</u> - Let $\omega$ be a state on a C*-algebra $\mathfrak{A}$, and let $(\mathcal{H}, \pi, \omega)$ be the associated GNS triple. Suppose $\rho$ is a positive linear functional on $\mathfrak{A}$ with $\rho \leq \omega$, i.e. $\omega - \rho$ is positive. Then there exists a unique $T \in \mathcal{B}(\mathcal{H})$ with
$\rho(b^*a) = (\pi(b)\Omega, T\pi(a)\Omega)$, $T \in \pi(\mathfrak{A})'$ and $0 \leq T \leq \mathbb{1}$.

Conversely, if $0 \leq T \leq \mathbb{1}$, $T \in \pi(\mathfrak{A})'$, then $\rho(a) \equiv (\Omega, T\pi(a)\Omega)$ is a positive linear functional with $\rho \leq \omega$.

<u>Proof</u> - Let $\rho \leq \omega$ be given. Let $\mathring{a}$ denote the class of a in $\mathfrak{A}/N$ as in 1.8.1 ( $N = \{x \in \mathfrak{A} \mid \omega(x^*x) = 0\}$ ).

Then

$$\begin{aligned} |\rho(b^*a)|^2 &\leq \rho(b^*b)\ \rho(a^*a) \\ &\leq \omega(b^*b)\ \omega(a^*a) \\ &= \|\mathring{b}\|_\omega\ \|\mathring{a}\|_\omega \end{aligned}$$

Hence, $\rho$ defines a bounded sesquilinear form on $K = \mathfrak{A}/N$, and so defines one, say $\ell$, on $\mathcal{H}$, the completion of K w.r.t. $\|\ \|_\omega$.

By Riesz' lemma, there exists a unique $T \in \mathcal{B}(\mathcal{H})$ with

$$\ell(\xi, \eta) = (\xi, T\eta) \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

But $\ell(\mathring{b}, \mathring{a}) = \rho(b^*a)$. Hence, taking $b = a$, we have $(\mathring{a}, T\mathring{a})_{\mathcal{H}} \geq 0$.
Using $\omega \geq \rho$, we get, as above,

$$|(\mathring{b}, T\mathring{a})_{\mathcal{H}}| \leq \|\mathring{b}\|_\omega\ \|\mathring{a}\|_\omega .$$

It follows that $0 \leq T \leq \mathbb{1}$, ( since K is dense in $\mathcal{H}$ ).

Furthermore, since $\mathring{a} = \pi(a)\Omega$, we have

$$\rho(b^*a) = (\mathring{b}, T\mathring{a}) = (\pi(b)\Omega, T\pi(a)\Omega).$$

Also, for a, b, c $\in \mathfrak{A}$, we have

$$(\pi(b)\Omega, \{T\pi(a) - \pi(a)T\}\pi(c)\Omega)$$
$$= \rho(b^* ac) - \rho((a^*b)^* c)$$
$$= 0$$

Since $\Omega$ is cyclic, we conclude that $T \in \pi(\mathfrak{A})'$.
The converse is trivial.

QED.

1.8.5) Theorem -Let $\omega$ be a state on a C*algebra $\mathfrak{A}$, and $(\mathcal{H}, \pi)$ the associated GNS representation.
Then $(\mathcal{H}, \pi)$ is irreducible if and only if $\omega$ is pure on $\mathfrak{A}$.
Proof - By 1.6.6, $(\mathcal{H}, \pi)$ is irreducible if and only if $\pi(\mathfrak{A})' = \mathbb{C}\mathbb{1}$. But by 1.8.4, $\pi(\mathfrak{A})' = \mathbb{C}\mathbb{1}$ if and only if $0 \leq \rho \leq \omega$ implies that $\rho = \lambda\omega$, some $0 \leq \lambda \leq 1$. However it is easily seen that this last statement is equivalent to $\omega$ being pure.

QED.

We can now improve 1.8.3.

1.8.6) Corollary - Any C*-algebra $\mathfrak{A}$ is isometrically isomorphic to a direct sum of irreducible representations of itself.
Proof - As in 1.8.3, but we take S to be the set of pure states on $\mathfrak{A}$. By 1.7.10, S separates points of $\mathfrak{A}$. Then we have that $\mathfrak{A}$ is isomorphic to $\bigoplus_{\omega \in S} \pi_\omega$. By 1.8.5, each $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible.

QED.

1.8.7) Theorem - Let $\omega$ be a state on a C*-algebra $\mathfrak{A}$, and let $\alpha \in \mathrm{Aut}\,\mathfrak{A}$. Suppose $\omega$ is invariant under $\alpha$, i.e. $\omega(\alpha(a)) = \omega(a)$, for all $a \in \mathfrak{A}$.
Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triple associated with $\omega$. Then there is a unitary operator U on $\mathcal{H}$ such that $U\Omega = \Omega$, and $U\pi(a)U^* = \pi(\alpha(a))$, for all $a \in \mathfrak{A}$. Moreover, U is unique.

<u>Proof</u> - Define a representation $(\mathcal{H}, \pi')$ of $\mathfrak{A}$ by $\pi'(a) = \pi(\alpha(a))$, $a \in \mathfrak{A}$, and consider the triple $(\mathcal{H}, \pi', \Omega)$. Since $\alpha(\mathfrak{A}) = \mathfrak{A}$, $\Omega$ is cyclic for $\pi'$.

Furthermore,

$$(\Omega, \pi'(a)\Omega) = (\Omega, \pi(\alpha(a))\Omega)$$
$$= \omega(\alpha(a)) = \omega(a) \quad \text{for all } a \in \mathfrak{A}.$$

By the uniqueness of the GNS triple (1.8.1) there is a unitary $U$ on $\mathcal{H}$ such that $U\Omega = \Omega$ and $U\pi(a)U^* = \pi'(a) = \pi(\alpha(a))$.

Suppose $V$ is another unitary with these same properties.

Then

$$U\pi(a)\Omega = U\pi(a)U^*\Omega = \pi(\alpha(a))\Omega$$

$$= V\pi(a)V^*\Omega = V\pi(a)\Omega .$$

Since $\Omega$ is cyclic, we have $U = V$.

QED.

1.8.8) <u>Corollary</u> - Let $\mathfrak{A}$ be a C*-algebra, and let $G$ be a topological group. Suppose $g \to \alpha_g$ is a representation of $G$ in Aut $\mathfrak{A}$. Let $\omega$ be a state on $\mathfrak{A}$ invariant under each $\alpha_g$, i.e. $\omega(\alpha_g(a)) = \omega(a)$ for all $g \in G$, $a \in \mathfrak{A}$. Suppose further that for any $a, b \in \mathfrak{A}$, the map $g \to \omega(b^* \alpha_g(a))$ is continnous. Then there is a strongly continnous unitary representation $g \to U(g)$ of $G$ on $\mathcal{H}$, where $(\mathcal{H}, \pi, \Omega)$ is the GNS triple associated with $\omega$, satisfying $U(g)\Omega = \Omega$ for all $g \in G$, and $U(g)\pi(a)U(g)^* = \pi(\alpha_g(a))$ for all $g \in G$, $a \in \mathfrak{A}$.

Moreover, the $U(g)$ are unique.

<u>Proof</u> - By 1.8.7, for each $g \in G$, we have a unique $U(g)$ satisfying $U(g)\Omega = \Omega$ and $U(g)\pi(a)U(g)^* = \pi(\alpha_g(a))$, for all $a \in \mathfrak{A}$. To see that $g \to U(g)$ is a representation of $G$, we compute

$$U(g)U(h)\pi(a)\Omega = U(g)\pi(\alpha_h(a))\Omega$$

$$= \pi(\alpha_g\, \alpha_h(a))\Omega = \pi(\alpha_{gh}(a))\Omega$$
$$= U(gh)\, \pi(a)\Omega\ , \text{ for all } a \in \mathfrak{A}\ , g, h \in G.$$

Since $\Omega$ is cyclic, we obtain $U(g)\, U(h) = U(gh)$.

It only remains to show that $U(\cdot)$ is strongly continnous. Let $a, b \in \mathfrak{A}$. Then

$$(\pi(b)\Omega\ , U(g)\, \pi(a)\Omega) = \omega(b^* \alpha_g(a))$$

is continnous in g, by assumption. Since U(g) is unitary, for each g, we see that $U(\cdot)$ is weakly continnous on $\mathcal{H}$, and therefore strongly continnous.

QED.

1.8.9) <u>Definition</u> - Let $\mathfrak{A}$ be a C*-algebra, and $g \to \alpha_g \in \operatorname{Aut} \mathfrak{A}$ a representation of a group G in Aut $\mathfrak{A}$. A state $\omega$ is called extremal invariant (with respect to $\alpha_g$) if $\omega$ is an extreme point of the convex set $\{\rho \text{ state on } \mathfrak{A} \mid \rho \circ \alpha_g = \rho$ for all $g \in G\}$.

1.8.10) <u>Corollary</u> - With the assumptions and notation of 1.8.8, we have $\mathcal{R} \equiv (\{U(g) \mid g \in G\} \cup \{\pi(\mathfrak{A})\})' = \mathbb{C}\mathbb{1}$ if and only if $\omega$ is extremal invariant.

<u>Proof</u> - Suppose $\omega$ is extremal invariant, but $\mathcal{R} \neq \mathbb{C}\mathbb{1}$. Let P be a non-trivial projection in $\mathcal{R}$. Then $P\Omega \neq 0$, and $P\Omega \neq \Omega$. Let $\omega_1$ be the state

$$\omega_1(a) = \frac{(P\Omega\ , \pi(a)\, P\Omega)}{\| P\Omega \|^2}, \quad a \in \mathfrak{A}$$

and $\omega_2$ the state

$$\omega_2(a) = \frac{(Q\Omega\ , \pi(a)\, Q\Omega)}{\| Q\Omega \|^2}\ , \quad a \in \mathfrak{A}$$

where $P + Q = \mathbb{1}$.

Then $\omega$ is given by the convex combination

$$\omega = \| P\Omega \|^2 \omega_1 + \| Q\Omega \|^2 \omega_2 .$$

Moreover, $\omega_1$ and $\omega_2$ are both invariant and are easily seen to be distinct. This contradicts the extremal invariance of $\omega$ .

Conveserly, suppose $\mathcal{R} = \mathbb{C}\mathbb{1}$, but $\omega$ is not extremal invariant. Then $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ with $\omega_1, \omega_2$ distinct invariant states, and $0 < \lambda < 1$.

Hence $\omega \geq \lambda\omega_1 = \rho$ , say. By 1.8.4, there is $T \in \pi(\mathfrak{A})'$ such that $\rho(b^*a) = (\pi(b)\Omega, T\pi(a)\Omega)$, $a,b \in \mathfrak{A}$ . Since $\rho$ is not a multiple of $\omega$ , $T \neq \mu\mathbb{1}$ . Furthermore, the invariance of $\rho$ under $\alpha_g$ implies that T commutes with each U(g), i.e. $T \in \{ U(g) \mid g \in G \}'$.

Hence $T \in \mathcal{R}$, a contradiction.

QED.

## 2. Operator Algebras.

The last chapter dealt mainly with the abstract structure of $C^*$-algebras. We want to consider now algebras of opetators on a Hilbert space, and to take advantage of the Hilbert space structure. We begin by defining several topologies and then prove the density theorems of von Neumann and Kaplansky. We also give a characterization of continuous functionals and states. Finally, in section 6, we discuss some of the theory of subrepresentations of $C^*$-algebras which will be useful for the algebric treatment of superselection sectors (chapter 7). Except for section 6, we have followed the lectures of Landford (1972). The standard text-books are those of Dixmier (1969 b) and Sakai (1971) to which we refer for further details and developments ( - see also Lanford (1972) and Naimark (1964)).

If $H$ is a Hilbert space, $B(H)$ denotes, as usual, the algebra of all bounded operators on $H$.

### 2.1) Topologies on B(H)

We shall consider five topologies on $B(H)$, the first three of which are probably more familiar.Let $A \in B(H)$, and let $(B_\alpha)$ be a net in $B(H)$. We shall define the various topologies in terms of a neighbourhood basis of $A$, and also in terms of the convergence of the net $(B_\alpha)$ to $A$ - the latter being usually the more convenient.

#### 2.1.1. Definition

The norm (or uniform) topology on $B(H)$ is that given by the open neighbourhood base

$$\mathcal{N}(A;\varepsilon) = \{B \in \mathcal{B}(\mathcal{H}) \mid \|A-B\| < \varepsilon\},\ \varepsilon > 0 .$$

$B_\alpha \to A$ in norm iff $\|B_\alpha - A\| \to 0$.

2.1.2. Definition

The strong topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A;(x_i)_1^n, \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}) \mid \sum_{i=1}^{n} \|(A-B)x_i\|^2 < \varepsilon\}$$

where $\varepsilon > 0$, and $(x_i)_1^n$ is a finite set of vectors in $\mathcal{H}$.

$B_\alpha \to A$ strongly iff $\|(B_\alpha - A)x\| \to 0$ for each $x \in \mathcal{H}$

2.1.3. Definition

The weak topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A;\ (x_i)_1^n,\ (y_i)_1^n, \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}) \mid |\sum_{i=1}^{n} (y_i, (A-B)x_i)| < \varepsilon\}$$

where $\varepsilon > 0$ and $(x_i)_1^n$, $(y_i)_1^n$ are finite sets in $\mathcal{H}$.

$B_\alpha \to A$ weakly iff $(y, (B_\alpha - A)x) \to 0$ for each $x, y$ in $\mathcal{H}$.

2.1.4. Definition

The ultrastrong topology on $\mathcal{B}(\mathcal{H})$ is given by the open neighbourhood base

$$\mathcal{N}(A;(x_i)_{i=1}^{\infty}, \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}) \mid \sum_{i=1}^{\infty} \|(A-B)x_i\|^2 < \varepsilon\}$$

where $\varepsilon > 0$, and $(x_i)$ is a sequence in $\mathcal{H}$ with $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$

$B_\alpha \to A$ ultrastrongly iff $\sum_{i=1}^{\infty} \|(A-B_\alpha)\, x_i\|^2 \to 0$

for each sequence $(x_i)$ with $\sum \|x_i\|^2 < \infty$.

2.1.5. Definition

The ultraweak topology on $\mathcal{B}(\mathcal{H})$ is given by the open neighbourhood base

$$\mathcal{N}(A; (x_i), (y_i), \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}) \mid |\sum_{i=1}^{\infty} (y_i, (A-B)x_i)| < \varepsilon\}$$

where $\varepsilon > 0$, and $(x_i)$, $(y_i)$ are sequences in $\mathcal{H}$ with

$$\sum \|x_i\|^2 + \|y_i\|^2 < \infty.$$

$B_\alpha \to A$ ultraweakly iff $\sum_{i=1}^{\infty} (y_i, (A-B_\alpha)x_i) \to 0$

for each pair of sequences $(x_i)$, $(y_i)$ with $\sum \|x_i\|^2 + \|y_i\|^2 < \infty$

$\mathcal{B}(\mathcal{H})$ equipped with the norm topology, 2.1.1., is, of course, a $C^*$-algebra. Moreover, the norm topology is characterized by convergence of <u>sequences</u>. For the other four topologies, however, one has to consider <u>nets</u>, not just sequences.

It can be shown that these five topologies are distinct if $\mathcal{H}$ is infinite dimensional - otherwise they are all the same.

Evidently, 2.1.1. defines a finer (or stronger) topology than the other four, and 2.1.4 is finer than the other three. Both 2.1.5 and 2.1.2 are finer than 2.1.3, but 2.1.5 and 2.1.2 are not comparable. It should be emphasized that the ultraweak topology is stronger than the weak topology, i.e. if $B_\alpha \to A$ ultraweakly, then $B_\alpha \to A$ weakly.

Let $\mathcal{K} = \oplus_{i=1}^{\infty} \mathcal{H}_i$, where each $\mathcal{H}_i = \mathcal{H}$.

Define, for each $A \in \mathcal{B}(\mathcal{H})$, the operator $\tilde{A}$ on $\mathcal{K}$ by $\tilde{A}(x_i) = (Ax_i)$. Then $\tilde{A} \in \mathcal{B}(\mathcal{K})$.

2.1.6. Proposition

A net $(B_\alpha)$ in $\mathcal{B}(\mathcal{H})$ converges ultrastrongly (resp. ultraweakly) to $A$ in $\mathcal{B}(\mathcal{H})$ if and only if $(\tilde{B}_\alpha)$ converges strongly (resp. weakly) to $\tilde{A}$ in $\mathcal{B}(\mathcal{K})$.

Proof: $(x_i) \in \mathcal{K}$ if and only if $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$

For such $(x_i)$,

$$\|(\tilde{B}_\alpha - \tilde{A})(x_i)\|_{\mathcal{K}}^2 = \sum_{i=1}^{\infty} \|(B_\alpha - A)x_i\|_{\mathcal{H}}^2 .$$

Similarly,

$$((y_i), (\tilde{B}_\alpha - \tilde{A})(x_i))_{\mathcal{K}} = \sum_{i=1}^{\infty} (y_i, (B_\alpha - A)x_i)_{\mathcal{H}} .$$

QED.

2.1.6 is a useful device for converting problems of "ultra-convergence" into ones of respectively strong or weak convergence.

The next proposition (whose proof is simple) says that on bounded sets "ultra-convergence" is equivalent to the corresponding strong or weak convergence.

2.1.7. Proposition

Let $(B_\alpha)$ be a bounded net in $\mathcal{B}(\mathcal{H})$, i.e. $\|B_\alpha\| \leq M$, some finite $M$, for all $\alpha$. Then $B_\alpha \to A$ ultrastrongly (resp. ultraweakly) if and only if $B_\alpha \to A$ strongly (resp. weakly).

Suppose $A_\alpha \to A$, $B_\alpha \to B$ in one of these topologies. Then we can ask whether $A_\alpha^* \to A^*$ or $A_\alpha B_\alpha \to AB$ in the same topology. That is, whether the maps $s: A \to A^*$, $m: (A,B) \to AB$ are continuous in these topologies. In general, they are not. The situation is as follows.

$s$ is continuous with respect to the norm, ultraweak and weak topologies, but <u>not</u> with respect to the ultrastrong and strong topologies. (The latter is familiar from the study of the scattering-matrix in which its unitarity is a non-trivial problem even though it is constructed from the strong-limit of unitary operators).

$m$ is jointly continuous w.r.t. the norm topology but only separately continuous w.r.t. the other topologies. On bounded sets $m$ is jointly continuous w.r.t. the ultrastrong and strong topologies, but <u>not</u> w.r.t. the ultraweak and weak topologies. For the relevant counterexamples, we refer to Lanford (1972).

## 2.2. Von Neumann's Density Theorem

We shall see that although the topologies 2.1.2.-2.1.5. are different, nevertheless many sets in $\mathcal{B}(\mathcal{H})$ have the <u>same closures</u> w.r.t. each of these topologies.

### 2.2.1. Definition

Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators containing $\mathbb{1}$. $\mathcal{R}$ is called a von Neumann algebra if and only if $\mathcal{R}$ is weakly closed in $\mathcal{B}(\mathcal{H})$.

Remark: von Neumann algebras are also called $W^*$ - algebras. Evidently, a von Neumann algebra is also a $C^*$-algebra.

2.2.2. Definition

Let $\mathcal{M}$ be a set in $\mathcal{B}(\mathcal{H})$. The commutant of $\mathcal{M}$ (in $\mathcal{B}(\mathcal{H})$), written $\mathcal{M}'$, is the set $\mathcal{M}' = \{B \in \mathcal{B}(\mathcal{H}) \mid AB=BA \ \forall A \in \mathcal{M}\}$.

2.2.3. Proposition

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. Then $\{\mathcal{M} \cup \mathcal{M}^*\}'$ is a von Neumann algebra.

Proof: Trivial.

2.2.4. Proposition

Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$. Then, if $\mathcal{M} \subset \mathcal{N}$, we have $\mathcal{M}' \supset \mathcal{N}'$ and $\mathcal{M} \subset \mathcal{M}''$.

Proof: Trivial.

2.2.5. Theorem (von Neumann's Density Theorem, Bicommutant Theorem).

Let $\mathcal{R}$ be a self-adjoint algebra in $\mathcal{B}(\mathcal{H})$ containing $\mathbb{1}$. Then the ultrastrong, ultraweak, strong and weak closures of $\mathcal{R}$ are all the same, and are equal to $\mathcal{R}''$.

Proof: First, we note that $\mathcal{R} \subset \mathcal{R}'' = \overline{\mathcal{R}''}^{w}$ since $\mathcal{R}''$ is a von Neumann algebra. Secondly,

$$\mathcal{R} \subset \overline{\mathcal{R}}^{us} \subset \overline{\mathcal{R}}^{uw}_{s} \subset \overline{\mathcal{R}}^{w} \subset \overline{\mathcal{R}''}^{w} = \mathcal{R}''$$

If we can show that $\overline{\mathcal{R}}^{us} = \mathcal{R}''$ then the proof is complete. This follows if we can show that $\mathcal{R}$ is ultrastrongly dense in $\mathcal{R}''$, i.e. given $B \in \mathcal{R}''$, a sequence $(x_i)$ in $\mathcal{H}$ such that $\sum \|x_i\|^2 < \infty$, and $\varepsilon > 0$, there exists $A \in \mathcal{R}$ such that $\sum \| Bx_i - Ax_i \|^2 < \varepsilon$. As in 2.1.6, we think of $(x_i)$ as an element $x$, say, of $\oplus_i \mathcal{H}_i$, each $\mathcal{H}_i = \mathcal{H}$. Then we must show that $\tilde{B}x$ is in the closed linear span of $\{\tilde{A}x \mid A \in \mathcal{R}\}$.

Let $P$ be the projection of $\oplus \mathcal{H}$ onto this closed linear set. We need only show that $P\tilde{B}x = \tilde{B}x$. Evidently, $\tilde{A}$ commutes with $P$ for any $A \in \mathcal{R}$, so $P \in \tilde{\mathcal{R}}'$ (the commutant being taken in $\mathcal{B}(\oplus \mathcal{H}_i)$, of course).

We want to show now that if $B \in \mathcal{R}''$, then $\tilde{B} \in \tilde{\mathcal{R}}''$. Let $C \in \mathcal{B}(\oplus \mathcal{H}_i)$. Then $C = \sum_{i,j} E_i C E_j$ where $E_i$ is the projection of $\oplus \mathcal{H}_i$ onto $\mathcal{H}_i$ considered as a subspace of $\oplus \mathcal{H}_i$. Writing $C_{ij} = E_i C E_j$, $C_{ij}$ is an operator from $\mathcal{H}_j$ to $\mathcal{H}_i$, i.e. $C_{ij} \in \mathcal{B}(\mathcal{H})$, since $\mathcal{H}_i = \mathcal{H}_j = \mathcal{H}$. We have, for $y = (y_i) \in \oplus \mathcal{H}_i$,

$$(Cy)_i = \sum_{j=1}^{\infty} C_{ij}\, y_j.$$

Hence, $C \in \tilde{\mathcal{R}}'$ if and only if $\sum_j A\, C_{ij}\, y_j = \sum_j C_{ij}\, Ay_j$, for all $i$ and $(y_j) \in \oplus \mathcal{H}_j$. That is, $C \in \tilde{\mathcal{R}}'$ if and only if $C_{ij} \in \mathcal{R}'$ for all $i,j$. Now let $B \in \mathcal{R}''$, and $C \in \tilde{\mathcal{R}}'$, i.e. $C_{ij} \in \mathcal{R}'$. Then clearly $\tilde{B}$ commutes with $C$, i.e. $\tilde{B} \in \tilde{\mathcal{R}}''$, as desired.

The proof is now complete because, for $B \in \mathcal{R}''$,

$$P\tilde{B}\,x = \tilde{B}P\,x, \text{ since } P \in \tilde{\mathcal{R}}',\ \tilde{B} \in \tilde{\mathcal{R}}'',$$
$$= \tilde{B}x$$

since $x \in \tilde{\mathcal{R}}x$ because $\mathbb{1} \in \mathcal{R}$.

QED.

As a consequence of this theorem, we see firstly that $\mathcal{R}$ is a von Neumann algebra if and only if $\mathcal{R}$ is closed w.r.t. each of the four topologies of the theorem, or if and only if $\mathcal{R} = \mathcal{R}''$. This gives several equivalent ways of defining a von Neumann algebra.

Secondly, we see that von Neumann algebras contain many projections, and are in fact determined by their projections. This is to be compared with the situation concerning $C^*$-algebras which may contain no non-trivial projections (e.g. $C[0,1]$, the $C^*$-algebra of continuous functions on the interval $[0,1]$). To see this, we note that any $A \in \mathcal{R}$ can be written as a linear combination of self-adjoint elements in $\mathcal{R}$. Now, by the spectral theorem, the spectral projections of any $B = B^*$ are given by strong limits of polymomials in B. So if $B \in \mathcal{R}$, so are all its spectral projections. Conversely, B is obtained as a norm limit of sums of its spectral projections.

We also note that $B \in \mathcal{R}'$ if and only if B commutes with all projections in $\mathcal{R}$, and so $A \in \mathcal{R}$ if and only if A commutes with all projections which commute with $\mathcal{R}$. Since any element of $\mathcal{R}'$, which is also a von Neumann algebra (and therefore a $C^*$-algebra), is a combination of unitaries, we see that $A \in \mathcal{R}$ if and only if A commutes with all unitaries which commute with $\mathcal{R}$. (This is sometimes useful since unitaries are invetible:

$$AU = UA \text{ is } \quad \text{equivalent to } UAU^{-1} = A).$$

## 2.3. Continuous Functions on an Operator Algebra

### 2.3.1. Proposition

Let $\mathcal{R}$ be a von Neumann algebra, and let $\phi : \mathcal{R} \to \mathbb{C}$ be a strongly continuous linear functional on $\mathfrak{R}$. Then there exist vectors $x_i$, $y_i$, $1 \leq i \leq n$, some finite n, such that $\phi$ is given by

$$\phi(A) = \sum_{i=1}^{n} (y_i, Ax_i)$$

In particular, $\phi$ is weakly continuous. In other words, $\phi$ is strongly continuous if and only if $\phi$ is weakly continuous if and only if $\phi$ has the above form.

<u>Proof</u>: The only non-trivial part is to show that if $\phi$ is strongly continuous, then $\phi$ has the above form. So let us suppose $\phi : \mathfrak{R} \to \mathbb{C}$ is strongly continuous. Then $\phi^{-1}(\{Z \mid |Z| < 1\}) \neq \emptyset$ is open in $\mathfrak{R}$ in the strong topology, and contains 0. Therefore there is a strong neigbourhood, $\mathcal{N}(0; (x_i)_{i=1}^{n}, \varepsilon)$, of 0 contained in $\phi^{-1}(\{Z \in \mathbb{C} \mid |Z| < 1\})$ = i.e. there exist $x_i$, $1 \leq i \leq n$, in $\mathcal{H}$, and $\varepsilon > 0$ such that $\sum_{i=1} \|Tx_i\|^2 < \varepsilon$ implies that $|\phi(T)| < 1$, $T \in \mathfrak{R}$.

Let $\mathcal{K} = \overset{n}{\oplus} \mathcal{H}$, and consider the set V in $\mathcal{K}$ given by

$$V = \{ (y_i)_1^n \mid \quad y_i = Tx_i,\ T \in \mathfrak{R} \} .$$

Evidently, V is a linear set in $\mathcal{K}$. We define the linear map $f : V \to \mathbb{C}$ by $f((y_i)) = \phi(T)$, where $(y_i) = (Tx_i)$. Now, if $\|(y_i)\|_{\mathcal{K}}^2 < \delta^2 \varepsilon$, we have $|\phi(T)| < \delta$, i.e. $|f((y_i))| < \delta$. This means that f is well-defined on V and is continuous, and so can be extended, by continuity, to $\overline{V}$ the closure of V in $\mathcal{K}$. But $\overline{V}$ is a Hilbert space, and a continuous linear functional on a Hilbert space is given by a vector in the space (Riesz'lemma). Hence, there is a vector $v \in \overline{V}$ such that $f(y) = (v, y)_{\mathcal{K}}$, for all $y \in \overline{V}$.

Let $A \in \mathfrak{R}$. Then $(Ax_i) \in V$, and so $f((Ax_i)) = (v, (Ax_i))_{\mathcal{K}} =$

$= \phi(A)$ by the definition of f. Writing v as $(y_i)$, we have

$$\phi(A) = \sum_{i=1}^{n} (y_i, Ax_i), \text{ all } A \in \mathcal{R}.$$

QED.

Remark: The assumption that $\mathcal{R}$ be a von Neumann algebra is clearly unnecessary — all one needs is for $\mathcal{R}$ to be a linear set of operators.

This result extends to ultrastrongly continuous linear functionals with the obvious modifications.

2.3.2. Proposition

Let $\mathcal{R}$ be a linear set in $\mathcal{B}(\mathcal{H})$, and let $\phi$ be a linear functional on $\mathcal{R}$. The following are equivalent:

(i) $\phi : \mathcal{R} \to \mathbb{C}$ is ultrastrongly continuous,

(ii) $\phi : \mathcal{R} \to \mathbb{C}$ is ultraweakly continuous,

(iii) there exist sequences $(x_i)$, $(y_i)$ in $\mathcal{H}$ with

$\sum \|x_i\|^2 < \infty$, $\sum \|y_i\|^2 < \infty$, such that

$$\phi(A) = \sum_{i=1}^{\infty} (y_i, Ax_i) \quad \text{for all } A \in \mathcal{R}.$$

Proof: The proof is trivial once we have shown that (i) $\Rightarrow$ (iii); but this is exactly as in 2.3.1 with $\overset{n}{\oplus}\mathcal{H}$ replaced by $\overset{\infty}{\oplus}\mathcal{H}$.

QED.

2.3.3. **Proposition** Let $K$ be a convex set in $\mathcal{B}(\mathcal{H})$. Then the strong and weak closures of $K$ are the same.

**Proof** We have $K \subset \overline{K}^s \subset \overline{K}^w$, and $\overline{\overline{K}^s}^w = \overline{K}^w$.

If we show that $\overline{K}^s$ is weakly closed, then we have the equality $\overline{K}^s = \overline{K}^w$. Replacing $K$ by $\overline{K}^s$, which is also convex, we assume that $K$ is strongly Closed but not weakly closed and will obtain a contradiction.

Let $A \in \overline{K}^w$, $A \notin K$. $K$ is strongly closed and convex, so by the Hahn-Banach theorem (see, for example, Lanford (1972) or Dunford and Schwartz (1966)) $A$ may be separated from $K$ by a strongly continuons functional: i.e. there is a strongly continuons functional $\phi$ on $\mathcal{B}(\mathcal{H})$ such that:

$$\operatorname{Re} \phi(A) > \sup \{ \operatorname{Re} \phi(B) \mid B \in K \}$$

Since $\phi$ is strongly continuons, by 2.3.1., it is weakly continuons, and, since $A \in \overline{K}^w$, there is a net $(B_\alpha)$ in $K$ with $B_\alpha \to A$ weakly, and so $\phi(B_\alpha) \to \phi(A)$. This clearly contradicts the above inequality. We conclude that $K$ is weakly closed.

QED.

## 2.4. Kaplansky's Density Theorem

Let $\mathcal{R}$ be a $*$-algebra of operators, and let $\overline{\mathcal{R}}^s$ be its strong closure in $\mathcal{B}(\mathcal{H})$. If $A \in \overline{\mathcal{R}}^s$ there is a net $A_\alpha$ in $\mathcal{R}$ which converges strongly to A. If A is self-adjoint, can we choose the $A_\alpha$ to be self-adjoint ? Can we choose $A_\alpha$ so that, in any case, $A_\alpha^*$ converges strongly to $A^*$, or such that $A_\alpha^2$ converges to $A^2$ ?

The possibility of making such choices is the content of Kaplansky's density Theorem.

2.4.1. <u>Theorem</u> (Kaplansky's Density Theorem)

Let $\mathcal{R}$ be a self-adjoint algebra in $\mathcal{B}(\mathcal{H})$, and let $\overline{\mathcal{R}}$ denote the strong closure of $\mathcal{R}$. For any element $A \in \overline{\mathcal{R}}$ there exists a net $A_\alpha$ in $\mathcal{R}$ such that:

(i) $\| A_\alpha \| \leq \| A \|$ for all $\alpha$,

(ii) $A_\alpha$ converges strongly to A,

(iii) $A_\alpha^*$ converges strongly to A*,

If A is self-adjoint, $A_\alpha$ may be taken to be self-adjoint.

<u>Proof</u> Let us assume firs that A is self-adjoint. We may also suppose that $\| A \| = 1$.

Consider the functions f, g; $[-1,1] \to \mathbb{R}$ given by $f(t) = 2t\,(1 + t^2)^{-1}$ and $g(t) = (1- \sqrt{1 - t^2})/t$. We see that $-1 \leq g(t) \leq 1$ and that $f(g(t)) = t$. Both f and g are continuons.

By realizing A as a real function in C(K) and noting that $\| A \| = 1$ implies that the function representing A has modulus less than or equal to one, we can define $B = g(A)$, which is self-adjoint, belongs to the C*-algebra generated by A, and satisfies $f(B) = f(g(A)) = A$, i.e. $A = 2B(\mathbb{1} + B^2)^{-1}$. Since B belongs to the C*-algebra generated by A, a fortiori B belongs to $\overline{\mathcal{R}}$. There is, therefore, a net $B_\alpha$ in $\mathcal{R}$ which converges strongly to B.

We claim that we may choose $B_\alpha$ self-adjoint.

Since $B_\alpha$ converges strongly to B, it also converges weakly to B. Hence $B_\alpha^*$ converges weakly to $B^* = B$, and so $\frac{1}{2}(B_\alpha + B_\alpha^*)$ converges weakly to B. That is, B is in the weak closure of the convex set of self-adjoint elements of $\mathcal{R}$. By 2.3.3., B is in the strong closure of the self-adjoint elements of $\mathcal{R}$, and so, as claimed, we may suppose that the $B_\alpha$ are self-adjoint.

Let $A_\alpha = 2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1}$. Then $A_\alpha$ is self-adjoint and $\| A_\alpha \| \leq 1$. To see that $A_\alpha$ converges strongly to A, consider

$$\begin{aligned} A - A_\alpha &= 2B(\mathbb{1} + B^2)^{-1} - 2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1} \\ &= 2(\mathbb{1} + B_\alpha^2)^{-1}\{(\mathbb{1} + B_\alpha^2)B - B_\alpha(\mathbb{1} + B^2)\}(\mathbb{1} + B^2)^{-1} \\ &= 2(\mathbb{1} + B_\alpha^2)^{-1}\{(B - B_\alpha) + B_\alpha(B_\alpha - B)B\}(\mathbb{1} + B^2)^{-1} \\ &= 2(\mathbb{1} + B_\alpha^2)^{-1}(B - B_\alpha)(\mathbb{1} + B^2)^{-1} + \\ &\quad + 2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1}(B_\alpha - B)(\mathbb{1} + B^2)^{-1} \end{aligned}$$

Now, $(B_\alpha - B)(\mathbb{1} + B^2)^{-1}$ converges strongly to zero. Moreover, $\| (\mathbb{1} + B_\alpha^2)^{-1} \| \leq 1$, and $\| 2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1} \| \leq 1$, and so $A - A_\alpha$ converges strongly to zero, as required. Thus, for A self-adjoint, the proof is complete.

For the general case, consider $\mathfrak{A}$ and $\overline{\mathfrak{A}}$, the self-adjoint algebras of operators on $\mathcal{H} \oplus \mathcal{H}$ given by matrices $(A_{ij})$, $1 \leq i,j \leq 2$, with $A_{ij} \in \mathcal{R}$ or $\overline{\mathcal{R}}$, respectively. It is easy to see that $\overline{\mathfrak{A}}$ is the strong closure of $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$.

Let $A \in \overline{\mathcal{R}}$. Then $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ is a self-adjoint element of $\overline{\mathfrak{A}}$, with norm equal to $\| A \|$. Hence, by the preceding proof, there is a net $(A_{ij}^{\alpha})$ of self-adjoint elements of $\mathcal{A}$ with norm less than or equal to $\|A\|$, converging strongly to $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$

That is, for each $x \oplus y \in \mathcal{H} \oplus \mathcal{H}$,

$$(A_{ij}^{\alpha})\ (x \oplus y) = (A_{11}^{\alpha}x + A_{12}^{\alpha}y) \oplus (A_{21}^{\alpha}x + A_{22}^{\alpha}y)$$

converges to $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = Ay \oplus A^* x$ in $\mathcal{H} \oplus \mathcal{H}$.

Now, $\|(A_{ij}^{\alpha})\| \leqslant \|A\|$ implies that

$$\|A_{12}^{\alpha}\| = \|E_1(A_{ij}^{\alpha})E_2\| \leqslant \|A\|$$

Where $E_1$ and $E_2$ are the projections in $\mathcal{H} \oplus \mathcal{H}$ onto the first and second components, respectevely.

Moreover taking $x = 0$, the above convergence implies that $A_{12}^{\alpha} \longrightarrow A$ strongly, and, taking $y=0$, implies that $A_{21}^{\alpha} = A_{12}^{\alpha *}$ (since $(A_{ij}^{\alpha})$ is self-adjoint) converges strongly to $A^*$, and the proof is complete.

QED.

__Remark__ . The conditions (i),(ii) and (iii) of 2.4.1. imply that $A_{\alpha}^2$ converges strongly to $A^2$, and that $A_{\alpha}^* A_{\alpha}$ converges strongly to $A^*A$.

In general, ~~we~~ will need a __net__ $A_{\alpha}$ to lead to the strong limit point, but if $\mathcal{H}$ is separable we can actually find a bounded sequence $A_n$ satisfying the conditions (i), (ii) and (iii).

**2.4.2.** __Proposition__ Let $\mathcal{H}$ be a separable Hilbert space and $S$ a bounded set in $\mathcal{B}(\mathcal{H})$. Then the topology induced on $S$ by the strong topology of is metrizable. In fact, it is given by a norm.

__Proof__ . We must find a norm $|||\cdot|||$ on $\mathcal{B}(\mathcal{H})$ such that a bounded net $A_\alpha$ converges strongly to A if and only if $\lim_\alpha |||A_\alpha - A||| = 0$ . To construct $|||\cdot|||$ , let $\{x_n\}$ be a countable dense set in $\mathcal{H} \setminus \{0\}$ . We define

$$|||A||| = \sum_{n=1}^{\infty} 2^{-n} \|Ay_n\| \qquad , \qquad y_n = \frac{x_n}{\|x_n\|}$$

Clearly $|||\cdot|||$ is a norm on $\mathcal{B}(\mathcal{H})$ : for example, $|||A||| = 0$ implies that $Ax_n = 0$ for all n, and so $A = 0$ since $\{x_n\}$ is dense. Also $|||A||| \leq \|A\|$.

We want to show that if $A_\alpha$ is a bounded net in $\mathcal{B}(\mathcal{H})$ then $A_\alpha$ converges strongly to A if and only if $|||A_\alpha - A||| \to 0$

Put $B_\alpha = A_\alpha - A$, and suppose $B_\alpha \longrightarrow 0$ strongly, and $\|B_\alpha\| \leq M$, for all $\alpha$; then

$$\overline{\lim_\alpha} |||B_\alpha||| = \overline{\lim_\alpha} \sum_{n=1}^{\infty} 2^{-n} \|B_\alpha y_n\|$$

$$\leq \overline{\lim_\alpha} \sum_{n=1}^{N} 2^{-n} \|B_\alpha y_n\| + \overline{\lim_\alpha} \sum_{n=N+1}^{\infty} 2^{-n} \|B_\alpha y_n\|$$

The first term is zero since $\|B_\alpha y_n\| \to 0$ for each $y_n$, and the second term is bounded by $M/2^N$, since $\|B_\alpha y_n\| \leq M$ for all $\alpha$, n.

Thus $\overline{\lim}\ |||B_\alpha||| \leq M/2^N$ for any N, and so $\overline{\lim}\ |||B_\alpha||| = 0$. That is, the strong convergence of $B_\alpha$ to 0 implies that $|||B_\alpha||| \to 0$.

Conversely, let $B_\alpha$ be a net with $\|B_\alpha\| \leq M$ and suppose $|||B_\alpha||| \longrightarrow 0$. This implies that $\|B_\alpha x_n\| \to 0$ for all n. Since the $B_\alpha$'s are uniformly bounded, and $\{x_n\}$ is dense, we conclude that $\|B_\alpha x\| \to 0$ for any $x \in \mathcal{H}$, i.e. $B_\alpha$ converges strongly to zero.

QED.

2.4.3. Theorem. Let $\mathcal{R}$ be a self-adjoint algebra of operators on a separable Hilbert space, and let A belong to the strong closure of $\mathcal{R}$. Then there is a sequence $A_n$ in $\mathcal{R}$ such that.

(i) $\|A_n\| \leq \|A\|$ for all n,

(ii) $A_n$ converges strongly to A,

(iii) $A_n^*$ converges strongly to $A^*$.

Proof. By Kaplansky's density theorem (2.4.1.), there is a net $A_\alpha$ in $\mathcal{R}$ with $\|A_\alpha\| \leq \|A\|$, and $A_\alpha \to A$, $A_\alpha^* \longrightarrow A^*$ strongly. By 2.4.2., $|||A_\alpha - A||| \to 0$ and $|||A_\alpha^* - A^*||| \to 0$. Thus, for each integer n, there is an $\alpha_n$ such that $|||A_{\alpha_n} - A||| \leq \frac{1}{n}$, and $|||A_{\alpha_n}^* - A||| \leq \frac{1}{n}$.

Setting $A_n = A_{\alpha_n}$, we have $\|A_n\| \leq \|A\|$, $|||A_n - A||| \to 0$, and $|||A_n^* - A||| \to 0$. Again, by 2.4.2, $A_n \to A$ and $A_n^* \longrightarrow A^*$ strongly.

QED.

## 2.5. Positive Continuons Functionals

We have already discussed the structure of ultrastrongly and strongly continuons functionals. We want to consider here positive such functionals. We will see that these correspond precisely to the set of density matrices.

2.5.1. Proposition Let $\mathcal{R}$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, containing $\mathbb{1}$, and let $\phi$ be a positive ultrastrongly continuons linear functional on $\mathcal{R}$. Then there exists a sequence $(x_i)$ of vectors in $\mathcal{H}$ such that $\sum_i \|x_i\|^2 < \infty$ and such that

$$\phi(A) = \sum_i (x_i, A x_i)$$

for all $A \in \mathcal{R}$. If $\phi$ is strongly continuons, the sequence of $x_i$'s may be chosen finite.

Proof. Let $\phi$ be ultrastrongly continuons and positive. Since the norm topology is finer than the ultrastrong topology, $\phi$ extends uniquely to an ultrastrongly continuons functional on $\overline{\mathcal{R}}^n$, the norm closure of $\mathcal{R}$. This extension is also positive, We may, therefore, without loss of generality, suppose that $\mathcal{R}$ is a C*-Algebra.

By 2.3.2, we may write $\phi(A) = (y, \tilde{A}x)$, where $x = (x_i)$, $y = (y_i) \in \bigoplus_i \mathcal{H}$, and $\tilde{A}x = (Ax_i)$.

We consider the positive linear functional on $\mathcal{R}$ given by $\psi(A) = (x + y, \tilde{A}(x + y))$. Let $A \in \mathcal{R}$ be positive. Then

$$\psi(A) = (x, \tilde{A}x) + (y, \tilde{A}y) + (y, \tilde{A}x) + (x, \tilde{A}y)$$

$$= (x, \tilde{A}x) + (y, \tilde{A}y) + 2\phi(A)$$

Since $(x, \tilde{A}y) = (y, \tilde{A}x) = \phi(A) \geqslant 0.$

Hence $\psi(A) \geqslant 2\phi(A)$, i.e. $\psi$ majorizes $\phi$.

Consider the cyclic subspace of $x + y$ in $\bigoplus_{i=1}^{\infty} \mathcal{H}$, i.e. the closure of $\{ \tilde{A}(x + y) \mid A \in \mathcal{R} \}$, and the representation of $\mathcal{R}$ defined by restricting $\tilde{A}$ to this cyclic subspace. By the uniqueness of the GNS construction, this representation is equivalent to the GNS representation associated with the functional $\psi$. Since $\psi$ majorizes $\phi$, we know, by 1.8.4, that there is a positive operator $T$ on the cyclic subspace, commuting with $\tilde{A}$ for each $A \in \mathcal{R}$, such that

$$\phi(A) = (T(x + y), \tilde{A}(x + y)) = (T^{1/2}(x + y), \tilde{A}\, T^{1/2}(x+y)).$$

Setting $z = (z_i) = T^{1/2}(x + y)$, we have

$$\phi(A) = \sum_{i=1}^{\infty} (z_i, A z_i)$$

as required.

If $\phi$ is strongly continuons, the proof is exactly the same except that $\bigoplus_{i=1}^{\infty} \mathcal{H}$ is replaced by a finite direct sum.

QED.

This result says $\phi$ looks rather like a trace. Indeed, we can now show which functionals are given by "density matrices".

2.5.2. <u>Theorem</u> Let $\mathcal{R}$ be self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, containing $\mathbb{1}$, and let $\phi$ be an ultrastrongly continuons positive linear functional on $\mathcal{R}$. Then there exists a positive linear operator of trace class such that

$$\phi(A) = \mathrm{Tr}(\rho A)$$

for all $A \in \mathcal{R}$. Conversely, if $\rho$ is a positive linear operator of trace class, $A \longrightarrow \mathrm{Tr}(\rho A)$ is an ultrastrongly continuons positive linear functional on $\mathcal{B}(\mathcal{H})$.

<u>Proof</u> By the preceding proposition, $\phi$ can be written as

$$\phi(A) = \sum_{i=1}^{\infty} (\zeta_i, A\zeta_i)$$

some $\zeta_i \in \mathcal{H}$ with $\sum_i \|\zeta_i\|^2 = \phi(\mathbb{1}) < \infty$

Define the linear operator $\rho$ on $\mathcal{H}$ by

$$\rho x = \sum_{i=1}^{\infty} \zeta_i (\zeta_i, x).$$

Then $\|\rho x\| \leqslant \sum_i \|\zeta_i\|^2 \|x\| = \phi(\mathbb{1}) \|x\|$ , so $\rho$ is bounded. Furthermore, $\rho$ is positive because

$$(x, \rho x) = \sum_i (x, \zeta_i)(\zeta_i, x) = \sum_i |(\zeta_i, x)|^2 \geqslant 0 .$$

Let ( $x_1, \dots, x_n$) be any finite orthonormal set in $\mathcal{H}$ .

Then

$$\sum_{j=1}^{n} (x_j, \rho x_j) = \sum_{j=1}^{n} \sum_i |(\zeta_i, x_j)|^2$$

$$= \sum_i \sum_{j=1}^{n} |(\zeta_i, x_j)|^2 \leqslant \sum_i \|\zeta_i\|^2$$

$$= \phi(\mathbb{1})$$

This bound, independent of n, implies that $\rho$ is of trace class. Hence, for any complete orthonormal set ($x_\alpha$ ) and any $A \in \mathcal{R}$ , we have

$$\phi(A) = \sum_i (\zeta_i, A\zeta_i) = \sum_{i,\alpha} (\zeta_i, x_\alpha)(x_\alpha, A\zeta_i)$$

$$= \sum_\alpha (x_\alpha, A\rho x_\alpha) = \mathrm{Tr}\,(A\rho)$$

i.e. $\phi(A) = \mathrm{Tr}\,(A\rho)$, as required.

Conversely, suppose $\rho$ is a positive trace class operator. Then $\rho$ can be written as $\rho = \sum_{i=1}^{\infty} \lambda_i^2 E_i$ , where $\lambda_i^2 \geqslant 0$ are the eigenvalues of $\rho$ and $E_i$ the corresponding projections onto the normalized eigenvectors, $\zeta_i$ , say. Then we have

$$\rho x = \sum_i \zeta_i \lambda_i^2 (\zeta_i, x)$$

Put $\xi_i = \lambda_i \zeta_i$, so that $\sum_i \| \xi_i \|^2 = \sum_i |\lambda_i|^2 < \infty$ since $\rho$ is trace class. Also, for any complete set $(x_\alpha)$ in $\mathcal{H}$

$$\mathrm{Tr}(\rho A) = \sum_\alpha (x_\alpha, \rho A x_\alpha) = \sum_{\alpha, i} (x_\alpha, \zeta_i) \lambda_i^2 (\zeta_i, A x_\alpha)$$

$$= \sum_{\alpha, i} (x_\alpha, \xi_i)(\xi_i, A x_\alpha) = \sum_{i, \alpha} (\xi_i, A x_\alpha)(x_\alpha, \xi_i)$$

$$= \sum_i (\xi_i, A \xi_i)$$

Thus $A \to \mathrm{Tr}(\rho A)$ is ultraweakly continuons, and, by (2.3.2), ultrastrongly continuons.

QED.

**Remark** The $\xi_i$'s constructed above are mutually orthogonal, so we have a refinement of 2.5.1 in that the $x_i$'s can be chosen mutually orthogonal.

$\rho$ is called a <u>density matrix</u>. We note, that, in general, $\rho$ is not uniquely determined by $\phi$, although this will be the case if $\overline{\mathcal{R}}^{us}$ is equal to $\mathcal{B}(\mathcal{H})$.

## 2.6. Disjoint representations of a C*-algebra

The purpose of this section is to prove the theorem of Glimm and Kadison (1960) which will be used in chapter 7. First we need the polar decomposition theorem.

2.6.1. <u>Definition</u> Let $\mathcal{H}$ be a Hilbert space, and let $W \in \mathcal{B}(\mathcal{H})$. W is called a partial isometry if there are subspaces K, L in $\mathcal{H}$ such that $W : K \longrightarrow L$ is isometric onto L, and $W : K^{\perp} \to \{0\}$. K is called the initial subspace, and L the final subspace.

Evidently, $W^*$ maps L isometrically onto K, and maps $L^{\perp}$ onto 0.
It is also easy to see that $W^*W = P_K$, the projection of $\mathcal{H}$ onto K, and $WW^* = P_L$, the projection onto L.

Conversely, if W is an operator such that $W^*W = P_K$, some K, then W is a partial isometry with initial space K and final space WK.

2.6.2. <u>Theorem</u> (Polar Decomposition)

Let $A \in \mathcal{B}(\mathcal{H})$. Then A can be written uniquely as $A = W|A|$ where $|A|$ is the positive square-root of $A^*A$ and W is a partial isometry with initial space equal to the closure of the range of $|A|$ and final space equal to the Closure of the range of A.

<u>Proof</u> For any $x \in \mathcal{H}$, we have

$$\|Ax\|^2 = (Ax, Ax) = (x, A^*Ax)$$

$$= (x, |A|^2 x) = \| \, |A| \, x \|^2$$

Thus there is a unique unitary operator, $W$, from the closure of the range of $|A|$ to the closure of the range of $A$ given by $W|A| = A$. We extend $W$ to a partial isometry by defining $Wy = 0$ for $y$ orthogonal to the range $|A|$.

QED.

<u>Remark</u> This way of writing $A$ is called the polar decomposition of $A$.

Suppose $U$ is unitary and commutes with $A$; then

$$A = U A U^* = U W U^* U |A| U^* = U W U^* |A|$$

is another polar decomposition. The uniqueness implies that $W = U W U^*$, i.e. $W$ also commutes with $U$.

**2.6.3.** <u>Definition</u> Let $(\mathcal{H}, \pi)$ be a representation of a $C^*$ - algebra, $\mathfrak{A}$. Suppose $\mathcal{H}_1 \subset \mathcal{H}$ is a subspace of $\mathcal{H}$ invariant under $\pi(\mathfrak{A})$. Then $(\mathcal{H}_1, \pi_1)$, where $\pi_1(\mathfrak{A})$ is defined to be $\pi(\mathfrak{A}) \restriction \mathcal{H}_1$, defines a representation of $\mathfrak{A}$ called a subrepresentation of $(\mathcal{H}, \pi)$

Evidently $(\mathcal{H}_1^{\perp}, \pi_1^{\perp})$, where $\pi_1^{\perp}(\mathfrak{A}) = \pi(\mathfrak{A}) \restriction \mathcal{H}_1^{\perp}$, is also a subrepresentation of $(\mathcal{H}, \pi)$.

**2.6.4.** __Definion__ Let $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ be any two representations of a $C^*$- algebra, $\mathfrak{A}$. They are said to be disjoint if no subrepresentation of one is unitarily equivalent to a subrepresentation of the other.

**2.6.5.** __Theorem__ Let $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ be two disjoint representations of a $C^*$- algebra, $\mathfrak{A}$. Then the von Neumann algebra $(\pi_1 \oplus \pi_2)(\mathfrak{A})''$ is equal to $\pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$.

__Proof__ We shall first show that $(\pi_1 \oplus \pi_2)(\mathfrak{A})' = \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$. Indeed, let $B \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ belong to $(\pi_1 \oplus \pi_2)(\mathfrak{A})'$. As in 2.2.5, we write $B = \begin{pmatrix} X & S \\ T & Y \end{pmatrix}$ with $X : \mathcal{H}_1 \to \mathcal{H}_1$, $Y : \mathcal{H}_2 \to \mathcal{H}_2$, $S : \mathcal{H}_2 \to \mathcal{H}_1$, $T : \mathcal{H}_1 \to \mathcal{H}_2$. Then $B \in \pi_1 \oplus \pi_2(\mathfrak{A})'$ implies that $X \in \pi_1(\mathfrak{A})'$, $Y \in \pi_2(\mathfrak{A})'$, and $S\,\pi_2(A) = \pi_1(A) S$, $T\,\pi_1(A) = \pi_2(A) T$, for all $A \in \mathfrak{A}$.

Consider $S\,\pi_2(A) = \pi_1(A) S$, $A \in \mathfrak{A}$. We extend $S$ to an operator in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by defining $S$ to be zero on $\mathcal{H}_1$, in $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then we can write: $S\,\pi_1 \oplus \pi_2(A) = \pi_1 \oplus \pi_2(A)\,S$, all $A \in \mathfrak{A}$.

Since $\mathfrak{A}$ is a $C^*$- algebra, it is generated by its unitary elements, so by 2.6.2. and the remark following it, we see that

$W\ \pi_1 \oplus \pi_2(A) = \pi_1 \oplus \pi_2(A)\,W$ for all $A \in \mathfrak{A}$, where $W$ is the partial isometry given by the polar decomposition of $S$. Taking adjoints, we obtain $\pi_1 \oplus \pi_2(A)\,W^* = W^*\ \pi_1 \oplus \pi_2(A)$, all $A \in \mathfrak{A}$.

Hence

$$\pi_1 \oplus \pi_2(\mathfrak{A})\, W^* W = W^*\, \pi_1 \oplus \pi_2(\mathfrak{A})\, W$$
$$= W^* W\, \pi_1 \oplus \pi_2(\mathfrak{A})$$

In other words, $\pi_1 \oplus \pi_2 \restriction W^*W\, \mathcal{H}_1 \oplus \mathcal{H}_2$ defines a subrepresentation of $(\mathcal{H}_1 \oplus \mathcal{H}_2,\ \pi_1 \oplus \pi_2)$. $W^*W\, \mathcal{H}_1 \oplus \mathcal{H}_2$ is the initial space of W which is contained in $\mathcal{H}_2$, and so $\pi_1 \oplus \pi_2 \restriction W^*W\, \mathcal{H}_1 \oplus \mathcal{H}_2$ is a subrepresentation of $(\mathcal{H}_2, \pi_2)$.

Similarly, $\pi_1 \oplus \pi_2 \restriction W W^* \mathcal{H}_1 \oplus \mathcal{H}_2$ is a subrepresentation of $(\mathcal{H}_1, \pi_1)$. But then W effects a unitary equivalence between these subrepresentations which is in contradiction with the assumed disjointness of $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$, unless $W = 0$.
We conclude that $S = 0$. In exactly the same way, we see that $T = 0$ and that therefore B belongs to $\pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$, i.e. $(\pi_1 \oplus \pi_2)(\mathfrak{A})' \subset \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$.
The converse inclusion is trivial and so we have

$$\pi_1 \oplus \pi_2(\mathfrak{A})' = \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$$

as claimed.

Now suppose $B \in \pi_1 \oplus \pi_2(\mathfrak{A})''$. Then, by von Neumann's density theorem, B is in the weak closure of $\pi_1 \oplus \pi_2(\mathfrak{A})$. It is easy to see that this implies that $B = C \oplus D$, with $C \in \pi_1(\mathfrak{A})''$ and $D \in \pi_2(\mathfrak{A})''$ (applying von Neumann's density theorem again to $\pi_1(\mathfrak{A})$ and $\pi_2(\mathfrak{A})$).
Hence

$$\pi_1 \oplus \pi_2(\mathfrak{A})'' \subset \pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$$

Conversely, take $A \oplus B \in \pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$. By the above argument, $\pi_1 \oplus \pi_2(\mathfrak{A})' = \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$ and so $A \oplus B$ commutes with $\pi_1 \oplus \pi_2(\mathfrak{A})'$, i.e. $A \oplus B \in \pi_1 \oplus \pi_2(\mathfrak{A})''$.

We conclude that

$$\pi_1 \oplus \pi_2(\mathfrak{A})'' = \pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$$

QED.

2.6.6 <u>Theorem</u> (Glimm and Kadison (1960))

Let $\omega_1$ and $\omega_2$ be states on a $C^*$-algebra, $\mathfrak{A}$, and suppose that their associated GNS representations $(\mathcal{H}_1, \pi_1)$, $(\mathcal{H}_2, \pi_2)$ are disjoint. Then $\| \omega_1 - \omega_2 \| = 2$.

<u>Proof</u>. By definition of the GNS representations, the form $\omega_1 - \omega_2$ on $\pi_1 \oplus \pi_2(\mathfrak{A})$ is given by the difference of two vector states.

$$(\omega_1 - \omega_2)\, \pi_1 \oplus \pi_2(A) = \omega_1(A) - \omega_2(A) =$$

$$= (\Omega_1, \pi_1(A)\Omega_1) - (\Omega_2, \pi_2(A)\Omega_2)$$

where $\Omega_i$ is the GNS cyclic vector in $\mathcal{H}_i$, $i = 1,2$.

It follows that $\omega_1 - \omega_2$ is weakly continuons and so extends, by continuity, to a form $\psi$ on $\pi_1 \oplus \pi_2(\mathfrak{A})''$. We claim that the norm of $\psi$ is the same as the norm of $\omega_1 - \omega_2$. It is clear that the norm of $\psi$ is not less than that of $\omega_1 - \omega_2$, so we need only show that $\| \psi(A) \| \leq \| \omega_1 - \omega_2 \|$ for any $A \in \pi_1 \oplus \pi_2(\mathfrak{A})''$ with $\| A \| = 1$. But von Neumann's density theorem implies that A is a strong limit point of $\pi_1 \oplus \pi_2(\mathfrak{A})$ and so, by Kaplansky's density theorem, there is a net $\mathcal{B}_\alpha$ in $\pi_1 \oplus \pi_2(\mathfrak{A})$, with $\|\mathcal{B}_\alpha\| \leq 1$, such that $\mathcal{B}_\alpha$ converges strongly to A. Hence $\psi(\mathcal{B}_\alpha)$ converges to. $\psi(A)$. But $\psi(\mathcal{B}_\alpha) = (\omega_1 - \omega_2)(\mathcal{B}_\alpha)$ and $|(\omega_1 - \omega_2)(\mathcal{B}_\alpha)| \leq \|\omega_1 - \omega_2\|$. The result follows.

Now, $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are disjoint, so by 2.6.5, $\pi_1 \oplus \pi_2(\mathfrak{A})'' = \pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$. This algebra contains $A = \mathbb{1} \oplus -\mathbb{1}$. Thus

$$2 \geq \|\omega_1 - \omega_2\| = \|\psi\| \geq |\psi(A)| = 2$$

since $\omega_1(\mathbb{1}) = \omega_2(\mathbb{1}) = 1$

Hence $\|\omega_1 - \omega_2\| = 2$

QED.

# 3. The Canonical Commutation Relations

## 3.1 The Heisenberg Relation

Probably the first thing one ever encounters in quantum mechanics in the commutation relation of Heisenberg. This relation is simply $[Q,P] \equiv QP - PQ = i\hbar$, where Q is an operator representing the position, and P the momentum of a single particle moving in one-dimension. For a single particle in, say, 3-dimensions, this relation becomes

$$[Q_k, P_\ell] = i\hbar\,\delta_{k\ell} \quad , \quad [Q_k, Q_\ell] = [P_k, P_\ell] = 0$$

with $1 \leq k, \ell \leq 3$ representing the spatial directions.

One can now ask what these relations mean mathematically, and whether such relations uniquely determine the operators $P$ and $Q$.
According to their physical interpretation, we require the P's and Q's to be self-adjoint operators, but to specify an operator, one must define its action <u>and</u> its domain of definition. For (densely-defined) bounded operators this is no problem as they always have a natural extension (defined by continuity) to the whole Hilbert space. We note that, in this case, there is a well-defined product PQ and QP which is also a bounded operator.

So let us suppose that we have two bounded operators P,Q satisfying

$$[Q, P] = i$$

(from now on we set $\hbar = 1$)

By induction, we see that

$$[Q^n, P] = in\, Q^{n-1}$$

for:

$$\begin{aligned}[Q^{n+1}, P] &= Q^{n+1}P - PQ^{n+1} \\ &= Q(Q^nP - PQ^n) + (QP - PQ)Q^n \\ &= Q\, in\, Q^{n-1} + [Q, P]Q^n \\ &= i(n+1)Q^n\end{aligned}$$

as required

But then

$$\begin{aligned} \| i n Q^{n-1} \| &= n \| Q^{n-1} \| \\ &= \| [Q^n, P] \| = \| Q^n P - P Q^n \| \\ &\leq 2 \|P\| \, \|Q^n\| \\ &\leq 2 \|P\| \, \|Q\| \, \|Q^{n-1}\| \end{aligned}$$

i.e. $n \leq 2 \|P\| \, \|Q\|$ for any n.

This is clearly a contradiction.We have:

3.1.1 Theorem

The relation $[Q,P] = i$ has no solution in $\mathcal{B}(\mathcal{H})$.

We are forced to consider unbounded operators and their delicate domain considerations.

3.1.2. Definition. We say that self-adjoint operators $(Q,D(Q))$, $(P,D(P)$ satisfy the Heisenberg commutation relation if there is a dense domain $\mathcal{D} \subset \mathcal{D}(Q) \cap \mathcal{D}(P)$ such that $Q\,D \subset D(P)$, $P\,D \subset D(Q)$, and such that, on D, we have.

$$QP - PQ = i$$

3.1.3. Definition. We say that symmetric operators $(Q,D(Q))$, $(P,D(P))$ satisfy the weak Heisenberg relation if there is a dense domain $D \subset D(Q) \cap D(P)$ such that

$$(Qf,Pg) - (Pf,Qg) = i\,(f,g)$$

for all f,g in D.

Do there exist Q and P satisfying these definitions? The answer is yes, as we see from the following Schrödinger representation of position and momentum. We take $\mathcal{H} = L^2(\mathbb{R}, dx)$, $(Qf)(x) = xf(x)$ on $\mathcal{D}(Q) = \{ f \mid \int |x f(x)|^2 dx < \infty \}$

P is defined in terms of Fourier transforms:

$$(Pg)^{\sim}(p) = p\tilde{g}(p) \text{ on } \mathcal{D}(P) = \{ g \mid \int |p \check{g}(p)|^2 dp < \infty \}.$$

Of course, if $g \in \mathcal{S}(\mathbb{R})$, for example, then $Pg = -i \frac{dg}{dx}$

Thus defined $(Q, D(Q))$, $(P, D(P))$ are self-adjoint and, if we take $D = \mathcal{S}(\mathbb{R})$, for example, then 3.1.2. (and hence 3.1.3) is easily seen to be satisfied.

We can now ask whether the Schrödinger representation is unique in some sense. The answer is no – there is an uncountable number of different solutions! Let us show this by constructing some of them.

For our Hilbert space, we take $\mathcal{H} = L^2([0,1], dx)$. Define $(Qf)(x) = x f(x)$, all $f \in \mathcal{H}$. Let $g$ be differentiable, with $g(0) = 0 = g(1)$, and $g' \in \mathcal{H}$. For such $g$ we define

$$(Pg)(x) = -ig'(x)$$

Obviously Q is a bounded, and it can be shown (see, for example, Robinson (1971)) that P has an uncountable number of distinct self-adjoint extensions. These correspond to specifying boundary conditions of the form $g(1) = e^{i\theta} g(0)$, $0 \leq \theta < 2\pi$. Call these extensions $P_\theta$. $P_\theta$ has a discrete spectrum with eigenvectors

$\exp i(2\pi n + \theta)x$

Take $D$ = all $C^\infty$- functions on $[0,1]$ which vanish at the end points. $P_\theta = P$ on $D$ and it is easy to see that $Q, P_\theta$ satisfy the Heisenberg relation on $D$.

We conclude that the Heisenberg relation does not uniquely specify $Q$ and $P$.

We remark in passing that from the above example we see how important it is to specify precisely the domain of an operator. In this example, this corresponds to specifying boundary conditions . Indeed, consider the operator $-id/dx$ with <u>no</u> boundary conditions. This has $e^{izx}$, $z \in \mathbb{C}$ ,as an eigenvector with eigenvalue $z$. So its spectrum is the whole complex plane!

3.1.4. <u>Definition</u> . Let $A$ be an operator, and let $f \in D(A)$. The variance of $A$ in $f$ is the non-negative number $v_f(A) \equiv \| (A - (f, A f)) f \|^2$

If $A$ is self-adjoint and $f \in D(A^2)$, then

$$v_f(A) = (f, (A^2 - (f, Af)^2) f).$$

If $f$ is a normalized eigenvector, then $v_f(A) = 0$ (and conversely, unless $Af = 0$ ).

3.1.5. <u>Proposition</u>

Let $Q, P, D$ satisfy the weak Heisenberg relation. Then, if $f \in D$, $\| f \| = 1$, we have

$$v_f(Q)\ v_f(P) \geqslant \frac{1}{4}$$

<u>Proof</u>. Let $f \in D$, $\| f \| = 1$ Then

$$\begin{aligned} i\,2\,\mathrm{Im}\,(Pf,Qf) &= (Pf,Qf) - \overline{(Pf,Qf)} \\ &= (Pf,Qf) - (Qf,Pf) \\ &= -\,i\,(f,f) \quad \text{by 3.1.3.} \end{aligned}$$

Hence $-\,2\,\mathrm{Im}\,(Pf,Qf) = \| f \|^2 = 1$, and so

$$\begin{aligned} 1 = 2\,\left| \mathrm{Im}\,(Pf,Qf) \right| &\leqslant 2\,\left| (Pf,Qf) \right| \\ &\leqslant 2\,\| Pf \|\,\| Qf \| \end{aligned}$$

i.e. $\| Pf \|^2 \, \| Qf \|^2 \geqslant \frac{1}{4}$.

If $P, Q$ satisfy the Heinsenberg relation, so do $P-\beta$ and $Q-\alpha$, for any real $\alpha, \beta$. Hence

$$\| (Q-\alpha)\, f \|^2 \, \| (P-\beta)\, f \|^2 \geqslant \frac{1}{4}$$

Taking $\alpha = (f,Qf)$, $\beta = (f,Pf)$ gives

$$v_f(Q)\, v_f(P) \geqslant \frac{1}{4}$$

as required

QED.

This, of course, is the well-khown uncertainty relation.

**Let us return** to our example on $L^2\,[0,1]$.

Let $f(x) = e^{i2\pi x}$. Then $v_f(P_o) = 0$, and so $v_f(Q)\, v_f(P_o) = 0$ – not much uncertainty here !

**This would appear to contradict the previons proposition**

However, $f(x) = e^{i2\pi x} \notin D$. Indeed, we cannot enlarge $D$ to include this vector - otherwise we <u>would</u> contradict the proposition. It is false that

$(Qf,Pf) - (Pf,Qf) = i\,(f,f)$ for this particular $f$.

More generally, if $Q,P,D$ satisfy the weak Heisenberg relation, then $D$ cannot contain any eigenvectors of $Q$ or $P$.

We have seen that, in general, we do not have uniqueness. However, under extra conditions this can be proved.

3.1.6. Theorem (Dixmier (1958))

Let $Q,P$ be closed symmetric operators on a separable Hilbert space. Then $(Q,P)$ is unitarily equivalent to a direct sum of Schrödinger representations of the Heinsenberg relations if and only if there is dense domain $D \subset D(Q) \cap D(P)$ such that

(i) $PD \subset D$, $QD \subset D$,

(ii) $(P^2 + Q^2) \upharpoonright D$ is essentially self-adjoint

(iii) $QP - PQ = i$ on $D$

These conditions ensure that $P$ and $Q$ are self-adjoint and $P \upharpoonright D$, $Q \upharpoonright D$ are essentially self-adjoint. A posteriori, (ii) says that the number operator for the harmonic oscillator is e.s.a.

3.1.7. Theorem (Tillman (1963,1964))

Let $Q,P$ be closed symmetric operators on a separable Hilbert space with $D(Q) \cap D(P)$ dense such that

(i) $(Qf,Pg) - (Pf,Qg) = i(f,g)$, all $f,g \in D(Q) \cap D(P)$,

(ii) $(Q + iP)^* = (Q - iP)$

Then P,Q are self-adjoint and equivalent to a direct sum of Schrödinger representations.

A posteriori, (ii) says that the harmonic oscillator creation operator is the adjoint of the annihilation operator.

For proofs of 3.1.6. and 3.1.7, we refer to the original papers (see also Putnam (1967)), and for further discussion see Emch (1972).

## 3.2. Von Neumann's Uniqueness Theorem

The previons paragraph should have sufficed to indicate the subtleties involved with the Heinsenberg relation. We shall recast these into a more convenient form.

We have seen that, formally, $[Q,P] = i$ implies $[Q^n,P] = in\,Q^{n-1}$, and so

$$\left[\frac{Q^n}{n!}, P\right] = i\,\frac{Q^{n-1}}{(n-1)!}$$

Summing over n gives $[e^{iaQ}, P] = -ae^{iaQ}$,

i.e. $Pe^{iaQ} = e^{iaQ}(P + a)$.

Hence $P^n e^{iaQ} = e^{iaQ}(P + a)^n$ and so

$$(ibP)^n \; e^{iaQ} = e^{iaQ} \; (ib\,(P + A)\,)^n$$

Thus $e^{ibP} \; e^{iaQ} = e^{iaQ} \; e^{ib(P+a)} = e^{iaQ} \; e^{ibP} \; e^{iab}$

We have shown that the Heisenberg relations imply, formally, that $e^{ibP} \; e^{iaQ} = e^{iaQ} \; e^{ibP} \; e^{iab}$.

Differentiating w.r.t. a and b, setting a=b=0, we see that these relations are formally equivalent.
The relation $e^{ibP} \; e^{iaQ} = e^{iaQ} \; e^{ibP} \; e^{iab}$ is called the Weyl relation. Putting the Heisenberg relations into this form is technically very convenient because we now only need to consider bounded, in fact, unitary operators. Let us formalize this.

3.2.1. <u>Definition</u> A representation of the Weyl relation (for one degree of freedom) is a pair of maps $s \longrightarrow U(s)$, $t \longrightarrow V(t)$ from $\mathbb{R}$ into unitary operators on a Hilbert space $\mathcal{H}$ such that

(i) $s \longrightarrow U(s)$ and $t \rightarrow V(t)$ are strongly continuons representations of $\mathbb{R}$.

(ii) $U(s)\,V(t) = e^{-ist} \; V(t) \; U(s)$, all s,t.

**Remark** – 1. For n degrees of freedom, we would have maps $\underline{s} \longrightarrow U(\underline{s})$, $\underline{t} \longrightarrow V(\underline{t})$ as representations of the group $\mathbb{R}^n$ and satisfying

$$U(\underline{s}) \; V(\underline{t}) = e^{-i\underline{st}} \; V(\underline{t}) \; U(\underline{s})$$

2. We have chosen U(s) to correspond to $e^{isQ}$, and V(t) to $e^{itP}$. This convention is not universal.

3. We have required that $U(s)$ and $V(t)$ be strongly continuous so that we can recover $Q$ and $P$ as their generators by Stones theorem. For unitaries, weak and strong continuity are equivalent.

3.2.2. <u>Definition</u> A representation $(U,V)$ of the Weyl relation is called <u>irreducible</u> if the only closed subspaces of $\mathcal{H}$ invariant under the $U(s)$'s and $V(t)$'s are $\{0\}$ and $\mathcal{H}$.

3.2.3. <u>Definition</u> The Schrödinger representation of the Weyl relation is that given by: $\mathcal{H} = L^2(\mathbb{R}, dx)$; for $f \in \mathcal{H}$ we set

$(U(s)f)(x) = e^{isx} f(x)$, $(V(t)f)(x) = f(x+t)$.

3.2.4. <u>Theorem</u> (von Neumann (1931))

Any representation $(U, V, \mathcal{H})$ of the Weyl relation is equivalent to a direct sum of irreducible representations.

<u>Proof</u> We set $W(s,t) = \exp\left(\frac{-ist}{2}\right) V(t)\, U(s)$, $s,t \in \mathbb{R}$.

Then it is easy see that

$$W(s,t)\, W(s',t') = \left[\exp \frac{i}{2}(ts' - t's)\right] W(s+s', t+t')$$

Putting $s = -s'$, $t = -t'$, and using $W(0,0) = \mathbb{1}$, we get

$$W(-s,-t) = W(s,t)^* = W(s,t)^{-1}, \quad s,t \in \mathbb{R}.$$

The strong continuinty of U and V implies that W is jointly continuons. Thus, for any $\rho \in \mathcal{S}(\mathbb{R}^2)$, say, we can define

$$A_\rho = \int \rho(s,t) \; W(s,t) \, ds \, dt$$

as a strong Riemann integral. We choose $\rho(s,t) = \exp - \frac{1}{4}(s^2+t^2)$,

and let us write A for $A_\rho$.

Clearly $A = A^*$ . We claim that A is not the zero operator. Suppose the contrary. Then

$$W(-s', -t') \; A \; W(s', t') = 0$$

$$\Rightarrow \int e^{-\frac{1}{4}(s^2+t^2)} \; W(-s', -t') \, W(s,t) \, W(s',t') \, ds \, dt = 0$$

$$\Rightarrow \int e^{-\frac{1}{4}(s^2+t^2)} \; e^{i(ts' - t's)} \; W(s,t) \, ds \, dt = 0$$

$$\Rightarrow \int e^{-\frac{1}{4}(s^2+t^2)} \; e^{its'} \, e^{-it's} \; (f, W(s,t) \, g) \, ds \, dt = 0$$

for all $f, g \in \mathcal{H}$ .

Hence $F(-t', s') = 0$ for all $s'$, $t'$, where $F(a,b)$ is the Fourier transform of $\exp(-\frac{1}{4}(s^2+t^2)) \times (f, W(s,t)g)$. But the only $L^2$-function with zero Fourier transform is zero; Thus $(f, W(s,t)\, g) = 0$ all s,t, and so $W(s,t) = 0$, which is impossible. We conclude that $A \neq 0$, as claimed.

A calculation with Gaussian integrals gives

$$A \;\; W(s,t) \;\; A = 2\pi \;\; A \exp \frac{-1}{4}(s^2+t^2)$$

Setting $s = t = 0$, we have $A^2 = 2\pi A$, or $E = \frac{1}{2\pi} A$ is a projection.

Let $\mathcal{M} = \operatorname{ran} E$. $\mathcal{M} \neq \{0\}$ since $A \neq 0$.

Let $\{f_\alpha\}$ be an orthonormal basis of $\mathcal{M}$ and let $\mathcal{H}_\alpha$ be the closed subspace of $\mathcal{H}$ spanned by vectors of the form $W(s,t) f_\alpha$.

Suppose $\alpha \neq \beta$. Then

$$
\begin{aligned}
(W(s,t)f_\alpha, W(s',t') f_\beta) &= (W(s,t) E f_\alpha, W(s',t')E f_\beta) \\
&= (f_\alpha, E W(-s,-t) W(s',t') E f_\beta) \\
&= c(f_\alpha, E W(s'',t'') E f_\beta) \\
&= c'(f_\alpha, Ef_\beta) \text{ since } E W E = \text{const}.E \\
&= c'(f_\alpha, f_\beta) \\
&= 0
\end{aligned}
$$

It follows that $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ are orthogonal.

We claim that $\bigoplus_\alpha \mathcal{H}_\alpha = \mathcal{H}$. To see this, let $\mathcal{H}' = \mathcal{H} \ominus \bigoplus_\alpha \mathcal{H}_\alpha$. Since $W : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$, we see that $W : \mathcal{H}' \to \mathcal{H}'$. Let $W' = W \restriction \mathcal{H}'$. As above, we can define $E'$ in terms of $W'$ and conclude that $E' \neq 0$. Let $f' \in \operatorname{ran} E'$. Then, since $W'$ is a restriction of $W$, we have that $E'$ is a restriction of $E$. Thus $Ef' = f'$. This contradicts $f' \in \mathcal{H}'$ which is orthogonal to $\bigoplus_\alpha \mathcal{H}_\alpha \supset \mathcal{M}$. Hence $\mathcal{H}' = \{0\}$.

Since $W_\alpha = W \restriction \mathcal{H}_\alpha$ gives rise to a representation of the Weyl relation on $\mathcal{H}_\alpha$, the proof is complete if we can show that $W_\alpha$ acts irreducibly on $\mathcal{H}_\alpha$ : This follows if we can show that any T in $\mathcal{B}(\mathcal{H}_\alpha)$ which commutes with every $W_\alpha$ (s,t) is a multiple of the identity But if T commutes with $W_\alpha$ , T also commutes with $E_\alpha$ . Hence $Tf_\alpha$ satisfies $E_\alpha \, Tf_\alpha = T f_\alpha$ . But $E_\alpha$ has a one dimensional range, and so $T_{f\alpha} = \lambda f_\alpha$ , some $\lambda$ . Since $\mathcal{H}_\alpha$ is generated by $W_\alpha f_\alpha$, and T commutes with $W_\alpha$ , it is clear that $T = \lambda \mathbb{1}_\alpha$ .

QED.

3.2.5. <u>Corollary</u> Let $(U,V,\mathcal{H})$ and $(U',V',\mathcal{H}')$ be two irreducibe representations of the Weyl relation. Then they are unitarily equivalent.

<u>Proof</u> As in the theorem, we construct E and $\mathcal{M}$ = ran E from the U and V on $\mathcal{H}$ . By the irreducibility, we conclude that $\mathcal{M}$ is one-dimensional, and if f is the normalized vector in $\mathcal{M}$, $\mathcal{H}$ is spanned by vectors of the form $W(s,t)f$, $s,t \in \mathbb{R}$ .

In the same way, $\mathcal{H}'$ is spanned by the $W'(s,t)f'$.

Let $g = \sum_{i=1}^{N} a_i \, W(s_i,t_i)f$ and define

$$Ig = \sum a_i \, W'(s_i,t_i) \, f'\text{. Then}$$

$$\begin{aligned}\| Ig \|^2 &= \sum \bar{a}_i \, a_j \, (W'(s_i,t_i)f', W'(s_j,t_j)f') \\ &= \sum \bar{a}_i \, a_j (f', E' W'(-s_i, -t_i) \, W(s_j,t_j)E'f) \\ &= \sum \bar{a}_i \, a_j \, C_{ij} \, (f',f')\end{aligned}$$

where $C_{ij}$ dependes only on $S_i,t_i,s_j,t_j$.

$$= \overline{a_i} \quad a_j \, C_{ij}$$
$$= \| g \|^2$$

Thus $I : \mathcal{H} \to \mathcal{H}'$ is isometric with a dense domain and range, and so extends to a unitary from $\mathcal{H}$ onto $\mathcal{H}'$.

Moreover,

$$I \quad W(s,t) g = W'(s,t) \sum a_i \, W'(s_i, t_i) \, f'$$
$$= W'(s,t) \quad Ig$$

i.e. $W'(s,t) = I \, W(s,t) \, I^{-1}$, and so $(U, V, \mathcal{H})$ and $(U', V', \mathcal{H}')$ are unitarily equivalent.

QED.

3.2.6. <u>Theorem</u> (Von Neumann Uniqueness Theorem)

Let $(U, V, \mathcal{H})$ be a representation of the Weyl relation. Then $(U, V, \mathcal{H})$ is equivalent to a direct sum of copies of the Schrődinger representation.

<u>Proof</u> By 3.1.4 and 3.1.5, we need only show that the Schrődinger representation is irreducible. As in 3.1.4, this follows if we can show that $\mathcal{M}_o = \operatorname{ran} E_o$ is one-dimensional, where $E_o$ is contructed from $U_o$ and $V_o$, the Schrődinger operators of 3.2.3.

Let $g \in L^2(\mathbb{R}, dx)$. Then $(W(s,t)g)(x) =$

$$= e^{is(x + \frac{1}{2}t)} \qquad g(x + t).$$

Thus

$$(A_o \quad g) \qquad (x) = \int e^{-\frac{1}{4}(s^2 + t^2)} \, e^{is(x + \frac{1}{2}t)} \, g(x+t) \, ds \, dt$$

$$= \sqrt{4\pi} \int e^{-\frac{1}{2}(x^2 + t^2)} \quad g(t)\ dt$$

$$= e^{-\frac{1}{2}x^2} \quad \sqrt{4\pi} \int e^{-\frac{1}{2}t^2} \quad g(t)\ dt$$

That is,

$$F_0\ g(x) = \frac{e^{-\frac{1}{2}x^2}}{\pi^{1/4}} \int \frac{e^{-\frac{1}{2}t^2}}{\pi^{1/4}} \quad g(t)\ dt.$$

$F_0$ is the projection onto the vector $\pi^{-1/4} \quad \exp \frac{-1}{2} x^2$ in $L^2$ ( $\mathbb{R}$ , dx) , i.e. ran $F_0$ is one-dimensional.

OED.

<u>Remark</u> We see that $E_0$ is the projection onto the vacuum or " no - mode" state of the harmonic oscilator.

3.2.7. <u>Corollary</u>: An irreducible representation of the Weyl relation is necessarily on a separable Hilbert space.

We can reformulate these results for a finite number of degrees of freedom. The only difference is that one must evaluate n - dimensional Gaussian integrals.

We have seen that, in contrast with the Heisenberg relation, the Weyl relation has an essentially unique solution - the Schrödinger representation.

To see what happens if we relax our requirements, consider the following :

$$\mathcal{H} = \{ f(x),\ x \in \mathbb{R} \mid \lim_{a \to \infty} \frac{1}{2a} \int_{-a}^{a} |f(x)|^2 dx < \infty \}$$

$\mathcal{H}$ has inner product

$$< f,g > = \lim \frac{1}{2a} \int_{-a}^{a} \bar{f}(x)\, g(x)\, dx$$

and defines a Hilbert space.

Define: $U(s)\, f(x) = e^{isx} f(x), \quad f \in \mathcal{H},$

$\qquad V(t)\, g(x) = g(x+t), \quad g \in \mathcal{H}.$

Evidently U and V are unitary and satisfy the Weyl relation $U(s)\, V(t) = e^{-ist}\, V(t)\, U(s)$. What are the generators Q and P ? They do not exist !

Indeed, U(s) and V(t) are <u>not</u> continuons. To see this, we compute $\|(U(s) - 1\!\!1)\, f\|^2$ for $f(x) = e^{ix} \in \mathcal{H}$. We find $\| U(s)f - f \|^2 = 2$ if $s \neq 0$, otherwise we have zero. Similarly, V(t) is not continuons. Therefore U(s) and V(t) cannot be written as the exponentials of self-adjoint operator on $\mathcal{H}$.

We also note that $\{ e^{i\lambda x} \mid \lambda \in \mathbb{R} \}$ is an uncountable collection of pairwise orthogonal vectors in $\mathcal{H}$, i.e. $\mathcal{H}$ is non-separable.

### 3.3. Infinitely-many degrees of freedom

We want to generalize the Weyl relation to allow for an infinite number of degrees of freedom.
We will not consider this in generality but will only consider representations over $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$.

### 3.3.1. Definition :

A representation of the canonical commutation relations (CCR) in Weyl form, over $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$, is a pair of maps $f \to U(f)$, $g \to V(g)$ from $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$ into unitary operators on a Hilbert space $\mathcal{H}$ such that

(i) $U(f_1)\, U(f_2) = U(f_1 + f_2)$, $V(g_1)\, V(g_2) = V(g_1 + g_2)$,

(ii) $s \to U(sf)$, $t \to V(tg)$ are strongly continuons for fixed $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$,

(iii) $U(f)\, V(g) = e^{-i(f,g)}\, V(g)\, U(f)$ where

$$(f,g) = \int f(x)\, g(x)\, d^n x.$$

The continuity assumption allows us to recover the generators of $U(sf)$ and $V(tg)$ which will satisfy the Heisenberg relations on a suitable domain.

<u>Example</u> Let $\mathcal{F}$ be the Foch space over $L^2(\mathbb{R}^n)$, and let $\Phi(f)$, $\Pi(g)$ be the operators

$$\Phi(f) = \frac{1}{\sqrt{2}}\,(a^*(f) + a(f)),\quad \Pi(g) = \frac{i}{\sqrt{2}}\,(a^*(g) - a(g))$$

which are essentially self-adjoint on $D_0$, the set of finite-particle vectors in $\mathcal{F}$. (For a definition of Fock space and the creation and annihilation operators see, for example, Emch (1972) or Hepp (1969).

Then $U(f) = \exp i\,\overline{\Phi(f)}$, $V(g) = \exp i\,\overline{\Pi(g)}$ define a representation of the CCR over $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$.
This is called the Fock representation.

It is natural to ask again whether there is only one (up to equivalence) representation. The answer in this case is no. That is, von Neumann's theorem does not extend to the case of infinitely-many degrees of freedom. We shall see this by constructing an uncountable number of inequivalent representations.

Let $\Phi(f)$ and $\Pi(g)$ be as above, and define

$$\Phi_a(f) = \Phi(f) + a\int f(x)\,dx, \quad a \in \mathbb{R},$$
$$\Pi_a(g) = \Pi(g)$$

Then $U_a(f) = \exp \overline{i\,\Phi_a(f)} = U(f)\,e^{ia\int f(x)dx}$ and $V_a(g) = V(g)$ define a representation of the CCR in the sense of 3.3.1.

### 3.3.2. Proposition

The representations $(U_a, V_a, \mathfrak{F})$, $(U_b, V_b, \mathfrak{F})$ are unitarily equivalent if and only if $a=b$.

**Proof**

If $a=b$ there is nothing to prove.

Suppose $a \neq b$, but we have equivalence.
That is, there is a unitary $T : \mathfrak{F} \to \mathfrak{F}$ such that

$T\, U_a(f)\, T^* = U_b(f)$, and $T\, V_a(g)\, T^* = V_b(g)$ for all $f$, $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$. By definition, we have

$$T\, U(f) T^*\; e^{ia\int f dx} = U(f)\; e^{ib\int f dx},$$

i.e. $$T\, U(f) T^* = U(f) \exp i(b-a) \int f(x)\, dx$$

Let $f_n \in \mathcal{S}$ such that $\| f_n \|_{L^2} \to 0$, and

$$\int f_n(x)\, dx \to \frac{\pi}{b-a}, \text{ as } n \to \infty$$

Then it is easy to see that $\Phi(f_n) \to 0$ strongly on $D_o$, and so (since $D_o$ is a domain of entire vectors for $\Phi(f_n)$) $\exp i\Phi(f_n)$ converges, on $D_o$, strongly to $\mathbb{1}$. This implies that $U(f_n) \to \mathbb{1}$ strongly on $\mathcal{F}$. Hence

$$T\, U(f_n) T^* \to T T^* = \mathbb{1} \quad \text{strongly}.$$

On the other hand, $U(f_n) \exp i(b-a) \int f_n\, dx$ converges strongly to $e^{i\pi} \mathbb{1} = -\mathbb{1}$. This is a contradiction. Hence $T$ does not exist, and the representations are inequivalent.

QED.

We have just explicitly constructed an uncountable number of inequivalent representations of the Weyl CCR. We remark here, without proof, that these are all irreducible.

At this point we might also mention that the relativistic time-zero free field of mass m, $\phi(f)$, and its conjugate momentum, $\pi(g)$, define a representation of the

Weyl CCR. This representation is inequivalent to the Fock representation above. This can be seen by showing that $\phi(f) + i\pi(f)$ annihilates no vector whereas $\Phi(f) + i\Pi(f)$ does.

We also note that each value of the mass gives an inequivalent representation. This is to be expected since each carries a representation of the Poincaré group with mass $m$. The energy operator has spectrum $\{0\} \cup \{m\} \cup [2m,\infty)$. Equivalence for different masses would imply the same energy spectrum, which clearly is not the case.

The Weyl relations can be generalized in many directions. We mention only the formulation of Mackey (1949) in which the U's and V's are defined over an abelian group and its dual, in which case there is a uniqueness theorem, and the formulation of Segal (1963,1967) in which one considers a symplectic form over a vector space and which yields a beautiful procedure for the quantization of free fields.

For further results and details we refer to Emch (1972) and the bibliography therein.

## 4 - The algebraic Approach to Quantum Theory

Rather than treat a theory in terms of fields and commutation relations, we want to consider a theory of observables. Of course, if we have fields, then these will define observables, but we want to consider the observables per se. In the conventional treatment of von Neumann, the observables are represented by the self-adjoint operators on a Hilbert space. It is this we wish to generalize. We should emphasize that our observables are mathematical or "ideal" observables. We do not pretend to consider the act of observation or the actual measurement of observables. Indeed this is a somewhat controversial subject. We refer the interested reader to the Varenna lectures of 1970 (D'Espagnat (1971)).

## 4.1 Segal's Postulates

We wish to describe a "system". This is supposed to consist of a collection of "observables", and the system is supposed to be capable of being in certain states. We shall consider the observables as being given, and we can then define the "state of the system" as the knowledge of the expected values of the observables. That is to say, a state is an assignment of an expected value to each observable.

If $A$ is an observable, then, for any $a \in \mathbb{R}$, we suppose $a \cdot A$ to be an observable - it has an expected value in any state equal to a times that value which $A$ has in the same state. In the same way, we assume that $A + B$ is an observables whenever $A$ and $B$ are.

$A^2$ is supposed to be that observable whose possible values are equal to the square of those of $A$.

Furthermore, it is simpler if we suppose our observables to be bounded - that is, they can only assume values from a bounded set of real number (depending on the observable, of course). This is no restriction inasmuch as unbounded observables can be considered as a limit or a collection of bounded ones.

Following Segal (1947, 1963), we make the following postulate.

4.1.1. <u>Phenomenological Postulate</u> : Algebraic Part.

A physical system is a collection of objects called (bounded) observables, for which operations of multiplication by a real number, squaring and addition are defined, and satisfy the usual assumptions for a linear space.

As remarked above, a state of the system assigns to each observable a real number, called the " expectation of the observable in the state". We define a state to <u>be</u> this assignment. We expect, intuitively, that a state $E$ should have the following properties :

1. Linearity, $E(A+B) = E(A) + E(B)$

   $E(aA) = a\,E(A)$

   for $A, B$ bounded observables, $a \in \mathbb{K}$.

2. Positivity, $E(A^2) \geq 0$

3. Boundedness, $|E(A)| \leq C_A$,

   where $C_A$ is the maximum value that $A$ can have.

Let us consider further this notion of maximum value.

### 4.1.2. <u>Phenomenological Postulate</u>: Analytical Part.

To each observable, A, is assigned a "bound", written $\|A\|$, in such a way that

(i) $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = 0$,

(ii) $\|aA\| = |a|\,\|A\|$ and $\|A+B\| \leq \|A\| + \|B\|$

(iii) $\|A^2\| = \|A\|^2$

The interpretation of $\|A\|$ is $C_A$. Then 4.1.2. is not unreasonable.

Segal makes some more postulates and is then able to recover many physical notions. However, we shall make the following postulate, which is stronger than Segal's, but which appears to be sufficient for all systems considered in practice.

### 4.1.3. <u>Postulate</u>

A physical system corresponds to the self-adjoint elements of a $C^*$ - algebra with identity, with the bound, $\|A\|$, given by the norm of the $C^*$ - algebra.

A few remarks are in order here.

1. It is mathematically convenient to allow an operation of multiplication by complex numbers.

2. The property 4.1.2 (iii) is just the $C^*$ - property for self-adjoint A. However, our postulate is much stronger than simply demanding that the observables form a <u>real</u> $C^*$ - algebra. Indeed, it is not known whether a real $C^*$ - algebra can always be considered as the self-adjoint elements of a complex $C^*$ - algebra. The study of real $C^*$ - algebra is much harder than for complex $C^*$ - algebras.

3. It is convenient to assume that the observables are complete with respect to the norm. If they were not, we could complete the algebra.

4. A $C^*$ - algebra has a product. There is <u>no</u> justification for this assumption. Moreover, the product of self-adjoint elements need not be self-adjoint, so the product is not even defined on the observables.

5. In any event, we have a more general scheme than that of von Neumann. In fact, we shall see that a generalization of von Neumann's sheme is necessary according to the theory of superselection rules.

6. For Segal's original postulates, we refer to Segal (1947,1963) and Emch (1972).

## 4.2. Exact Values of Observables

**4.2.1. Definition** A state of a system, $\mathfrak{A}$, is a state on the $C^*$ - algebra, $\mathfrak{A}$. That is, a positive linear functional with norm one.

The set of states is denoted by $\mathfrak{A}^{*+}$.

We recall that a state is a mixture if it is a convex combination of two different states. A state is pure if it is not a mixture.

As an example, consider the $C^*$ - algebra (without identity) $\mathcal{C}(\mathcal{H})$ of all compact operators on $\mathcal{H}$. Then every positive continuons functional, $\omega$, on $\mathcal{C}(\mathcal{H})$ has the form

$$\omega(A) = \mathrm{Tr}(DA)$$

for some $D \geq 0$, $D \in \mathcal{C}(\mathcal{H})$, $\mathrm{Tr}\, D < \infty$. That is, all states are given by density matrices. The pure states are the vector states, i.e. those of the form

$$\omega(A) = (\zeta, A\zeta), \quad \zeta \in \mathcal{H}.$$

If $\mathfrak{A}$ is not equal to $\mathcal{C}(\mathcal{H})$ then there will be states which are not given by desity matrices and pure states not given by vectors states. (See Segal (1947) for an example). So we see that this Scheme is more general than von Neumann's in which the states are assumed to be given by density matrices.

**4.2.2. Definition** Let $A = A^* \in \mathfrak{A}$ be an observable, and let $\omega \in \mathfrak{A}^{*+}$ be a state. The variance of

$A$ in $\omega$ is defined to be

$$\mathcal{V}_\omega(A) = \omega(A^2) - \omega(A)^2$$

We say that $A$ has an exact value in $\omega$ if $\mathcal{V}_\omega(A) = 0$, the exact value being $\omega(A)$.

We call the set of exact values of an observables its physical spectrum.

4.2.3. <u>Theorem</u>

The physical spectrum of an observable $A$ is equal to $\sigma(A)$, the spectrum of $A$.

<u>Proof</u> Let $\mathcal{A}$ be the $C^*$- algebra generated by the observable $A$. Then $\mathcal{A}$ is commutative and so is isometrically isomorphic to the uniform algebra $C(K)$ over the compact Hausdorff space $K = \mathrm{Sp}\,\mathcal{A}$

Let $\omega \in \mathfrak{A}^{*+}$ , and suppose $\mathcal{V}_\omega(A) = 0$. Then if $\mu_\omega$ is the measure on $K$ induced by $\omega$ , we have

$$\mathcal{V}_\omega(A) = 0 = \omega((A - \omega(A)\mathbb{1})^2)$$
$$= \int_K |\hat{A}(\kappa) - \omega(A)|^2 \, d\mu_\omega(\kappa)$$

where $\hat{A}$ is the Gelfand transform of $A$.

It follows that $\hat{A}(\kappa) = \omega(A)$, $\mu_\omega$ - almost every where. In particular, there is $\kappa \in K$ such that $\hat{A}(\kappa) = \omega(A)$. But $\sigma(A) = \mathrm{ran}\,\hat{A}$, i.e. $\omega(A) \in \sigma(A)$.

Conversely, suppose $\lambda \in \sigma(A)$. Then $\lambda = \hat{A}(\kappa)$ for some $\kappa \in K$. For any $B \in \mathcal{A}$ we define

$$\omega_\kappa(B) = \hat{B}(\kappa).$$

It is clear that $\omega_\kappa$ defines a state on $\mathcal{A}$, and $\vartheta_{\omega_\kappa}(A) = 0$. Any state on $\mathcal{A}$ can be extended to a state on $\mathfrak{A}$, and so $\lambda = \omega_\kappa(A)$ belongs to the physical spectrum.

QED.

4.2.4. <u>Corollary</u> With the notation of the theorem, let $\omega$ (A) be exact. Then $\omega$ is pure on $\mathcal{A}$.

<u>Proof</u>

As in the theorem, $\hat{A}(\kappa) = \omega(A)$, $\mu_\omega$-almost everywhere. Suppose there were $\kappa_1, \kappa_2 \in K$ such that $\hat{A}(\kappa_1) = \omega(A) = \hat{A}(\kappa_2)$. Then for any polynomial $\mathcal{P}$, $\mathcal{P}(\hat{A})(\kappa_1) = \mathcal{P}(\hat{A})(\kappa_2)$. But polynomials in $\hat{A}$ are dense in C(K) which contains functions which have different values at $\kappa_1$ and $\kappa_2$. Thus there in only one $\kappa \in K$ with $\hat{A}(\kappa) = \omega(A)$. In other words, the singleton $\{\kappa\}$ has $\mu_\omega$ – measure one. Hence $\mu_\omega$ is a "delta-function" at $\kappa$ and so $\omega$ is pure on $\mathcal{A}$.

QED.

4.2.5. <u>Corollary</u> The physical spectrum of A is equal to the set $\{\omega(A) \mid \omega \text{ pure on } \mathcal{A}\}$.

Obvions from 4.2.4.

QED.

4.2.6. <u>Corollary</u> Let $\mathcal{B}$ be a commutative $C^*$-algebra containing A . Then the physical spectrum of A is equal to the set $\{\omega(A) \mid \omega \text{ pure on } \mathcal{B}\}$.

<u>Proof</u>

If $\omega$ is pure on $\mathcal{B}$, then $\omega$ is a character on $\mathcal{B}$, and so is a character on $\mathcal{A}$. Therefore $\omega(A)$ is an exact value. Conversely, by 4.2.4, if $\omega(A)$ is exact, then $\omega$ is pure on $\mathcal{A}$. But a pure state on $\mathcal{A}$ can always be extended to a pure state on $\mathcal{B}$.

QED.

4.3. <u>Simultaneous Measurability</u> (Segal (1947), Emch (1972)).

4.3.1. <u>Definition</u> Let $\Gamma$ be a collection of observables and $\omega$ a state. We say that $\omega$ is dispersion free on $\Gamma$ if $\vartheta_\omega(A) = 0$ for all A in $\Gamma$.

We have seen that if $\vartheta_\omega(A) = 0$, then $\omega$ is dispersion free on $\mathcal{A}$.

Suppose that we have two observables. To say that they are different is to say that we can find a state in which they differ. We can say that a collection of states specifies the system if it distinguishes between the observables.

4.3.2. <u>Definition</u> Let $\mathcal{S}$ be a family of states on $\mathfrak{A}$, and $\mathfrak{B} \subset \mathfrak{A}$ a subset. $\mathcal{S}$ is said to be separating for $\mathfrak{B}$ if, for $A, B \in \mathfrak{B}$, $\omega(A) = \omega(B)$ for all $\omega \in \mathcal{S}$ implies that $A=B$.

Suppose A and B are two observables. What does it mean to say that they are simultaneonsly observable? It is natural to require that their exact values can be simultaneously realized : i.e. We can find states $\omega$ such that $\vartheta_\omega(A) = \vartheta_\omega(B) = 0$ . This should hold for sufficienthy many states. Furthermore, if A and B are simultaneonsly measurable we would expect the same to be true for $A^2$, $(A+B)^2$, etc.

4.3.3. <u>Definition</u> Let $\Gamma$ be a collection of observables. We say that $\Gamma$ is a collection of simultaneonsly measurable observables if and only if there exists a set $\mathcal{S}$ of states separating for and dispersion free on $\mathcal{A}_{\mathbb{R}}(\Gamma)$, the hermitian elements of $\mathcal{A}(\Gamma)$, the $C^*$-algebra generated by $\Gamma$.

4.3.4. __Theorem__

Let $\Gamma$ be a collection of observables. Then $\Gamma$ is simultaneonsly measurable if and only if $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is commutative.

__Proof__

Suppose $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is commutative. Then $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is isometrically isomorphic to $C_{\mathbb{R}}(K)$. For each $\kappa \in K$, we define $\omega_\kappa : \mathcal{A}(\Gamma) \to \mathbb{C}$ by $\omega_\kappa(A) = \hat{A}(\kappa)$, where $\hat{A}$ is the Gelfand transform of A. Clearly $\omega_\kappa$ is a state on $\mathcal{A}(\Gamma)$, and so has an extension to $\mathfrak{A}$. Evidently, $\omega_\kappa$ is dispersion free on $\mathcal{A}_{\mathbb{R}}(\Gamma)$. Moreover, the set $\{\omega_\kappa \mid \kappa \in K\}$ is separating for $\mathcal{A}_{\mathbb{R}}(\Gamma)$.

Conversely, suppose that $\mathcal{S}$ is a dispersion free and separating family for $\mathcal{A}_{\mathbb{R}}(\Gamma)$.

Let $A, B \in \mathcal{A}_{\mathbb{R}}(\Gamma)$, and define $A \circ B \in \mathcal{A}_{\mathbb{R}}(\Gamma)$ by

$$A \circ B = \frac{1}{4}\left[(A+B)^2 - (A-B)^2\right]$$

Then, for $\omega \in \mathcal{S}$, we have

$$\begin{aligned} \omega(A \circ B) &= \frac{1}{4}\,\omega\left((A+B)^2 - (A-B)^2\right) \\ &= \frac{1}{4}\left\{\omega((A+B)^2) - \omega((A-B)^2)\right\} \\ &= \frac{1}{4}\left\{\omega(A+B)^2 - \omega(A-B)^2\right\} \end{aligned}$$

since $\omega$ is dispersion free on $\mathcal{A}_{\mathbb{R}}(\Gamma)$

$$= \omega(A)\ \omega(B)$$

Hence, for $A,B,C \in \mathcal{A}_{\mathbb{R}}(\Gamma)$, we have

$$\omega((A \circ B) \circ C) = \omega(A \circ B)\ \omega(C) = \omega(A)\,\omega(B)\,\omega(C)$$

$$= \omega(A)\ \omega(B \circ C) = \omega(A \circ (B \circ C))$$

This holds for all $\omega \in \mathcal{S}$, which is separating, so

$$(A \circ B) \circ C = A \circ (B \circ C),$$

i.e. " $\circ$ " is an associative product on $\mathcal{A}_{\mathbb{R}}(\Gamma)$.

Now, without loss of generality, we may suppose that $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is an algebra of operators on a Hilbert space. It is easy to see that by taking strong limits " $\circ$ " is associative on $\overline{\mathcal{A}_{\mathbb{R}}(\Gamma)}^{s}$, the strong closure of $\mathcal{A}_{\mathbb{R}}(\Gamma)$. But, by the spectral theorem, $\overline{\mathcal{A}_{\mathbb{R}}(\Gamma)}^{s}$ contains the spectral projections of $\mathcal{A}_{\mathbb{R}}(\Gamma)$, and the associativity of " $\circ$ " implies that these projections commute with each other. Hence by the spectral theorem, $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is commutative.

QED.

<u>Remark</u> One might think that if A and B are simultaneonsly observable, then whenever A has an exact value, so does B. However, this need not be the case. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and set $\omega(\cdot) = \frac{1}{2}(e_1, (\cdot)\, e_1) + \frac{1}{2}(e_2, (\cdot)\, e_2)$.

Then $\omega(A) = 1$ is an exact value, but $\omega(B) = 3/2$ is not.

What is true, however, is the following.

**4.3.5. Theorem**

Let $\Gamma$ be a set of simultaneonsly measurable observables. Suppose $A \in \Gamma$ and $\omega(A)$ is an exact value. Then there exists $\rho$ such that $\rho(A) = \omega(A)$, and $\rho(B)$ is an exact value for all $B \in \Gamma$.

**Proof**

By 4.3.4, $\mathcal{A}(\Gamma)$ is commutative. By 4.2.4, $\omega$ is pure on $\mathcal{A}$, the C*-algebra generated by A, so can be extended to a pure state $\rho$ on $\mathcal{A}(\Gamma)$

By 4.2.6, $\rho(B)$ is an exact value for any $B \in \Gamma$.

QED.

## 4.4. Probabilistic Description

It is now straightforward to introduce the notion of joint probability distributions for simultaneous observations. Indeed, let $\Gamma$ be a set of simultaneously measurable observables, and let $\mathcal{A}(\Gamma)$ be the $C^*$-algebra they generate. Let $\omega$ be any state of the system, $\mathcal{O}$.

By 4.3.4, $\mathcal{A}(\Gamma)$ is commutative and so is isomorphic to $C(K)$, $K$ compact. By restriction, $\omega$ defines a state on $C(K)$ which can be written (by the Riesz - Markow theorem) as

$$\omega(A) = \int_K A(K) \quad d\mu_\omega(K), \quad A \in \mathcal{A}(\Gamma),$$

for some regular probability measure $\mu_\omega$ on $K$.

Let $A_1, \ldots\ldots, A_n \in \Gamma$, and let $I_1 \ldots\ldots\ldots, I_n$ be Borel sets on the real line. We define the joint probability distribution of the observables $A_1, \ldots, A_n$ in the state $\omega$ to be

$$\mathbb{P}_{A_1, \ldots\ldots, A_n; \omega}(I_1, \ldots, I_n) = \mu_\omega(A_1^{-1}(I_1) \cap \ldots \cap A_n^{-1}(I_n)).$$

This is the probability that $A_1$ has values in $I_1$, $A_2$ in $I_2$, etc, in the state $\omega$.

For the case of one observable, $A$, $\mathbb{P}_{A;\omega}(I)$ is just the probability that $A$ has values in $I$ in the state $\omega$. If we write $\mathbb{P}_{A;\omega}(\lambda)$ for the case $I = (-\infty, \lambda]$, one can show that the expected value of $A$ in $\omega$ is given by

$$\mathbb{E}_\omega(A) = \int \lambda \, d\mathbb{P}_{A;\omega}(\lambda) = \omega(A)$$

as we would expect.

We note that the probability distribution $\mathbb{P}_{A;\omega}$ is indepedent of any realization of $A$ as a function. Indeed, the characteristic function of $\mathbb{P}_{A;\omega}$ is given by $\omega(e^{itA})$. In the same way, the joint distribution is uniquely determined.

# 5 - Local Quantum Theory

We shall present here the axiomatic scheme of Haag and Kastler (1964), which is concerned with a relativistic quantum theory of observables.

## 5.1 The Haag-Kastler axioms

We shall, as in the last Chapter, assume that our observables generate a $C^*$-algebra, $\mathfrak{A}$ . The first axiom represents the idea that each region of space-time gives rise to a family of obervables.

### 5.1.1 Postulate

To each region $\mathcal{O}$ in Minkovoski space, $\mathbb{M}$ , there corresponds a sub $C^*$- algebra, $\mathfrak{A}(\mathcal{O})$ , of $\mathfrak{A}$ . Moreover, $\mathfrak{A}$ is generated by the $\mathfrak{A}(\mathcal{O})$ as $\mathcal{O}$ runs over $\mathbb{M}$ .

By definition, a region is a bounded open set in $\mathbb{M}$ (- identified with $\mathbb{R}^4$ ). On physical grounds, such regions could be considered as being too general. For this reason, we may restrict $\mathcal{O}$ to be a double-cone :i.e. the intersection of a backward cone with a forward cone.

Strictly speaking, only the self-adjoint of $\mathfrak{A}$ are observables, but we shall use the word for any element of $\mathfrak{A}$ . Thus $\mathfrak{A}(\mathcal{O})$ is the algebra of observables associated with the region $\mathcal{O}$ . The elements of $\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ are called local observables, whilst those of $\mathfrak{A}$ are called quasi-local.

The next axiom has an obvious interpretation.

5.1.2. **Postulate** (Isotony) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are regions in $\mathbb{M}$ whith $\mathcal{O}_1 \subset \mathcal{O}_2$, then

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2).$$

Einstein's principle of causality states that no physical influence can propoqate faster thas the speed of ligth. That is, observables associated with space-like regions should be simultaneously measurable. We have seen that this is equivalent to saying that they commute.

5.1.3. **Postulate** (Einstein-Causality)

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated regions, then $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ commute : that is, if $A \in \mathfrak{A}(\mathcal{O}_1)$ and $B \in \mathfrak{A}(\mathcal{O}_2)$, then $AB = BA$.

Poincaré covariance of the theory is expressed as follows.

5.1.4. **Postulate** (Poincaré Covariance)

There is a representation $\alpha$ of $\mathcal{P}_+^\uparrow$, the restricted Poincaré group, in Aut. $\mathfrak{A}$, the automorphism group of $\mathfrak{A}$, such that

$$\alpha(L)\,\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\Lambda\mathcal{O} + a)$$

for any region $\mathcal{O}$, and $L = (a,\Lambda) \in \mathcal{P}_+^\uparrow$

The next axiom is technical, and excludes classical field theory ( - in which case $\mathfrak{A}$ would be commutative).

5.1.5. <u>Postulate</u> $\mathfrak{A}$ is primitive. That is, $\mathfrak{A}$ possesses a faithful, irreducible representation.

Suppose that we have a (Wightman) field theory. Then one can consider the set of fields smeared with test-functions with support in some region $\mathcal{O}$. By forming bounded functions of these fields we could define a local algebra $\mathfrak{A}(\mathcal{O})$. (In this way it is not difficult to show that these axioms are obeyed by free fields - with 5.1.3. suitable reformulated for fermi-fields). The point is that there may be many field-theories which lead to the same $\mathfrak{A}(\mathcal{O})$'s (cf. Borchers (1960)). These axioms put the emphasis on the abstract structure of the $\mathfrak{A}(\mathcal{O})$'s. The elements of $\mathfrak{A}$ can be considered as "observable" as the fields in a field theory are "observable" ( - suitable modified for fermi-fields).

We should remark here that the $\mathfrak{A}(\mathcal{O})$ are often taken to be von Neumann algebras (Araki (1963, 1964a, 1946, 1964 c, 1969), Borchers (1967), Haag and Schroer (1962)). One reason for this lies in the difficulty of a satisfactory formulation of the "positivity of energy" within the abstract approach. An attempt was made by Doplicher (1965), but there is an implicit continuity assumption which does not hold in the case of a free bose field. The point is that, by 5.1.4, we have a representation $\alpha(a)$ of Space-time translations, but no energy-momentum operator. The automorphisms $\alpha(a)$ may be implemented in some representations of $\mathfrak{A}$, but not in others. Moreover, even when it is implemented, the generators may not satisfy the spectrum condition of positive energy (see 5.4).

We note here that $\alpha(L)$ can never be an inner automorphism of $\mathfrak{A}$ (Haag and Kastler (1964), Emch (1972)). This reflects the global nature of Poincaré transformations, and the essentially local nature of $\mathfrak{A}$.

For further discussion of the axioms and their intuition, we strongly recommend the lectures of Haag (1966, 1970, 1972). (See also Araki (1969)).

## 5.2. Superselection Rules

Consider a field theory describing fields with spin zero and one-half, say and let $\xi_0$ and $\xi_{1/2}$ be vector states with spin zero and one-half, respectively. Let $\eta$ be the vector state given by the superposition of

$\xi_0$ and $\xi_{1/2}$ ,

$$\eta = \frac{1}{\sqrt{2}} ( \xi_0 + \xi_{1/2} )$$

Under a notation of $2\pi$ , the physics should be unchanged. However, $\eta$ is transformed into

$$\eta \to \eta' = \frac{1}{\sqrt{2}} ( \xi_0 - \xi_{1/2} )$$

To say that a notation of $2\pi$ has no observable effect, is to say that $\eta$ and $\eta'$ describe the same state:

$$( \eta , A \eta ) = ( \eta' , A \eta' )$$

for all observables, A.

This is clearly a restriction on the operators which are supposed to be observable.

Wick, Wightman and Wigner (1952) proposed that the Hilbert space of states, $\mathcal{H}$ ,could be decomposed into a direct sum, $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ such that each $\mathcal{H}_\alpha$ is mapped into itself under the algebra of observables. In the example above, we would decompose $\mathcal{H}$ as $\mathcal{H}_o \oplus \mathcal{H}_e$ , where $\mathcal{H}_o$ is the subspace of states with odd half-integer spin, and $\mathcal{H}_e$ that with even half-integer spin. Then one supposes that $( \xi_o , A \xi_e ) =$ $= 0$ for every observable A , and states $\xi_o \in \mathcal{H}_o$ , $\xi_e \in \mathcal{H}_e$ .

The subspaces $\mathcal{H}_\alpha$ are called superselection sectors, and the statement that the observables have such a direct sum structure is called a superselection rule*.

By a consideration of gauge invariance one is lead to the charge superselection rule. The $\mathcal{H}_\alpha$ will correspond to the subspaces of different charge **. We will consider this in detail for the free charged bose field in the next chapter.

The concept of superselection rule can be "explained" * in the algebraic framework - the sectors correspond to inequivalent representations of the algebra of obsevables, $\mathfrak{A}$ . (We will see this in the next chapter).

We are thus led to a study of the representations of $\mathfrak{A}$ .

* It must be noted that the concept of superselection rule is not universally accepted: see, for example, Mirman (1970) and, however, Wick, Wightman and Wigner (1970).

** It has recenthy been shown by Strocchi and Wightman (1974) that this follows from the usual laws of quantum electrodynamics.

## 5.3. Physical Equivalence

Consider the measurement of the state $\omega$ on $\mathfrak{A}$.

An experiment will correspond to the measurement of a finite number of observables $A_1, \ldots A_n$, with resulting experimental values $p_1, \ldots p_n$, and with some error $\varepsilon$ , say. Then

$$| \omega (A_i) - p_i | < \varepsilon \text{ for } i = 1, \ldots, n.$$

We cannot determine $\omega$ uniquely from this data. Indeed, as far as this experiment is concerned, we can only conclude that the system is in some state $\omega'$ with

$$| \omega' (A_i) - p_i | < \varepsilon$$

Thus

$$| \omega' (A_i) - \omega (A_i) | < 2\varepsilon, \ i=1,\ldots,n$$

We see that an experiment corresponds to a $W^*$ - neighbourhood of the state $\omega$.

Now, associated naturally to any representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$, is the set of states given by convex combinations of vectors states. We could say that two representations are physically equivalent if we cannot distinguish between them experimentally.

**5.3.1. Definition** (Haag and Kastler (1964)).

Two representations $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ of $\mathfrak{A}$ are said to be physically equivalent iff any $W^*$ - neighbourhood of a convex combination of vector states in one representation contains a convex combination of vector states in the other representation.

It turns out (Fell (1960)) that any two faithful representations are physically equivalent.

This is perhaps a justification for emphasizing the abstract approach. However, these representations should correspond to different superselection sectors, so these are all physically equivalent. Does this mean that we should forget about the study of the various sectors ? The answer (Haag (1972)) is that, in principle, we could consider, only the vacuum sector, but, for mathematical convenience, we make idealizations which mean that we must consider many sectors. For example, we make the idealization that outside the "laboratory" there is only the vacuum. Inside the laboratory we may have, say, an overall charge +3. We therefore have to consider the +3 sector. Of course, there is presumably a total charge of -3 outside the laboratory, but we suppose this to be so far away as not to have any effect on experiments within the laboratory. As we have said, we make the idealization that there is only the vacuum outside our laboratory. Again, we recommend the lectures of Haag (1966, 1970, 1972) for a discussion of this point.

## 5.4. Energy and Momentum as Observables

So far, we have not introduced the so-called spectrum condition, and the notion of a vacuum state

### 5.4.1. Postulate (Spectrum Condition)

There exists a state $\omega$ on $\mathfrak{A}$ invariant under space-time translations $\alpha(a)$, $a \in \mathbb{R}^4$, and such that $a \to \omega(A\,\alpha(a)\,B)$ is continuons for all $A, B \in \mathfrak{A}$. If $(\mathcal{H}_\omega, \pi_\omega)$ is the GNS representation of $\mathfrak{A}$ given by $\omega$, and if $U_\omega(a)$ is the unitary operator in $\mathcal{H}_\omega$ implementing $\alpha(a)$, then the joint spectrum of the generators of $U_\omega(a) = e^{iaP}$, should be in the closed forward light-cone,

$$\overline{V_+} = \{ p \in \mathbb{M} \mid (p,p) \geq 0 \ , \ p^0 \geq 0 \} .$$

Such a state $\omega$ is called a vacuum state.

<u>Remark</u> The invariance of $\omega$ guarantees the existence of the $U_\omega(a)$, and the continuity condition guarantees the existence of $P = (P^0, \underline{P})$.

Clearly, if $\Omega$ is the GNS cyclic vector, then $U_\omega(a)\,\Omega = \Omega$, i.e. $P\,\Omega = 0$. This is why $\omega$ is called a vacuum state.

5.4.2. <u>Proposition</u> (Araki (1964 b), Borchers (1966))

Let $\omega$ be a vacuum state on $\mathfrak{A}$, with associated GNS constructs $(\mathcal{H}, \pi, U, \Omega)$. Then $U(a) \in \pi(\mathfrak{A})''$

<u>Proof</u>

Let $A \in \mathfrak{A}$, $X \in \pi(\mathfrak{A})'$. Then we have,

$$\begin{aligned}(\Omega, \pi(A)\,U(a)\,X\,\Omega) &= (\Omega, U(-a)\,\pi(A)\,U(a)\,X\,\Omega)\\ &= (\Omega, \pi(\alpha(-a)A)\,X\,\Omega)\\ &= (\Omega, X\pi(\alpha(-a)A)\,\Omega)\end{aligned}$$

$$= (\Omega, X\, U(-a)\, \pi(A)\Omega).$$

Hence, for any $\rho \in \mathcal{S}(\mathbb{R}^4)$

$$\int \rho(a)\, (\Omega, \pi(A)\, U(a)\, X\, \Omega)\, da$$

$$= \int \rho(a)\, (\Omega, X\, U(-a)\, \pi(A)\, \Omega)\, da$$

By the spectrum condition, the first integral is zero if $\tilde{\rho}$, the Fourier transform of $\rho$, has support outside $\overline{V}_+$, whereas the second is zero if $\tilde{\rho}$ has support outside $\overline{V}_- = -\overline{V}_+$.

It follows that $(\Omega, \pi(A)\, U(a)\, X\Omega)^{\sim}$, considered as a distribution, has support in $\overline{V}_+ \cap \overline{V}_- = \{0\}$. Since $(\Omega, \pi(A)\, U(a)\, X\, \Omega)$ is bounded and continuons in a, we conclude that it is a constant.

Thus, for $S, T \in \mathfrak{A}$,

$$\begin{aligned}(\Omega, \pi(S^* T)\, U(a)\, X\, \Omega) &= (\pi(S)\Omega, \pi(T)\, U(a)\, X\, \Omega) \\ &= (\pi(S)\, \Omega, U(a)\, X\, U(-a)\, \pi(T)\Omega) \\ &= (\Omega, \pi(S^*T)\, X\, \Omega), \text{setting } a = 0 \\ &= (\pi(S)\Omega, X\, \pi(T)\, \Omega)\end{aligned}$$

That is,

$$(\pi(S)\, \Omega, (U(a)\, X\, U(-a) - X)\, \pi(T)\, \Omega) = 0$$

Since $\Omega$ is cyclic, we conclude that

$$U(a)\,X = X\,U(a), \quad \text{all } X \in \pi(\mathfrak{A})'$$

and so $U(a) \in \pi(\mathfrak{A})''$.

QED.

<u>Remark</u> This theorem tells us in which sense energy and momentum can be considered as observables.

5.4.3. <u>Corollary</u> With the notation of the theorem, the following are equivalent.

(i) $\omega$ is extremal invariant

(ii) $U(a)\,\xi = \xi$, all $a$, $\xi \in \mathcal{H}$ implies $\xi = \lambda\,\Omega$, $\lambda \in \mathbb{C}$.

(iii) $\pi(\mathfrak{A})$ is irreducible

<u>Proof</u>

(i) $\Longleftrightarrow$ (iii): We have $U(\mathbb{R}^4) \subset \pi(\mathfrak{A})''$, and so $\pi(\mathfrak{A})' \subset U(\mathbb{R}^4)'$, i.e. $\pi(\mathfrak{A})' \cap U(\mathbb{R}^4)' = \pi(\mathfrak{A})'$

But $\omega$ is extremal invariant iff $\pi(\mathfrak{A})' \cap U(\mathbb{R}^4)' = \mathbb{C}\,\mathbb{1}$.
This holds iff $\pi(\mathfrak{A})$ is irreducible.

(ii) $\Rightarrow$ (iii).

Let $X \in \pi(\mathfrak{A})'$. Then, since $U(a) \in \pi(\mathfrak{A})''$, we have

$$U(a)\,X\,\Omega = X\,U(a)\,\Omega = X\,\Omega, \text{ all } a \in \mathbb{R}^4.$$

Hence $X\,\Omega = \lambda\,\Omega$, some $\lambda \in \mathbb{C}$.

Thus

$$X\,\pi(A)\,\Omega = \pi(A)\,X\,\Omega = \pi(A)\lambda\,\Omega,$$

for all $A \in \mathcal{A}$.

Since $\Omega$ is cyclic, we have $X = \lambda\,\mathbb{1}$, and so $\pi(\mathcal{A})$ is irreducible.

(iii) $\Longrightarrow$ (ii) We will only sketch the proof. For the details see Kastler (1967).

By 5.1.3, it is not difficult to show that

$$B\,\pi(\alpha(a)A) - \pi(\alpha(a)A)B \underset{|\underline{a}|\to\infty}{\to} 0 \quad \text{weakly,}$$

for any $B \in \pi(\mathcal{A})''$.

Since $\pi(\mathcal{A})$ is irreducible (by hypothesis), we have $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$. Thus for any unit vectors $\xi, \eta \in \mathcal{H}$, we have, taking $B$ to be the projection onto $\xi$,

$$(\xi, \pi(\alpha(a)A)\eta) - (\xi, \pi(\alpha(a)A)\xi)(\xi, \eta) \to 0$$

as $|\underline{a}| \to \infty$.

It follows that for any unit vectors $\xi, \eta \in \mathcal{H}$,

$$(\xi, \pi(\alpha(a)A)\xi) - (\eta, \pi(\alpha(a)A)\eta) \to 0$$

as $|\underline{a}| \to \infty$.

Let $\xi$ be such that $U(a)\xi = \xi$, and take $\eta = \Omega$ Then we have

$$(\xi, \pi(A)\xi) = (\Omega, \pi(A)\Omega), \text{ all } A \in \mathcal{A}.$$

Since $\pi(\mathfrak{A})$ is irreducible, it follows that $\xi$ is proportional to $\Omega$.

QED.

## 5.5. The Reeh - Schlieder Theorem

### 5.5.1 Postulate (Additivity)

Let $\{\mathcal{O}_i\}$ be a cover of $\mathbb{M}$ by regions $\mathcal{O}_i$. Then $\mathfrak{A}$ is generated by the $\mathfrak{A}(\mathcal{O}_i)$.

This axiom implies that the algebras $\mathfrak{A}(\mathcal{O})$ do not become trivial if $\mathcal{O}$ is made small.

The following is a weaker version:

if $\omega$ is a vacuum state on $\mathfrak{A}$, then

$\pi(\mathfrak{A})$ is generated by the $\pi(\mathfrak{A}(\mathcal{O}_i))$

where $\pi$ is the GNS representation given by $\omega$.

In this case, we say that additivity holds in the vacuum sector.

Consider the set of vectors $\pi(\mathfrak{A}(\mathcal{O}))\Omega$. It is natural to think of such states as being "localized" in the region $\mathcal{O}$ in some sense.

Taking the closure, we obtain a Hilbert Space $\mathcal{H}(\mathcal{O})$. As we vary $\mathcal{O}$, we might expect to get a collection of Hilbert spaces giving the various localized states. This is not, the case, as the following theorem of Reeh and Schlieder shows:

$\mathcal{H}(\mathcal{O}) = \mathcal{H}$ .

5.5.2 <u>Theorem</u> (Reeh-Schlieder (1961), Araki(1964 b)).

Let $\omega$ be a vacuum state, with associated GNS constructs $(\mathcal{H}, \pi, U, \Omega)$, and suppose additivity holds in the vacuum sector. Then, for any region $\mathcal{O}$ in $\mathbb{M}$, $\Omega$ is cyclic and separating for $\pi(\mathfrak{A}(\mathcal{O}))$.

<u>Proof</u>

Let $\mathcal{O}$ be a region. To show that $\Omega$ is cyclic for $\pi(\mathfrak{A}(\mathcal{O}))$, we need only show that $(\xi, \pi(A)\Omega) = 0$, for all $A \in \mathfrak{A}(\mathcal{O})$, implies $\xi = 0$.
Since $\Omega$ is cyclic for $\pi(\mathfrak{A})$, we need only show that $(\xi, \pi(A)\Omega) = 0$ for all $A \in \mathfrak{A}(\mathcal{O})$, implies that $(\xi, \pi(A)\Omega) = 0$ for all $A \in \mathfrak{A}$.

Let $\mathcal{O}_0 \subset \mathcal{O}$ be such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}$ . Let $A_1, \ldots, A_n$ belong to $\mathfrak{A}(\mathcal{O}_0)$ , and let $a_1, \ldots, a_n \in \mathbb{R}^4$ . Set

$$F(a_1, a_2 - a_1, \ldots, a_n - a_{n-1}) = \left(\xi, \pi\big((\alpha(a_1)A_1)\ldots(\alpha(a_n)A_n)\big)\Omega\right).$$

Since $\overline{\mathcal{O}}_0$ is closed, and contained in $\mathcal{O}$, which is open, there is a neighbourhood $N_1 \times \ldots \times N_n$ in $\mathbb{R}^{4n}$ such that

$\mathcal{O}_0 + a_i \subset \mathcal{O}$ for $a_i \in N_i$, $1 \leq i \leq n$.

But then $\alpha(a_1)(A_1) \ldots \alpha(a_n)(A_n) \in \mathfrak{A}(\mathcal{O})$ and so, by hypothesis, $F(a_1, a_2 - a_1, \ldots\ldots) = 0$ for all $(a_1, a_2, \ldots a_n) \in N_1 \times \ldots\ldots \times N_n$.

However,

$$F = (\xi,\ e^{ia_1 P}\ \pi(A_1)\ e^{i(a_2 - a_1)P}\ \pi(A_2) \ldots e^{i(a_n - a_{n-1})P}\ \pi(A_n)\,\Omega)$$

Let $z_i \in \mathbb{C}^4$, $1 \leqslant i \leqslant n$, with $\mathrm{Im}\, z_1 \in V_+$, $\mathrm{Im}(z_2 - z_1) \in V_+, \ldots\ldots, \mathrm{Im}(z_n - z_{n-1}) \in V_+$. Then, since $P$ has its spectrum in $\overline{V_+}$, we have $\mathrm{Im}\, z_1\, P \geqslant 0$, $\mathrm{Im}(z_2 - z_1)\, P \geqslant 0, \ldots$etc. Thus $F(z_1, z_2 - z_1, \ldots)$ defines an anlytic function in the region $\mathrm{Im}\, z_1 \in V_+$, $\mathrm{Im}(z_2 - z_1) \in V_+$, etc., which vanishes on the boundary $\mathrm{Im}\, z_1 = 0$, etc, $\mathrm{Re}\, z_1 \in N_1$, $\mathrm{Re}\, z_2 \in N_2$, etc.

It follows that $F$ is identically zero, as an analytic function, in $\mathrm{Im}\, z_1 \in V_+$, etc., and so $F(a_1, a_2 - a_1, \ldots) = 0$ for all $a_i \in \mathbb{R}^4$.

Now, by the additivity in the vacuum sector, $\pi(\mathfrak{A})$ is generated by the $\pi(\mathfrak{A}(\mathcal{O}_0 + a))$, $a \in \mathbb{R}^4$.

We conclude that

$$(\xi, \pi(A)\Omega) = 0$$

for all $A \in \mathfrak{A}$ and so $\xi = 0$, and $\pi(\mathfrak{A}(\mathcal{O}))\Omega$ is dense in $\mathcal{H}$, as required.

To show that $\Omega$ is separating for $\pi(\mathfrak{A}(\mathcal{O}))$, we note that, by the preceding, $\Omega$ is separating for $\pi(\mathfrak{A}(\mathcal{O}_1))'$, for any region $\mathcal{O}_1$. If we choose $\mathcal{O}_1$ space-like with respect to $\mathcal{O}$, then, by locality, $\pi(\mathfrak{A}(\mathcal{O})) \subset \pi(\mathfrak{A}(\mathcal{O}_1))'$. Thus $\Omega$ is separating for $\pi(\mathfrak{A}(\mathcal{O}))$.

QED.

Remark The last part of the theorem has important consequences. If $A \in \mathfrak{A}(\mathcal{O})$ and $\pi(A)\Omega = 0$, then $\pi(A) = 0$. If $\pi$ is faithful, then $A = 0$. In other words, no local observables can annihilate the vacuum. This is a severe restriction on the algebras $\mathfrak{A}(\mathcal{O})$. Indees, this means that we cannot talk about the charge for a region $\mathcal{O}$ as an element of $\mathfrak{A}(\mathcal{O})$. For, presumably, such an observable should give zero on the vacuum, which is impossible unless it is zero

In the same way, one has difficulty in formulating the notion of a particle detector (see Haag (1972)). Intuitively, such a detector should correspond to an observable C such that

(i) $C = C^* = C^2$, i.e. C is a projection and so says "yes" or "no".

(ii) $C \in \mathfrak{A}(\mathcal{O})$, some $\mathcal{O}$.

(iii) $C\,\Omega = 0$, "no" on the vacuum.

We see by the Reeh-Schlieder theorem that the only possibility is $C = 0$ !

## 6 - The Charged Bose Field and its Sectors

In this chapter, we shall discuss in detail the free charged field and its charge sectors. We will see that these give rise to inequivalente, irreducible represen tations of the algebra of observables, and are physically equivalent. The observables will be defined as the gauge invariant elements of the field algebra.

Our first Objective will be to give a precise formulation of the charged field.

### 6.1. Definition of the charged field

The charged field can be thonght of as a pair of fields representing the "particle" and "antiparticle", respectively. We choose the "particle" to carry a charge +1, and the "antiparticle" charge -1. The Fock space on which the charged field acts should contain vectors of all charges.

Let us recall the formalism for the free neutral field. It acts on $\mathcal{F}$, the symmetric Fock space over $L^2(\mathbb{R}^3, d^3K)$.

The creation and annihilation operators $a^*(f)$, $a(g)$ are defined, as usual: e.g.

$$a(g)\ \Psi(\underline{k}_1,\ldots\ldots\underline{k}_n) = \sqrt{n}\int g(\underline{k})\ \Psi(\underline{k},\underline{k}_2,\ldots,\underline{k}_n)\,d\underline{k}$$

The free neutral scalar field of mass m is given as the operator-value distribution

$$\phi(\underline{x},t) = (2\pi)^{-3/2}\int \left(e^{i(k,x)}\ a^*(\underline{k}) + e^{-i(k,x)}a(\underline{k})\right)\frac{d^3k}{\sqrt{2\omega}}$$

where $(k,x) = k^0 t - \underline{k}\cdot\underline{x}$, and $k^0 = \omega(\underline{k}) = \sqrt{\underline{k}^2 + m^2}$.

In smeared form, this becomes, for $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$,

$$\phi(f) = 2^{-1/2}\left(a^*(F) + a(\bar{F})\right)$$

where $F(\underline{k}) = \sqrt{\frac{2\pi}{\omega}}\ \tilde{f}(\omega(\underline{k}),\underline{k})$.

We have used the Minkowski convention for the definition of the Fourier transform :

$$\tilde{f}(p) = (2\pi)^{-2}\int e^{i(p,x)}\ f(x)\ d^4x.$$

Let $\mathcal{F}^+$ and $\mathcal{F}^-$ be two distinguished Fock spaces over $\mathbb{L}^2(\mathbb{R}^3, d^3k)$.

The Fock space for the charged field is

$$\mathcal{K} = \mathcal{F}^+ \otimes \mathcal{F}^-$$

Let $a^*_\pm(\cdot)$ and $a_\pm(\cdot)$ be the creation and annihilation operators on $\mathcal{F}^\pm$, respectively.

We interpret $a_+^* \,(\cdot) \otimes 1\!\!1$ as the operator in $\mathcal{K}$ creating a particle with charge + 1, and $a_+ \,(\cdot) \otimes 1\!\!1$ as that destroying a particle with charge + 1. Similarly we interpret $1\!\!1 \otimes a_-^* \,(\cdot)$ and $1\!\!1 \otimes a_- \,(\cdot)$ as creating and destroying charge -1.

Let $D\pm$ be the set of finite particle vectors in $\mathcal{F}^{\pm}$. For $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, we define the charged field $\phi(f)$ on $D \equiv D+ \otimes D-$ to be

$$\phi(f) = 2^{-1/2} \left( a_+^* (F) \otimes 1\!\!1 + 1\!\!1 \otimes a_- (\bar{F}) \right)$$

where $F(\underline{k}) = \sqrt{\dfrac{2\pi}{\omega(\underline{k})}}\; \tilde{f}\,(\omega(\underline{k}), \underline{k})$.

Its "complex conjugate" $\phi_c(f)$ is defined on $D \equiv D^+ \otimes D^-$ as

$$\phi_c(f) = 2^{-1/2} \left( a_+ (\bar{F}) \otimes 1\!\!1 + 1\!\!1 \otimes a_-^* (F) \right).$$

It is clear that $\phi(f)^*$, the adjoint of $\phi(f)$, is an extension of $\phi_c(f)$. We will define $\phi(f)$ and $\phi_c(f)$ on bigger domains so that they become adjoints of each other. (the notation $\phi_c$ is temporary).

We note that $\mathcal{K}$ is spanned by vectors of the form $a_+^*(h_1) \ldots\ldots a_+^*(h_n)\, \Omega_+ \otimes a_-^*(g_1) \ldots a_-^*(g_m)\, \Omega_-$. If we write $\Omega = \Omega_+ \otimes \Omega_-$, then this vector is just

$$a_+^*(h_1) \ldots a_+^*(h_n) \otimes a_-^*(g_1) \ldots a_-^*(g_m)\, \Omega$$

It is clear that $\Omega$ is cyclic for the $a_+^*(\cdot) \otimes 1\!\!1$ and $1\!\!1 \otimes a_-^*(\cdot)$.

6.1.1. Defintion.

Let $N_{\pm}$ be the number operators in $\mathcal{F}^{\pm}$. The number operator on $\mathcal{K}$ is

$$N = N^{+} \otimes \mathbb{1} + \mathbb{1} \otimes N^{-}$$

The total charge operator on $\mathcal{K}$ is

$$Q = N^{+} \otimes \mathbb{1} - \mathbb{1} \otimes N^{-}.$$

Evidently D is a domain of analytic vectors for N and Q, on which they are therefore essentially self-adjoint.

N has eigenvalues 0,1,2,....., whereas Q has eigenvalues $0, \pm 1, \pm 2, \ldots$.

Clearly, $a_{+}^{*}(h_1)\ldots a_{+}^{*}(h_n) \otimes a_{-}^{*}(g_1)\ldots a_{-}^{*}(g_m)\,\Omega$ is an eigenvector for N and Q with eigenvalues n+m and n-m, respectively.

We see, also, that $\phi(f)$ creates a charge of +1 and destroys a charge -1.
Accordingly, we say that $\phi(f)$ carries charge +1.
Similarly, $\phi_c(f)$ carries charge -1.

6.1.2. Proposition

Let $\Psi \in D$, and suppose $\Psi$ contains less than n particles. Then, for $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$,

$$\| ( \phi(f) \pm \phi_c(f)) \Psi \| \leq 2\sqrt{2}\sqrt{n+1}\, \|F\|_2 \|\Psi\|,$$

where $F(\underline{k}) = \sqrt{\frac{2\pi}{\omega}}\ \tilde{f}(\omega, \underline{k})$

<u>Proof</u>.

We will show that

$$\| (a_+^*(G) \otimes \mathbb{1}) \Psi \| \leq \sqrt{n+1}\ \|G\|_2 \|\Psi\|$$

and

$$\| (a_+(G) \otimes \mathbb{1}) \Psi \| \leq \sqrt{n+1}\ \|G\|_2 \|\Psi\|$$

for $G \in L^2(\mathbb{R}^3, d^3k)$. The same proof holds for $\mathbb{1} \otimes a_-^*$ and $\mathbb{1} \otimes a_-$.

Consider the first inequality. Let $\mathcal{H}_m^{\pm} = \mathcal{F}_0^{\pm} \oplus \mathcal{F}_1^{\pm} \oplus \dots \oplus \mathcal{F}_m^{\pm}$. We have

$$\| a_+^*(G) \otimes \mathbb{1}\ \Psi \|^2 = (\Psi, (a_+^*(G) \otimes \mathbb{1})^* (a_+^*(G) \otimes \mathbb{1}) \Psi)$$
$$= (\Psi, a_+(\bar{G})\, a_+^*(G) \otimes \mathbb{1}\ \Psi).$$

By hypothesis, $\Psi$ has at most n particles, and so $\Psi \in \mathcal{H}_n^+ \otimes \mathcal{H}_n^-$. Moreover, $a_+(\bar{G}) a_+^*(G)$ is a bounded self-adjoint operator from $\mathcal{H}_n^+$ into $\mathcal{H}_n^+$ with norm less than or equal to $(n+1)\ \|G\|_2^2$.

Similarly, $a_+^*(\bar{G})\, a_+(G)$ is bounded $\mathcal{H}_n^+ \to \mathcal{H}_n^+$, with norm less than or equal to $n\ \|G\|_2^2$.
The proof is therefore completed once we have proved the following lemma.

6.1.3. <u>Lemma</u>

Let $A = A^*$ be a bounded operator on a Hilbert space $\mathcal{H}_1$. Let $\mathcal{H}_2$ be a Hilbert space. Then $A \otimes \mathbb{1}$ is bounded on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and

$$\| A \otimes \mathbb{1} \| = \| A \| .$$

<u>Proof</u>

By the spectral theorem, there are measure spaces $(X, \mu)$, $(y, \nu)$ such that $\mathcal{H}_1 \simeq L^2(X, \mu)$, $\mathcal{H}_2 \simeq L^2(y, \nu)$ and $A \simeq A(x) \in L^\infty(X, \mu)$.
Then $\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq L^2(X \times Y, \mu \otimes \nu)$. For $z \in \mathcal{H}_1 \otimes \mathcal{H}_2$ we have

$$\| A \otimes \mathbb{1} \; z \|^2 = \int_{X \times Y} | A(x) \, z(x,y) |^2 \, d\mu(x) \, d\nu(y)$$
$$\leq \| A \|_\infty^2 \; \| z \|^2 .$$

But $\| A \|_\infty = \| A \|$, and so $\| A \otimes \mathbb{1} \| \leq \| A \|$.
Taking $z$ of the form $z_1 \otimes z_2$ it is easy to see that

$$\| A \otimes \mathbb{1} \| = \| A \|$$

QED.

The proof of 6.1.2. is now complete.

6.1.4. <u>Proposition</u>

The operators $\phi(f) + \phi_c(f)$ and $i(\phi(f) - \phi_c(f))$ are essentially self-adjoint on D for all $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$.

Proof.

By 6.1.2, D is domain of entire vectors for $\phi(f) \pm \phi_c(f)$.

QED.

(For a discussion of analytic vectors, essential self--adjointness etc. see, for example, Simon (1972)).

Let us denote by $\xi(f)$ the self-adjoint operator $\frac{1}{2}(\phi(f) + \phi_c(f))^*$ and by $\eta(f)$ the self-adjoint operator $-\frac{i}{2}(\phi(f) - \phi_c(f))^*$.

6.1.5. Proposition

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, then the unitary groups generated by $\xi(f)$ and $\eta(f)$ commute.

<u>Proof</u>.

$\xi$ (f) and $\eta$ (f) commute on D.

By taking expectation values in elements of D, we can write the unitaries as exponential power series, because D is a domain of entire vectors. By the first remark, the unitaries commute on D. Since D is dense in $\mathcal{K}$, the result follows.

QED.

We note than on D, we have

$$\phi(f) = \xi(f) + i\eta(f)$$

and

$$\phi_c(f) = \xi(f) - i\eta(f).$$

We can now give a precise definition of $\phi(f)$.

6.1.6. <u>Definition</u>

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$. The charged field $\phi(f)$ and its conjugate, $\phi^*(f)$, are the operators on $\mathcal{K}$ with domain

$$D(\phi(f)) = D(\phi^*(f)) = D(\xi(f)) \cap D(\eta(f))$$

given by

$$\phi(f) = \xi(f) + i\,\eta(f)$$

and

$$\phi^*(f) = \xi(f) - i\,\eta(f).$$

6.1.7. <u>Theorem</u>

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$. Then $\phi(f)$ and $\phi^*(f)$ are normal operators, and are adjoints of each other.

<u>Proof</u>

By 6.1.5, and the spectral theorem, there is a measure space $(X, \mu)$ such that $\mathcal{K} \simeq L^2(X,\mu)$ and $\xi$ and $\eta$ are equivalent to multiplication by real measurable functions. Let us denote these also by $\xi$ and $\eta$.

Then the operator $(D\ \phi(f)),\ \phi(f))$ is equivalent to $(D(\xi) \wedge D(\eta),\ \xi + i\eta)$, and $(D(\phi^*(f)),\ \phi^*(f))$ is equivalent to $(D(\xi) \wedge D(\eta),\ \xi - i\eta)$.

But $D(\xi) \wedge D(\eta)$ is the set

$$\{u \in L^2 \mid (\xi + i\eta)\,u \in L^2\} = \{u \in L^2 \mid (\xi - i\eta)u \in L^2\}.$$

That is, $D(\xi) \wedge D(\eta)$ is the domain of the multiplication operators $\xi \pm i\eta$, Which are normal and adjoints of each other. Such properties are preserved under unitary equivalence.

QED.

Remark. This theorem justifies the notation $\phi^*(f)$..

A technical result we shall need is the following.

6.1.8. Theorem

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, $f \neq 0$. Then $\ker \xi(f) = \{0\}$, i.e. $\xi(f)\psi = 0$ implies $\psi = 0$.

Proof

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, $f \neq 0$ be given. Let us define $G(\underline{k}) = F(\underline{k}) / 2 \| F \|_2^2$, where F is as in 6.1.2.

Let $\phi_{\pm}$ denote the time-zero free neutral field on $\mathcal{F}^{\pm}$, and $\pi_{\pm}$ the time-zero momentum.

Set

$$B = 2^{-1/2} \{ \pi_+(G) \otimes 1\!\!1 + 1\!\!1 \otimes \pi_-(G)$$

on D.

Then B is e.s.a. on D (as in 6.1.4).

Moreover, on D, we have

$$[\xi(f), B] = i$$

Because D is a domain of entire vectors for $\xi$ and B, we can "exponentiate" this relation to conclude that $U(s) = \exp is\,\xi(f)$ and $V(t) = \exp it\,\overline{B}$ give a representation of the Weyl relations for one degree of freedom.

By the von Neumann uniqueness theorem, we have

$$\mathcal{K} \simeq \bigoplus_j L^2 ( \mathbb{R}_j, dx_j ), \text{ and}$$

$$U(s) \simeq \bigoplus_j e^{isx_j} , \text{ for } j \in I, \text{ some index set.}$$

Now $\xi(f) \Psi = 0$ is equivalent to $U(s) \Psi = \Psi$ for all $s \in \mathbb{R}$ , i.e. 1 is an eigenvalue of $V(s)$.

This implies 1 is an eigenvalue of $e^{isx_j}$, which is false.

QED.

Suppose $V : L^2 ( \mathbb{R}^3 ) \longrightarrow L^2 ( \mathbb{R}^3 )$ is a "one--particle operator". There is a natural action on $\mathcal{F}^{\pm}$ given by

$$\Gamma_{\pm} (V) \restriction \mathcal{F}_n^{\pm} = \otimes^n V$$

That is $\Gamma_{\pm} (V) : \mathcal{K} \longrightarrow \mathcal{K}$ is given by

$$\Gamma_{\pm} (V) = \mathbb{1} \oplus V \oplus (V \otimes V) \oplus \quad \ldots\ldots$$

If $\| V \| \leqslant 1$, then $\| \Gamma_{\pm} (V) \| \leqslant 1$, otherwise $\Gamma_{\pm}(V)$ is unbounded.

Moreover, if $V$ is unitary, so is $\Gamma_{\pm} (V)$.

We can then define $\Gamma_{+}(V) \otimes \Gamma_{-} (V)$ on $\mathcal{K}$ which is also unitary.

Let $(a, \Lambda ) \in \mathcal{P}_+^{\uparrow}$ be a Poincaré transformation.

We define the action of $(a, \Lambda)$ on $L^2 ( \mathbb{R}^3 , d^3k)$ by

$$u(a, \Lambda ) : \Psi (\underline{k}) \to e^{i(a,k)} \frac{\omega(\underline{\Lambda^{-1}k})^{1/2}}{\omega(\underline{k})^{1/2}} \Psi (\underline{\Lambda^{-1}k} ) \Big|_{k^0 = \omega (\underline{k})}$$

where $(a,k) = a^\circ k^\circ - \underline{a} \cdot \underline{k}$ and $\underline{\Lambda^{-1} k}$ is the spatial componente of the 4 - vector $\Lambda^{-1}k$, $k = (\omega(\underline{k}), \underline{k})$.

One verifies that $u(a, \Lambda)$ is a strongly continuons unitary representation of $\mathcal{P}_+^\uparrow$ in $L^2(\mathbb{R}^3, d^3\underline{k})$.

In $\mathcal{K}$, we define the action of $\mathcal{P}_+^\uparrow$ by $(a, \Lambda) \to \Gamma_+(u(a, \Lambda)) \otimes \Gamma_-(u(a, \Lambda)) \equiv U(a, \Lambda)$.

6.1.9. <u>Proposition</u>

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, and let $(a, \Lambda) \in \mathcal{P}_+^\uparrow$. Let $f_{a,\Lambda}(x) = f(\Lambda^{-1}(x-a))$. Then, on D, we have

$$U(a, \Lambda)\ \phi^{\#}(f)\ U(a, \Lambda)^{-1} = \phi^{\#}(f_{a,\Lambda})$$

where $\phi^{\#}(f)$ denotes $\phi(f)$ or $\phi^{*}(f)$.

<u>Proof</u>

The proof is straightforward.

QED.

6.2. <u>The Field Algebra</u>

We would like to define local algebras associated with the fields $\phi$ and $\phi^*$. However , these are not self-adjoint - we must use their real and imaginary parts.

6.2.1. <u>Definition</u>

Let $\mathcal{O} \subset \mathbb{M}$ be a region. We define the local field algebra, $\mathcal{F}(\mathcal{O})$ , to be the von Neumann algebra generated by the unitary operators with generators $\xi$ (f) and $\eta$(f) as f varies over $\mathcal{D}_{\mathbb{R}}(\mathcal{O})$.

It is a clear that if $\mathcal{O}_1 \subset \mathcal{O}_2$ , then $\mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2)$ . It is not difficult to see that if $\mathcal{O}_1$ , and $\mathcal{O}_2$ are space-like separated, then $\mathcal{F}(\mathcal{O}_1)$ and $\mathcal{F}(\mathcal{O}_2)$ commute. Moreover, we see that

$$\alpha(a, \Lambda)\, A \equiv U(a, \Lambda)\, A\, U(a, \Lambda)^*$$

defines an automorphism of $\mathcal{B}(\mathcal{K})$ which satisfies

$$\alpha(a, \Lambda)\, \mathcal{F}(\mathcal{O}) = \mathcal{F}(\Lambda\mathcal{O} + a)$$

6.2.2. <u>Definition</u>

The field algebra $\mathcal{F}$ is the norm closure of the union of all the $\mathcal{F}(\mathcal{O})$, where $\mathcal{O}$ is a region in $\mathbb{M}$.

Clearly $\alpha(a, \Lambda) : \mathcal{F} \to \mathcal{F}$ .

We should remark that there is another localization which is also Poincaré covariant but which is anti-local with respect to the one defined above (Wilde (1971)).

6.2.3. <u>Theorem</u>

$\mathcal{F}$ is irreducible.

<u>Proof</u>

We shall only sketch the proof.

Let $\dot{\phi}$ be the operator-valued distribution obtained from $\phi$ by taking its time-derivative. If $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, let $f_1 \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$ satisfy

$$\tilde{f}_1(\omega(\underline{k}), \underline{k}) = \tilde{f}(\omega(\underline{k}), \underline{k}) / \omega(\underline{k})$$

Using the fact the D is a domain of entire vectors, we see that

$$\exp i(\phi(f) + \phi^*(f))^- \text{ and } \exp(\dot{\phi}(f_1) - \dot{\phi}^*(f_1))^-$$

commute and that

$$\exp i\,\phi_+(f) \otimes \mathbb{1} = \exp \tfrac{i}{2}(\phi(f) + \phi^*(f))^- \exp \tfrac{1}{2}(\dot{\phi}(f_1) - \dot{\phi}^*(f_1)).$$

Since D is a core for the self-adjoint operators involved, and since $\dot{\phi}(f_1)$ is a limit of operators of the form $\frac{1}{t}(\phi(g_t) - \phi(g))$, it follows by the semigroup convergence theorem (see e.g. Kato (1966)) that $\mathcal{F}''$ contains all the operators of the form $\{\exp i\,\phi_+(f)\otimes \mathbb{1}$, $\mathbb{1}\otimes\exp i\,\phi_-(f)\}$ where $f\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$.

Let $\mathcal{A}=\{e^{i\phi_+(f)} \mid f\in\mathcal{S}\}''$,

$\mathcal{B}=\{e^{i\phi_-(f)} \mid f\in\mathcal{S}\}''$.

(the commutants taken in $\mathcal{B}(\mathcal{F}^{\pm})$ , resp.)

Then

$$\mathcal{F}\supset(\mathcal{A}\otimes\mathcal{B}).$$

But $\phi_{\pm}$ act irreducibly on $\mathcal{F}^{\pm}$, i.e.

$\mathcal{A}=\mathcal{B}(\mathcal{F}^{+})$, $\mathcal{B}=\mathcal{B}(\mathcal{F}^{-})$.

It follows that $\mathcal{F}''=\lambda\mathbb{1}$,

i.e. $\mathcal{F}$ is irreducible.

QED.

## 6.3. Gauge Transformations and the Observables

Let $0\leq\theta<2\pi$, and define the unitary operators $e^{\pm i\theta}$ on $L^2(\mathbb{R}^3, d^3k)$. We define $U(\theta)$ on $\mathcal{K}$ to be

$$U(\theta)=\Gamma_+(e^{i\theta})\otimes\Gamma_-(e^{-i\theta}).$$

Clearly $\theta\longrightarrow U(\theta)$ is a strongly continuons representation of the torus.

### 6.3.1. Proposition

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, $0 \leq \theta < 2\pi$. Then, on D, we have

$$U(\theta)\ \phi(f)\ U(\theta)^* = e^{i\theta}\ \phi(f).$$

and

$$U(\theta)\ \phi^*(f)\ U(\theta)^* = e^{-i\theta}\ \phi^*(f)$$

<u>Proof</u>

Obvious.

QED.

6.3.2. <u>Definition</u>

The transformation $\phi(f) \rightarrow e^{i\theta}\ \phi(f)$, $\phi^*(f) \rightarrow e^{-i\theta}\ \phi^*(f)$ is called a gauge transformation of the first kind. The gauge group is the torus.

We note that the generator of $U(\theta)$ is nothing other than Q the charge operator.

6.3.3. <u>Proposition</u>

Let $\mathcal{O} \subset \mathbb{M}$ be a region. Then $U(\theta)\,\mathcal{F}(\mathcal{O})\,U(\theta)^* = \mathcal{F}(\mathcal{O})$, all $0 \leq \theta < 2\pi$.

**Proof**

This follows from the fact that, on D, we have

$$U(\theta)\ \xi(f)\ U(\theta)^{*} = 2\cos\theta\ \ \xi(f)$$

and

$$U(\theta)\ \eta(f)\ U(\theta)^{*} = 2\sin\theta\ \ \eta(f)$$

and then by exponentiating.

QED.

Now, Lagrangian field theory suggests that gauge transformations should have no physical consequences. In other words, the observables should be invariant under a gauge transformation. This leads us to the next definition.

6.3.4. **Definition**

Let $\mathcal{O} \subset \mathbb{M}$ be a region. The local observables associated with $\mathcal{O}$ are the elements

$$\mathfrak{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \wedge \{U(\theta) \mid 0 \leq \theta < 2\pi\}'.$$

We define $\mathfrak{A}$ to be the C*-algebra generated by the $\mathfrak{A}(\mathcal{O})$.

### 6.3.5. Proposition

For each $\mathcal{O}$, $\mathfrak{A}(\mathcal{O})$ is a Von Neumann algebra. The algebras $\{\mathfrak{A}, \mathfrak{A}(\mathcal{O})\}$ satisfy the Haag-Kastler axioms 5.1.1. - 5.1.4., where $\alpha(a, \Lambda)$ is given by

$$\alpha(a, \Lambda)\, A = U(a, \Lambda)\, A\, U(a, \Lambda)^*.$$

**Proof**

It is clear that $\mathfrak{A}(\mathcal{O})$ is a weakly closed * - algebra containing $\mathbb{1}$, i.e. it is a von Neumann algebra.

The axioms 5.1.1, 5.1.2, 5.1.3 hold because they hold for the $\mathring{\mathcal{F}}(\mathcal{O})$. Axiom 5.1.4. follows because it holds for the $\mathcal{F}(\mathcal{O})$ and $U(\theta)$ commutes with $U(a, \Lambda)$.

QED.

## 6.4. The Charge Sectors

As previously remarked, the charge operator Q is the generator of $U(\theta)$, and has eigenvalues 0, $\pm 1, \pm 2, \ldots$

Let $\mathcal{K}_q$ be the subspace of $\mathcal{K}$ with Charge $q$.

Then

$$\mathcal{K} = \bigoplus_{q=-\infty}^{+\infty} \mathcal{K}_q$$

Since $\mathfrak{A}$ commutes with $U(\theta)$ we see that $\mathfrak{A}$ maps each $\mathcal{K}_q$ into itself. Therefore we can define a representation $(\mathcal{K}_q, \pi_q)$ of $\mathfrak{A}$ by

$$\pi_q \mathfrak{A} = \mathfrak{A} \restriction \mathcal{K}_q$$

6.4.1. <u>Definition</u>

The representations $(\mathcal{K}_q, \pi_q)$, $-\infty < q < \infty$, are called the charge sectors of the charged field.

Suppose we have a vector in $\mathcal{K}_q$. This defines a state on $\mathfrak{A}$. Suppose now we add some charge to our state, but in a very remote region of space. This should not make very much difference as far as local observables are concerned. We might expect, then, that the different charge representations are physically equivalent. This is the "particle behind the moon" argument of Haag and Kastler (1964). We shall prove a somewhat stronger statement, but first we need two lemmas.

6.4.2. <u>Lemma</u>

Let $A \in \mathfrak{A}(\mathcal{O})$, and $f \in \mathcal{D}_{\mathbb{R}}(\mathcal{O}_1)$ where $\mathcal{O}$ and $\mathcal{O}_1$ are space-like. Then, for any $z$, $z' \in D(\phi(f)) = D(\phi^*(f))$ we have

$$(z, A\, \phi(f)\, z') = (\phi^*(f)\, z, Az'),$$

i.e. $\phi(f)$ and $A$ weakly commute on $D(\phi(f))$.

<u>Proof</u>

We know that both $e^{is\xi(f)}$ and $e^{it\eta(f)}$ commute with $A$, for all $s,t \in \mathbb{R}$.

Hence

$$(e^{-is\xi(f)} z, A z') = (z, A e^{is\xi(f)} z')$$

and

$$(e^{-it\eta(f)} z, A z') = (z, A e^{it\eta(f)} z').$$

The result follows by taking derivatives w.r.t. s and t at $s=t=0$, and adding.

QED.

6.4.3. <u>Lemma</u>

Let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, and let $f_{\underline{a}}$ be the space-translate of f, i.e. $f_{\underline{a}}(t,\underline{x}) = f(t, \underline{x} - \underline{a})$. Suppose $\| F(\underline{k}) \|_2^2 = 2$, where $F(\underline{k}) = (2\pi)^{1/2} \omega^{-1/2} \tilde{f}(\omega,\underline{k})$. Then, for $z \in D$, $\phi^*(f_{\underline{a}}) \phi(f_{\underline{a}}) z$ converges weakly to $z$, as $|\underline{a}| \to \infty$.

<u>Proof</u>

Let $z \in D$, Since $\phi^*(f_{\underline{a}}) \ \phi(f_{\underline{a}}) \ z$ is uniformly bounded in $\underline{a}$ (e.g. by 6.1.2), we need only show that

$$( z', \phi^* (f_{\underline{a}}) \ \phi(f_{\underline{a}}) \ z) \longrightarrow (z',z) \text{ as } |\underline{a}| \to \infty,$$

for $z'$ in some dense set. We choose $z' \in D$.

Writing $\phi^*$ and $\phi$ in terms of creation and annihilation operators, we obtain

$$2( z', \phi^*(f_{\underline{a}}) \ \phi(f_{\underline{a}}) \ z) = (z', a_+(\overline{F_{\underline{a}}}) \ a_+^*(F_{\underline{a}}) \otimes \mathbb{1} \ z)$$

$$+ ( z', a_+(\overline{F_{\underline{a}}}) \otimes a_-(\overline{F_{\underline{a}}}) z) + (z', a^*_+(F_{\underline{a}}) \otimes a_-^* (F_{\underline{a}}) \ z)$$

$$+ ( z', \mathbb{1} \otimes a_-^* (F_{\underline{a}}) \ a_- (\overline{F_{\underline{a}}}) \ z)$$

where $F_{\underline{a}}(\underline{k}) = e^{-i \underline{k} \underline{a}} F(\underline{k})$.

The second, third and fourth terms all converge to zero as $|\underline{a}| \longrightarrow \infty$ by the Riemann-Lebesgue Lemma (because they all contain a term of the form $a_{\underline{+}}(F_{\underline{a}})\ z$ or $a_{\underline{+}}(F_{\underline{a}})\ z'$). The first term can be written as

$$(z', a^*_+(F_{\underline{a}}) \ a_+ (\overline{F_{\underline{a}}}) \otimes \mathbb{1} \ z) + (z',z) \int F_{\underline{a}} \overline{F}_{\underline{a}} \, d^3k$$

using the commutation relations.

Once again, the first term converges to zero by the Riemann-Lebesgue lemma. The last term is equal to $2(z',z)$ because of our normalisation $\| F \|_2^2 = 2$.

QED.

6.4.4. <u>Theorem</u>

If $\omega$ is a vector state of $\mathfrak{A}$ in the representation $(\mathcal{K}_q, \pi_q)$ then any W*-neighbourhood of $\omega$ contains a vector state $\rho$ in the representation $(\mathcal{K}_{q'}, \pi_{q'})$, any q, q'. In particular, the charge sectors are physically equivalent.

<u>Proof</u>

By a $|q - q'|\varepsilon$ - argument, it is enough to consider $q' = q + 1$.

Let $\omega$ be a vector of $\mathfrak{A}$ in the representation $(\mathcal{K}_q, \pi_q)$. That is, $\omega$ has the form

$$\omega(\cdot) = (z, \pi_q(\cdot)\, z)$$

for some $z \in \mathcal{K}_q$, $\|z\| = 1$.

Let $\mathcal{N}(\omega; A_1 \ldots, A_p, \varepsilon)$ be a W*-neighbourhood of $\omega$;

$$\mathcal{N}(\omega; A_1, \ldots, A_p, \varepsilon) = \{\omega' \in \mathfrak{A}^{*+} \mid |\omega'(A_\ell) - \omega(A_\ell)| < \varepsilon,\ \ell = 1 \ldots p\}$$

We can choose $h \in D \cap \mathcal{K}_q$ such that

$$\omega'(\cdot) = (h, \pi_q(\cdot)\, h)$$

belongs to $\mathcal{N}(\omega; A_1, \ldots, A_p, \varepsilon/2)$. This is possible because $D \wedge \mathcal{K}_q$ is dense in $\mathcal{K}_q$ and p is finite.

Assume, for the moment, that $A_1, \ldots, A_p \in \mathfrak{A}(\mathcal{O})$ some region $\mathcal{O}$. We define a positive linear functional on $\mathfrak{A}$ by

$$\rho_{\underline{a}}(\cdot) = (\phi(f_{\underline{a}})h, \Pi_{q+1}(\cdot)\,\phi(f_{\underline{a}})h)$$

where $f \in \mathcal{D}(\mathcal{O}_1)$, some region $\mathcal{O}_1$, and f satisfies the normalization of 6.4.3.

Now, by 6.4.2, $\rho_{\underline{a}}(A_\ell)$ can be written as

$$\rho_{\underline{a}}(A_\ell) = (\phi^*(f_{\underline{a}})\,\phi(f_{\underline{a}})h, \Pi_q(A_\ell)h),$$

$\ell = 1, \ldots p$, and $|\underline{a}|$ sufficiently large.

By 6.4.3, we see that

$$\rho_{\underline{a}}(X) \to \omega'(X) \quad \text{as} \quad |\underline{a}| \to \infty,$$

for $X = 1\!\!1, A_1, \ldots, A_p$.

In other words, for large $|\underline{a}|$, the state $\sigma(\cdot) = \rho_{\underline{a}}(\cdot)/\rho_{\underline{a}}(1\!\!1)$ belongs to $\mathcal{N}(\omega', A_1, \ldots A_p, \varepsilon/2)$, i.e. $\sigma$ belongs to $\mathcal{N}(\omega, A_1, \ldots, A_p, \varepsilon)$.

It remains to remove the restriction $A_1, \ldots A_p \in \mathfrak{A}(\mathcal{O})$. Let $A_1, \ldots, A_p \in \mathfrak{A}$. Then, by definition of $\mathfrak{A}$, there is a region $\mathcal{O}$, and elements $A_1', \ldots, A_p' \in \mathfrak{A}(\mathcal{O})$ such that $\| A_\ell - A'_\ell \| < \varepsilon$, for $1 \leq \ell \leq p$.

Thus, given $\omega$, we construct a $\sigma$ as above, and deduce that it belongs to

$\eta(\omega; A'_1, \dots, A'_p, \varepsilon)$. But

$$|\omega(A_\ell) - \sigma(A_\ell)| \leq |\omega(A'_\ell) - \sigma(A'_\ell)| + 2\,\|A'_\ell - A_\ell\| \leq 3\varepsilon .$$

That is, $\sigma \in \eta(\omega; A_1, \dots, A_p, 3\varepsilon)$.
The result follows.

QED.

Remark It has been shown by Fell (1960) that two representations of a C*-algebra are physically equivalent if and only if they have the same kernel. Using this result, we see that $\Pi_q$ and $\Pi_{q'}$ have the same kernel. But then $\oplus_q \Pi_q$ has the same kernel, which is zero since it is the identity representation of $\mathfrak{A}$.

Hence each $(\mathcal{K}_q, \Pi_q)$ is faithful. We have thus proved the following.

6.4.5. Corollary

The representations $(\mathcal{K}_q, \Pi_q)$ are faithful representations of $\mathfrak{A}$.

This also follows from the strong local equivalence of the sectors (see 6.5.)

We would like to discuss the irreducibility of the charge representations. To do this we shall use the notion of a " mean" (See Doplicher, Haag and Roberts (1969)

We recall that our gauge group is T, the torus, which is represented on $\mathcal{K}$ by $U(\theta)$.

6.4.6. <u>Definition</u>

The mean of an operator $X \in \mathcal{B}(\mathcal{K})$ with respect to the unitary representation U of the gauge group T is the operator m(X), where

$$m(X) = \int_T U(\theta)\, X\, U(\theta)^* \, d\theta$$

where the integral is a weak integral in $\mathcal{B}(\mathcal{K})$. ( $\int_T \cdot \, d\theta$ is the normalized integral over T).

6.4.7. <u>Lemma</u>

(a) $U(\theta)m(X)\, U(\theta)^* = m(\, U(\theta)XU(\theta)^*) = m(X)$.

(b) $m : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ is weakly continuons on bounded sets.

<u>Proof</u>

(a) Obvious.

(b) We shall give an explicit alternative proof to that of Doplicher et al (1969).

Let $X_\nu \longrightarrow X$ weakly, with $\| X_\nu \| \leqslant K$ for all $\nu$, some K. Let $A_\nu = X - X_\nu$. We must show that $m(A_\nu) \longrightarrow 0$ weakly.

Let $z, z' \in \mathcal{K}$. Then, for fixed $\alpha \in T$, it is easy to see that, for given $\varepsilon > 0$,

$$|(z', U(\beta)\ A_\nu\ U(\beta)^*\ z)| < \varepsilon$$

for all $\beta$ in some neighbourhood $N(\alpha)$ of $\alpha$ and all $\nu > \nu(\alpha)$, some $\nu(\alpha)$.

Now, by varying $\alpha$ over T, we obtain a family of $\nu(\alpha)$'s and $N(\alpha)$'s. The $N(\alpha)$'s cover T, which is compact, so there exists a finite collection $\alpha_1, \ldots, \alpha_k$ such that $T = \bigcup_{j=1}^{k} N(\alpha_j)$

Let $\nu > \nu(\alpha_j)$, $1 \leqslant j \leqslant k$. Then, for any $\alpha \in T$, we have

$$|(z', U(\alpha)\ A_\nu\ U(\alpha)^*\ z)| < \varepsilon$$

because $\alpha \in N(\alpha_j)$, some $1 \leqslant j \leqslant k$.

Hence,

$$|(z', m\ (A_\nu)\ z)| < \varepsilon ,$$

and the result follows

QED.

6.4.8. <u>Lemma</u>

Let $\mathcal{B}$ be a C*-algebra in $\mathcal{B}(\mathcal{K})$ such that $m(\mathcal{B}) \subset \mathcal{B}$. Then $\{\mathcal{B} \cap U(T)'\}^{-} = \mathcal{B}^{-} \cap U(T)'$, where the bar denotes the weak closure.

Proof

It is clear that $\{\mathcal{B} \cap U(T)'\}^{-} \subset \mathcal{B}^{-} \cap U(T)'$. Let $A \in \mathcal{B}^{-} \cap U(T)'$. Then $A \in \mathcal{B}^{-}$ and so by Kaplansky's density theorem, there is a net $A_\nu$ in $\mathcal{B}$, with $\|A_\nu\| \leq \|A\|$, such that $A_\nu \longrightarrow A$ weakly.

By 6.4.7.(b), $m(A_\nu) \longrightarrow m(A)$ weakly. But $A \in U(T)'$, and so $m(A) = A$. Hence $m(A_\nu) \longrightarrow A$ weakly. Since $m(A_\nu) \in \mathcal{B} \cap U(T)'$ we conclude that $A \in \{\mathcal{B} \cap U(T)'\}^{-}$.

QED.

6.4.9. Theorem

The representations $(\mathcal{K}_q, \pi_q)$, $q = 0, \pm 1, \ldots$ of $\mathcal{A}$ are irreducible.

Proof

By definition of $\mathcal{A}(\mathcal{O})$,

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{F}(\mathcal{O}) \wedge U(T)' = m(\mathfrak{F}(\mathcal{O}))$$

(using $U(\theta)\ \mathfrak{F}(\mathcal{O})\ U(\theta) = \mathfrak{F}(\mathcal{O})$), and so

$$\mathfrak{A} = m(\mathfrak{F}) = \mathfrak{F} \wedge U(T)'.$$

By 6.4.8, we have

$$\mathfrak{A}^- = \mathfrak{F}^- \wedge U(T)'.$$

But, by 6.2.3, $\mathfrak{F}^- = \mathcal{B}(\mathcal{K})$ and so

$$\mathfrak{A}^- = U(T)'.$$

We see therefore that $\mathfrak{A}^- : \mathcal{K}_q \rightarrow \mathcal{K}_q$ for each $q$.
Moreover, $U(T)' = \bigoplus_q \mathcal{B}(\mathcal{K}_q)$, and so

$$\mathfrak{A}^- \upharpoonright \mathcal{K}_q = \mathcal{B}(\mathcal{K}_q).$$

But $\mathfrak{A}^- \upharpoonright \mathcal{K}_q$ is in the weak closure of $\Pi_q(\mathfrak{A})$.
Hence $(\mathcal{K}_q, \Pi_q)$ is irreducible.

QED.

<u>Remark</u> We have now proved that the $\mathfrak{A}(\mathcal{O})$, $\mathfrak{A}$ satisfy the axioms 5.1.1 - 5.1.5.

6.4.10. <u>Corollary</u>

The representations $(\mathcal{K}_q, \Pi_q)$, $q = 0, \pm 1, \ldots$ of $\mathfrak{A}$ are unitarily inequivalent.

Proof

We have, for any $\theta$, $U(\theta) \in \bar{\mathcal{O}\!\iota}$.
Let $A_\nu \in \mathcal{O}\!\iota$ be such that $A_\nu \longrightarrow U(\theta)$ weakly.
Then, for $z \in \mathcal{K}_q$, we have

$$(z, A_\nu z) \longrightarrow e^{i\theta q}(z,z)$$

If $(\mathcal{K}_q, \pi_q)$ and $(\mathcal{K}_{q'}, \pi_{q'})$ were unitarily equivalent, we would have, for $z \in \mathcal{K}_q$,

$$(z, \pi_q(A_\nu) z) = (Wz, \pi_{q'}(A_\nu) Wz)$$

$$\longrightarrow e^{i\theta q'}(Wz,Wz) = e^{i\theta q'}(z,z)$$

where $Wz \in \mathcal{K}_{q'}$, and W effects the equivalence between $\pi_q$ and $\pi_{q'}$. On the other hand,

$$(z, \pi_q(A_\nu) z) \longrightarrow e^{i\theta q}(z,z),$$

which gives a contradiction.

QED.

Remark It is not difficult to show that the representations $(\mathcal{K}_q, \pi_q)$ are mutually disjoint.

To summarize the last few results : the charge sectors are irreducible, physically equivalent, but unitarily inequivalent representations of the algebra of observables, $\mathcal{O}\!\iota$.

## 6.5. Strong Local Equivalence

Another notion of equivalence has been introduced by Borchers (1967) which is stronger than physical equivalence but weaker than unitary equivalence.

6.5.1. <u>Definition</u>

Let $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ be two representations of a quasilocal algebra, $\mathfrak{A}$. They are said to be locally equivalent if and only if for each region $\mathcal{O}$ there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that $U \pi(A) U^* = \pi'(A)$ for all $A \in \mathfrak{A}(\mathcal{O})$. That is, $\pi(\mathfrak{A}(\mathcal{O}))$ and $\pi'(\mathfrak{A}(\mathcal{O}))$ are unitarily equivalent for each region $\mathcal{O}$.

6.5.2. <u>Definition</u>

$(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ are said to be strongly locally equivalent if and only if for each region $\mathcal{O}$, the representations $\pi(\mathfrak{A}(\mathcal{O}^s))$ and $\pi'(\mathfrak{A}(\mathcal{O}^s))$ of $\mathfrak{A}(\mathcal{O}^s)$ are unitarily equivalent ; where $\mathfrak{A}(\mathcal{O}^s)$ is the C*-algebra generated by $\{ \mathfrak{A}(\mathcal{O}_1) \mid \mathcal{O}_1$ is a region space-like w.r.t. $\mathcal{O} \}$.

<u>Remark</u> In general, the unitary operator effecting the equivalence will depend on the region $\mathcal{O}$.

Clearly, two representations are locally equivalent if they are strongly locally equivalent.

We shall see now that, allthough they are unitarily inequivalent, the charge sectors ( $\mathcal{H}_q, \Pi_q$ ) of the charged field are strongly locally equivalent.

The idea of the proof is simple : we write $\phi(f)$ as

$\phi(f) = VM$ where $M^2 = \phi(f)^* \phi(f)$ and V is unitary.

Since $\phi^*(f) \phi(f)$ carries no charge, the charge must be carried by V. Choosing f with supp $f \subset \mathcal{O}$ and using local commutativity, we see that V effects the equivalence between ( $\mathcal{K}_q$, $\Pi_q(\mathcal{R}(\mathcal{O}^s))$ ) and ( $\mathcal{K}_{q+1}, \Pi_{q+1}(\mathcal{R}(\mathcal{O}^s))$ ).

6.5.3. Theorem

The representations ( $\mathcal{K}_q, \Pi_q$ ), $q = 0, \pm 1..$ are strongly locally equivalent

Proof

Since strong local equivalence is an equivalence relation, we need only prove that ( $\mathcal{K}_q$ , $\Pi_q$ ) and ( $\mathcal{K}_{q+1}$, $\Pi_{q+1}$ ) are strongly locally equivalent for any $q = 0, \pm 1, \pm 2, \ldots$.

Let $\mathcal{O}$ be a given region, and let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$ with supp $f \subset \mathcal{O}$.

Let $\phi(f) = VM$, $M^2 = \phi(f)^* \phi(f)$ be the polar decomposition of $\phi(f)$. Since $\phi(f)$ has no kernel (6.1.8) we see that V is unitary on $\mathcal{K}$ .

It is easy to see that $\phi(f)$ maps $D(\phi(f)) \cap \mathcal{K}_q$ into $\mathcal{K}_{q+1}$ , and by using the fact that D is a domain of analytic (but <u>not entire</u>) vectors for $M^2$ (by the estimates of 6.1.2.). We see that $M^2$ commutes with $U(\theta)$, all $0 \leq \theta < 2\pi$ . Hence M maps $D(\phi(f)) \cap \mathcal{K}_q$ into $\mathcal{K}_q$ .
Moreover, M is self-adjoint and $M > 0$ (by 6.1.8) and so $\operatorname{ran}(M \restriction D(\phi(f)) \cap \mathcal{K}_q)$ is dense in $\mathcal{K}_q$ . It follows that V maps $\mathcal{K}_q$ into $\mathcal{K}_{q+1}$ .

By the same argument applied to $\phi^*(f) = V^*M$, we see that V* maps $\mathcal{K}_{q+1}$ into $\mathcal{K}_q$ .

Thus V maps $\mathcal{K}_q$ unitarily onto $\mathcal{K}_{q+1}$ .

Now, by the spectral theorem, V commutes with $\mathcal{F}(\mathcal{O})'$, i.e. $V \in \mathcal{F}(\mathcal{O})'' = \mathcal{F}(\mathcal{O})$. In particular, V commutes with $\mathfrak{A}(\mathcal{O}^s)$, and, since $\mathfrak{A}(\mathcal{O}^s)$ leaves each $\mathcal{K}_q$ invariant, we have

$$V \; \pi_q(\mathfrak{A}(\mathcal{O}^s)) = \pi_{q+1}(\mathfrak{A}(\mathcal{O}^s)) \; V \restriction \mathcal{K}_q ,$$

all $q$.

QED.

<u>Remark</u> It is easy to see from this result that the representations $(\mathcal{K}_q, \pi_q)$ of $\mathfrak{A}$ all have the same kernel, and so are faithful. This gives an alternate proof of this fact without appealling to Fell's theorem. On the other hand, by Fell's theorem, 6.5.3. implies that the charge sectors are physically equivalent.

Let $A \in \mathcal{O}\!\mathcal{l}$ . Then, since $V \in \mathcal{F}$ , we have $V^*A V \in \mathcal{F}$ , and $V^*A V$ maps each $\mathcal{K}_q$ into itself ; i.e. $V^* AV$ commutes with the gauge group. Hence $V^*AV \in \mathcal{O}\!\mathcal{l}$ . We see, therefore, that the mapping $A \longrightarrow \gamma(A) = V^*AV$ is an automorphism of $\mathcal{O}\!\mathcal{l}$ .
Moreover , if $A \in \mathcal{O}\!\mathcal{l}(\mathcal{O}^s)$, we have $\gamma(A) = A$ ; i.e.

$\gamma \restriction \mathcal{O}\!\mathcal{l}((\mathcal{O}^s)$ is the identity automorphism of $\mathcal{O}\!\mathcal{l}(\mathcal{O}^s)$.

We could therefore call $\gamma$ an automorphism localized in $\mathcal{O}$ .

Consider now the representation $\pi$ of $\mathcal{O}\!\mathcal{l}$ acting on $\mathcal{K}_o$ given by $\pi(A) = \pi \circ \gamma(A)$.
Then, for $z \in \mathcal{K}_o$ , we have

$$\pi(A)\, z = \pi_0 \circ \gamma(A)\, z = \gamma(A)\, z$$
$$= V^* AV z = V^* \pi_1(A)\, Vz$$

since $Vz \in \mathcal{K}_1$ . So we have proved that $\pi_1$ is unitarily equivalent to $\pi$ , i.e. $\pi_1 \simeq \pi_0 \circ \gamma$.
Similarly, $\pi_q \simeq \pi_0 \circ \gamma^q$ .

In other words, up to unitary equivalence, <u>the sectors $(\mathcal{K}_q, \pi_q)$ are given by localized automorphisms acting in the charge zero sector</u>.

A general discussion of this situation is the subject of the next chapter.

## 7. <u>The General Structure of Sectors</u>

In the last chapter, we constructed the observable algebra $\mathcal{O}\!\mathcal{l}$ from the field algebra $\mathcal{F}$ and found all the sectors which occurred. The same analysis has been carried ont by Doplicher, Haag and Roberts (1969 a) for a general field algebra and gauge group, G. They find that there is a one-one correspondence between the sectors ( i.e. unitary equivalence classes of representations of the algebra of observables) occurring and inequivalent irreducible unitary representations of G.

We should like to consider the converse problem of constructing the sectors given the algebra of observables in the vacuum sector. As in the case of the charged field, we think of the sectors being obtained from the vacuum sector by acting with charge carrying fields. These must be constructed, and one can ask whether they are bose or fermi fields or neither. This analysis was initiated by Borchers and re-examined by Doplicher et al.
We shall follow the treatment of Doplicher, Haag and Roberts (1969 b, 1971, 1974). (See also Haag (1970)).

## 7.1. States of Interest for Strong Interaction Physics

The aim is to find the sectors, given an algebra of observables, $\mathfrak{A}$ . That is, what are the irreducible representations of $\mathfrak{A}$ ? This is a very difficult question to answer, so to make the problem more tractable we appeal to physical arguments to single out some of these representations. To any representation ( $\mathcal{H}$ , $\pi$ ) of $\mathfrak{A}$ , we can associate a family of states - the vector states given by ( $\mathcal{H}$ , $\pi$ )

( which are pure if ( $\mathcal{H}$ , $\pi$ ) is irreducible), or, more generally, the density matrices ( or equivalently , ultraweakly continuons states) of ( $\mathcal{H}$ , $\pi$ ) Evidently, a unitarily equivalent representation will give the same set of such states.

Conversely, any pure state on $\mathfrak{A}$ defines, by the GNS construction, an irreducible representation of $\mathfrak{A}$ . We see then, a certain correspondence between representations of $\mathfrak{A}$ and states on $\mathfrak{A}$ . We shall single out a family of representations of $\mathfrak{A}$ by appealling to physical arguments to single out a collection of states of interest ( in elementary particle physics).

Since we are not trying to describe a theory of cosmology, but rather elementary particle physics, it is natural to assume that our "laboratory" is isolated in an otherwise empty universe - that is, we require our states of interest to behave like the vacuum in remote regions of space. Let us formulate this concept more precisely.

From now on, we suppose that we are given a quasi-local C*-algebra $\mathfrak{A}$ of observables : $\mathfrak{A}$ is generated by C*-algebras $\mathfrak{A}(\mathcal{O})$ , $\mathcal{O}$ a region. The $\mathfrak{A}(\mathcal{O})$'s are supposed to satisfy the axioms of isotony and causality and Poincaré covariance .

Let $\omega_0$ be a vacuum state on $\mathfrak{A}$ in the sense of 5.4.1, and let ( $\mathcal{H}_0$ , $\pi_0$ ) be the associated GNS representation. We shall suppose that ( $\mathcal{H}_0$, $\pi_0$ ) is irreducible, or equivalently, that $\omega_0$ is pure, and also that ( $\mathcal{H}_0$ , $\pi_0$ ) is faithful. The representation ( $\mathcal{H}_0$, $\pi_0$) is called the vacuum sector.

The preceding discussion leads us to the following.

### 7.1.1. Definition

Let $\omega$ be a state on $\mathfrak{A}$. We say that $\omega$ agrees asymptotically with the vacuum, $\omega_0$, if, for any sequence $\{\mathcal{O}_n\}$ of increasing regions with $\bigcup_n \mathcal{O}_n = \mathbb{M}$, we have

$$\lim_{n \to \infty} \| (\omega - \omega_0) \restriction \mathfrak{A}(\mathcal{O}_n^s) \| = 0$$

where, we recall, $\mathfrak{A}(\mathcal{O}^s)$ is the C*-algebra generated by the $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1$ space - like w.r.t. $\mathcal{O}$.

In other words, $\omega$ begins to look like the vacuum far away - the convergence being in norm. This requirement is too stringent to apply in a theory of electromagnetism. Indeed, by Gauss' law the charge within a region is given by the flux of electric field strength through any surrounding sphere - however large. We would not therefore expect a state with non-zero electric charge to approximate the vacuum in the sense of 7.1.1. However, an analysis of the sectors given by states satisfying 7.1.1. (together with some further assumptions) can be carried out. One can hope that such states are enough to describe purely strong interaction physics.

However, within this realm one must also consider non-abelian gauge groups such as SU(2) ( Drühl, Haag and Roberts (1970), Doplicher and Roberts (1972), Haag (1970)).

It is an open problen as to what happens if 7.1.1. is replaced by a weaker condition, such as ; for each local observable A (A $\in \mathfrak{A}(\mathcal{O})$, some $\mathcal{O}$ )

$$\lim_{a \to \infty} (\omega - \omega_0)(\alpha_a(A)) = 0$$

where a is a space-like vector.

Let ( $\mathcal{H}_1$ , $\pi_1$ ) and ( $\mathcal{H}_2$ , $\pi_2$ ) be representations of $\mathfrak{A}$ . We have already defined the notion of strong local equivalence (6.5.2). We shall say that the representations are strongly locally equivalent for a given region $\mathcal{O}$ if and only if $\pi_1 \restriction \mathfrak{A}(\mathcal{O}')$ and $\pi_2 \restriction \mathfrak{A}(\mathcal{O}')$ are unitarily equivalent.

7.1.2. <u>Theorem</u>

Let ( $\mathcal{H}$ , $\pi$ ) be a representation of $\mathfrak{A}$ and suppose ( $\mathcal{H}$ , $\pi$ ) is strongly locally equivalent to ( $\mathcal{H}_0$ , $\pi_0$ ) for $\mathcal{O}$ , some region $\mathcal{O}$ . Let $\omega$ be any ultraweakly continuous state in ( $\mathcal{H}$ , $\pi$ ). Then $\omega$ asymptotically agrees with $\omega_0$ .

<u>Proof</u>

Suppose $\omega$ and $\omega_0$ do not agree asymptotically. Then there exists a sequence $\{\mathcal{O}_n\}$ of regions with $\mathcal{O}_n \subset \mathcal{O}^{\circ}_{n+1}$ and $\bigcup_n \mathcal{O}_n = \mathbb{M}$ , and $\varepsilon > 0$ such that

$$\| (\omega - \omega_0) \upharpoonright \mathfrak{A}(\mathcal{O}_n^s) \| \geq \varepsilon$$

for all n.

By the definition of the norm of a functional, there exists $B_n \in \mathfrak{A}(\mathcal{O}_n^s)$ with $\| B_n \| = 1$ such that

$$| (\omega - \omega_0) B_n | > \varepsilon/2$$

for all n.

Now, $\| \pi_0(B_n) \| = \| B_n \|$ and so $\{ \pi_0(B_n) \}$ is a sequence in the unit ball of $\mathcal{B}(\mathcal{H}_0)$, which is weakly compact. Hence there is an operator $B \in \mathcal{B}(\mathcal{H}_0)$ with $\| B \| \leq 1$ and a net $B_\alpha$ in $\{ B_n \}$ such that $\pi_0(B_\alpha)$ converges weakly to B. Since $\mathcal{O}_n^s \supset \mathcal{O}_{n+1}^s$, and $B_n \in \mathfrak{A}(\mathcal{O}_n^s)$, we see that B commutes with each $\pi_0(\mathfrak{A}(\mathcal{O}_n))$. That is,

$$B \in \left\{ \bigcup_n \pi_0(\mathfrak{A}(\mathcal{O}_n)) \right\}'$$

This implies that $B \in \pi_0(\mathfrak{A})' = \mathbb{C}\mathbb{1}$ since $(\mathcal{H}_0, \pi_0)$ is irreducible. Write $B = c\, \pi_0(\mathbb{1})$, $c \in \mathbb{C}$, $|c| \leq 1$. Then $\pi_0(B_\alpha)$ converges weakly to $\pi_0(c\mathbb{1})$.

Now, by hypothesis, there is a unitary $V : \mathcal{H}_0 \to \mathcal{H}$ such that

$$\pi(A) = V \pi_0(A) V^*$$

for all $A \in \mathfrak{A}(\mathcal{O}^s)$.

For sufficiently large n, $\mathcal{O} \subset \mathcal{O}_n$, i.e. $\mathcal{O}_n^s \subset \mathcal{O}^s$, and so, for sufficiently large $\alpha$, $B_\alpha \in \mathfrak{A}(\mathcal{O}^s)$. By the above unitary equivalente, it follows that $\pi(B_\alpha)$ converges weakly to $c\,\pi(\mathbb{1})$.

Since $\| \pi(B_\alpha) \| \leq 1$, the convergence is also in the ultraweak topology, and so $\omega(B_\alpha)$ converges to **c**.

On the other hand, is a vector state in ( , ) so

( B ) converges to C.

Hence ( - ) ( B ) = 0, which contradicts

QED.

## 7.2. Borchers' Property

In order to obtain a converse to 7.1.2, we need to make a technical assumption concerning the representation ( $\mathcal{H}$, $\pi$ ) of $\mathfrak{A}$ . Following Doplicher et al (1971) we shall call this property B.

If $\mathcal{O}$ is a region, we denote by $\mathcal{C}_\pi(\mathcal{O})$ the von Neumann algebra $\pi(\mathfrak{A}(\mathcal{O}^s))'$.

### 7.2.1. Definition

A representation ( $\mathcal{H}$, $\pi$ ) of $\mathfrak{A}$ is said to satisfy property B if, for any region $\mathcal{O}$ included in the interior of another region $\mathcal{O}_1$ , and for any non-zero projection E in $\mathcal{C}_\pi(\mathcal{O})$ , there is an isometry W $\in \mathcal{C}_\pi(\mathcal{O}_1)$ such that WW* = E and W*W = $\mathbb{1}$.

This property was shown to hold by Borchers (1967b) under the assumptions of the spectrum condition and additivity in ( $\mathcal{H}$, $\pi$ ) (Also $\pi$ must be locally regular : since $\pi_0$ is faithful, we may consider $\pi(\mathfrak{A}(\mathcal{O}))$ as a representation of $\pi_0(\mathfrak{A}(\mathcal{O}))$. We need to know that $\pi$ extends to a representation of $\pi_0(\mathfrak{A}(\mathcal{O}))''$. We also need irreducibility of $\pi$ ; this allows $W^*W = \mathbb{1}$, otherwise we would have $W^*W = F$, some projection F). Property B holds in ( $\mathcal{H}_0$, $\pi_0$ ).

Suppose, then, that ( $\mathcal{H}$, $\pi$ ) satisfies property B. Let ( $\mathcal{H}_1$, $\pi_1$ ) be a subrepresentation of ( $\mathcal{H}$, $\pi \restriction \mathfrak{A}(\mathcal{O}^s)$ ) Now consider ( $\mathcal{H}_1$, $\pi_1 \restriction \mathfrak{A}(\mathcal{O}_1^s)$ ) as a representation of $\mathfrak{A}(\mathcal{O}_1^s)$. ( $\mathcal{O}$ is contained in the interior of $\mathcal{O}_1$ ). We claim that this is unitarily equivalent to ( $\mathcal{H}$, $\pi \restriction \mathfrak{A}(\mathcal{O}_1^s)$ ). Indeed, the projection E onto $\mathcal{H}_1$ belongs to $\pi(\mathfrak{A}(\mathcal{O}^s))' = \mathcal{E}_\pi(\mathcal{O})$. Hence, there is an isometry W as above in $\mathcal{E}_\pi(\mathcal{O}_1)$, i.e. commuting with $\pi(\mathfrak{A}(\mathcal{O}_1^s))$. We have then,

$$W \pi \restriction \mathfrak{A}(\mathcal{O}_1^s) = \pi \restriction \mathfrak{A}(\mathcal{O}_1^s)\, W$$

$$= \pi_1 \restriction \mathfrak{A}(\mathcal{O}_1^s)\, W$$

since $\mathcal{H}_1$, is the final space of W. We see that W effects the Claimed equivalence.

### 7.2.2. <u>Proposition</u>

Let ( $\mathcal{H}_1$, $\pi_1$ ) and ( $\mathcal{H}_2$, $\pi_2$ ) be representations $\mathfrak{A}$ satisfying property B. Let $\mathcal{O}$ be a region and suppose that $(\mathcal{H}_1, \pi_1 \upharpoonright \mathfrak{A}(\mathcal{O}^s))$ and $(\mathcal{H}_2, \pi_2 \upharpoonright \mathfrak{A}(\mathcal{O}^s))$ are non-disjoint. Then, for any region $\mathcal{O}_1$ containing $\mathcal{O}$ in its interior, we have $(\mathcal{H}_1, \pi_1 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ and $(\mathcal{H}_2, \pi_2 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ are unitarily equivalent.

Proof

By assumption (see 2.6.4), there are sub representations $(\hat{\mathcal{H}}_1, \hat{\pi}_1 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ and $(\hat{\mathcal{H}}_2, \hat{\pi}_2 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ of $(\mathcal{H}_1, \pi_1 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ and $(\mathcal{H}_2, \pi_2 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$, respectively, which are unitarily equivalent.

By the previons remark, however, $(\hat{\mathcal{H}}_1, \hat{\pi}_1 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ is unitarily equivalent to $(\mathcal{H}_1, \pi_1 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ for any region $\mathcal{O}_1$ containing $\mathcal{O}$ in its interior. Similary, $(\hat{\mathcal{H}}_2, \hat{\pi}_2 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s)) \simeq (\mathcal{H}_2, \pi_2 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ and the result follows.

QED.

7.2.3. Theorem

Let $\omega$ be a state on $\mathfrak{A}$, and suppose that $\omega$ asymptotically agrees with the vacuum, $\omega_0$. If the GNS representation $(\mathcal{H}, \pi)$, associated to $\omega$, satisfies property B, then there is a region $\mathcal{O}$ such that

$$\pi \upharpoonright \mathfrak{A}(\mathcal{O}^s) \simeq \pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}^s),$$

i.e. ( $\mathcal{H}$, $\pi$ ) and ( $\mathcal{H}_0$, $\pi_0$ ) are strongly locally equivalent for $\mathcal{O}$ .

<u>Proof</u>

Let $\{\mathcal{O}_n\}$ be a sequence of increasing regions which exhaust $\mathbb{M}$ . Since $\omega$ and $\omega_0$ asymptotically agree, we have

$$\|(\omega - \omega_0) \upharpoonright \mathfrak{A}(\mathcal{O}_n^s)\| < 2$$

for large n.

Hence, by the theorem of Glimm and Kadison (2.6.6), the states $\omega \upharpoonright \mathfrak{A}(\mathcal{O}_n^s)$ and $\omega_0 \upharpoonright \mathfrak{A}(\mathcal{O}_n^s)$ induce non-disjoint representations of $\mathfrak{A}(\mathcal{O}_n^s)$ . These representations are sub representations of $\pi_\omega \upharpoonright \mathfrak{A}(\mathcal{O}_n^s)$ and $\pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}_n^s)$ , respectively, hence a fortiori, the latter are not disjoint.

Let $\mathcal{O}$ be a region containing $\mathcal{O}_n$ in its interior. Then, by 7.2.2, ( $\mathcal{H}$, $\pi \upharpoonright \mathfrak{A}(\mathcal{O}^s)$) is unitarily equivalent to ( $\mathcal{H}_0$, $\pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}^s)$).

QED.

## 7.3. <u>Duality</u>

Suppose $\mathcal{O}_1$ is a region, and suppose $(\mathcal{H}, \pi)$ is a representation of $\mathfrak{A}$ satisfying

$$\pi \restriction \mathfrak{A}(\mathcal{O}^s) \simeq \pi_0 \restriction \mathfrak{A}(\mathcal{O}^s)$$

for some double cone $\mathcal{O}$. Then, if $(\mathcal{H}, \pi)$ carries a representation of the translation group, we can translate $\mathcal{O}_1$ into the set $\mathcal{O}^s$. It follows that

$$\pi \restriction \mathfrak{A}(\mathcal{O}_1) \simeq \pi_0 \restriction \mathfrak{A}(\mathcal{O}_1).$$

Since $\pi_0$ is faithful, we can think of $\pi$ as a representation of $\pi_0(\mathfrak{A})$. The above unitary equivalence implies that $\pi$ extends to a representation of $\pi_0(\mathfrak{A}(\mathcal{O}_1))''$, this being unitarily equivalent to $\pi_0(\mathfrak{A}(\mathcal{O}_1))''$.

We see then, that if we restrict our attention to representations $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ which are strongly locally equivalent to $(\mathcal{H}_0, \pi_0)$ for some $\mathcal{O}$, then (if we also have translations implemented in $(\mathcal{H}, \pi)$) $\pi$ defines a representation of $\pi_0(\mathfrak{A}(\mathcal{O}_1))''$ for any region $\mathcal{O}_1$.
We may, therefore, always consider local von Neumann algebras (See Haag, Kadison and Kastler (1970)).

From now on, we shall consider $\mathfrak{A}$ as being defined by its vacuum representation. More, precisely, we suppose that $\mathfrak{A}$ is a C*-algebra of operators on a Hilbert space $\mathcal{H}_0$ containing a vacuum vector $\Omega_0$.
We shall suppose that the local algebras $\mathfrak{A}(\mathcal{O})$ are weakly closed, i.e. von Neumann algebras. The symbol $\pi_0$ is then redundant, but we often use it for emphasis.

To proceed further, we shall need some more structure. The concept of duality plays a central rôle in the work of Doplicher, Haag and Roberts.

Consider the statement of locality. If $\mathcal{O}$ is a region, then $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}^s)$ commute. In the vacuum representation, this can be expressed as

$$\pi_0(\mathfrak{A}(\mathcal{O})) \subset \pi_0(\mathfrak{A}(\mathcal{O}^s))'.$$

Suppose $X \in \pi_0(\mathfrak{A}(\mathcal{O}^s))'$, does X belong to $\pi_0(\mathfrak{A}(\mathcal{O}))$ ? In general, the answer is no (Indeed, this could never be true for all $X \in \pi_0(\mathfrak{A}(\mathcal{O}^s))'$ if $\pi_0(\mathfrak{A}(\mathcal{O}))$ were not a von Neumann algebra).

7.3.1. <u>Definition</u>

We say that a representation ( $\mathcal{H}$, $\pi$ ) of $\mathfrak{A}$ satisfies duality for a region $\mathcal{O}$ if and only if

$$\pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}^s))'.$$

In particular, this implies that $\pi(\mathfrak{A}(\mathcal{O}))$ is a von Neumann algebra.

<u>We shall assume that duality holds for all regions in the vacuum sector</u>. This means that

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}^s)'$$

for any region $\mathcal{O}$ . It is important here that we adopt the definition of a region as a <u>double cone</u>. Indeed, for the free field, Araki (1964a) has shown that duality does hold for double cones but that it does <u>not</u> hold for a region given by two double cones, one on top of the other. We should also note that duality fails even for double cones for some generalized free fields (Landau (1974)).

In can be seen that duality implies a maximality of the local algebras $\mathfrak{A}(\mathcal{O})$ (Haag (1970)). Indeed, let $\mathcal{R}(\mathcal{O}) \supset \mathfrak{A}(\mathcal{O})$. Then, if $\mathcal{R}(\mathcal{O})$ is to be interpreted as an algebra of observables within $\mathcal{O}$, we must have, by locality, that $\mathcal{R}(\mathcal{O})$ commutes with $\mathfrak{A}(\mathcal{O}^s)$, i.e. $\mathcal{R}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}^s)'$. But then duality gives $\mathcal{R}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O})$, and so $\mathcal{R}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})$. It is unknown whether maximality implies duality (see Haag (1970)).

## 7.4. Localized Monomorphisms

We recall that a monomorphism of a C*-algebra is an injective * - homomorphism.

Given a representation $(\mathcal{H}, \pi)$ of a C*- algebra, $\mathcal{R}$, and a monomorphism $\rho$, we can define another representation $(\mathcal{H}, \hat{\pi})$ on the same Hilbert space by $\hat{\pi}(A) = \pi \circ \rho(A)$, $A \in \mathcal{R}$.

### 7.4.1. Definition

Let $\rho$ be a monomorphism on $\mathfrak{A}$. We say that $\rho$ is localized in a region $\mathcal{O}$ if

$\rho(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}^s)$. We write $\mathcal{M}(\mathcal{O})$ for the collection of such monomorphisms.

The importance of localized monomorphisms lies in the following.

7.4.2. <u>Theorem</u>

Let $(\mathcal{H}, \pi)$ be a faithful representation of $\mathfrak{A}$. Then $(\mathcal{H}, \pi)$ and $(\mathcal{H}_o, \pi_o)$ are strongly locally equivalent for a double cone $\mathcal{O}_1$ if and only if there is a monomorphism $\rho$, localized in $\mathcal{O}_1$, such that $(\mathcal{H}, \pi)$ is unitarily equivalent to $(\mathcal{H}_o, \pi_o \circ \rho)$.

<u>Proof</u>

Suppose $(\mathcal{H}, \pi)$ and $(\mathcal{H}_o, \pi_o)$ are strongly locally equivalent for $\mathcal{O}_1$. Then there is a unitary operator $V : \mathcal{H}_o \rightarrow \mathcal{H}$ such that

$$\pi(A)\, V = V\, \pi_o(A)$$

for all $A \in \mathfrak{A}(\mathcal{O}_1^s)$.

For $A \in \mathfrak{A}$, define $\rho(A) = V^* \pi(A) V$. Then $\rho(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}_1^s)$.

Let $B \in \mathfrak{A}(\mathcal{O}^s)$ where $\mathcal{O} \supset \mathcal{O}_1$. Then, for $A \in \mathfrak{A}(\mathcal{O})$ we have

$$\begin{aligned}
\rho(A)\ \pi_o(B) &= V^*\ \pi(A)\ V\ \pi_o(B) \\
&= V^*\ \pi(A)\ V\ \pi_o(B)\ V^*V \\
&= V^*\ \pi(A)\ \pi(B)\ V \text{ since } B \in \mathfrak{A}(\mathcal{O}^s) \subset \mathfrak{A}(\mathcal{O}_1^s) \\
&= V^*\ \pi(AB)\ V = V^*\ \pi(BA)\ V \\
&= V^*\ \pi(B)\ V\ V^*\ \pi(A)\ V \\
&= \pi_o(B)\ V^*\ \pi(A)\ V \\
&= \pi_o(B)\ \rho(A).
\end{aligned}$$

Thus $\rho(A)$ commutes with $\mathfrak{A}(\mathcal{O}^s)$, and so, by duality for $(\mathcal{H}_o, \pi_o)$, $\rho(A) \in \mathfrak{A}(\mathcal{O})$. In other words, $\rho : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(\mathcal{O})$. Moreover, $\pi$ faithful implies $\rho$ is injective, and so, by continuity, $\rho$ extends to a monomorphism of $\mathfrak{A}$. Since $\rho(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}_1^s)$, we have $\rho \in M(\mathcal{O}_1)$.

By construction, we have $\pi(\mathfrak{A}) = V\ \rho(\mathfrak{A})\ V^*$, i.e. $\pi(\mathfrak{A}) = V\ \pi_o \circ \rho(\mathfrak{A})\ V^*$, and so $\pi \simeq \pi_o \circ \rho$.

Conversely, let $\rho \in M(\mathcal{O}_1)$, and set $\hat{\pi} = \pi_o \circ \rho$. Then, for $A \in \mathfrak{A}(\mathcal{O}_1^s)$, we have

$$\hat{\pi}(A) = \pi_o \circ \rho(A) = \pi_o(A),$$

and so $\hat{\pi}$ and $\pi$ define the same representation of $\mathfrak{A}(\mathcal{O}_1^s)$. Hence, if $(\mathcal{H}, \pi) \simeq (\mathcal{H}_o, \hat{\pi})$, we clearly have the required strong local equivalence.

QED.

7.4.2. says that sectors strongly locally equivalent to the vacuum sector are given by localized monomorphisms.

## 7.5. Localized Automorphisms

### 7.5.1 Definition

We denote by $\Gamma(\mathcal{O})$ those automorphisms of $\mathfrak{A}$ localized within $\mathcal{O}$; $\Gamma(\mathcal{O}) = M(\mathcal{O}) \cap \mathrm{Aut}\,\mathfrak{A}$.

We can improve upon 7.4.2. under the additional hypothesis of duality in $(\mathcal{H}, \pi)$.

### 7.5.2. Theorem

Let $(\mathcal{H}, \pi)$ be a faithful representation of $\mathfrak{A}$, and suppose that $(\mathcal{H}, \pi)$ and $(\mathcal{H}_0, \pi_0)$ are strongly locally equivalent for some double cone $\mathcal{O}_1$. Suppose that duality holds in $(\mathcal{H}, \pi)$ for all $\mathcal{O} \supset \mathcal{O}_1$, i.e.

$$\pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}^{s}))'$$

for all $\mathcal{O} \supset \mathcal{O}_1$. Then there is an automorphism $\gamma$ localized in $\mathcal{O}_1$, such that $(\mathcal{H}, \pi)$ is unitarily equivalent to $(\mathcal{H}_0, \pi_0 \circ \gamma)$.

**Proof**

By 7.4.2, $(\mathcal{H}, \pi)$ is unitarily equivalent to ( $\mathcal{H}_o , \pi_o \circ \gamma$ ) for some $\gamma$ in $M(\mathcal{O}_1)$. We must show that $\gamma \in \mathrm{Aut}\, \mathfrak{A}$ , i.e. that $\gamma$ is onto $\mathfrak{A}$. With the notation of 7.4.2. (but with $\gamma$ replacing $\rho$ , we have

$$\gamma(A) = V^* \pi(A) V$$

for all $A \in \mathfrak{A}$.

Let $\mathcal{O} \supset \mathcal{O}_1$ and consider

$$\pi(\mathfrak{A}(\mathcal{O}^s))' = \{ V \pi_o(\mathfrak{A}(\mathcal{O}^s)) V^* \}' \quad \text{since} \quad \mathcal{O}^s \subset \mathcal{O}_1^s$$
$$= V \quad \pi_o(\mathfrak{A}(\mathcal{O}^s))' \quad V^*$$
$$= V \quad \pi_o(\mathfrak{A}(\mathcal{O})) \quad V^* \text{ by duality for } \pi_o .$$

But, by hypothesis,

$$\pi(\mathfrak{A}(\mathcal{O}^s))' = \pi(\mathfrak{A}(\mathcal{O})) .$$

Hence

$$V^* \pi(\mathfrak{A}(\mathcal{O})) V = \pi_o(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$$

for all $\mathcal{O} \supset \mathcal{O}_o$.

In other words, $\gamma(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$ for all $\mathcal{O} \supset \mathcal{O}_o$ . By isotony, such $\mathfrak{A}(\mathcal{O})$ are dense in $\mathfrak{A}$ , and we conclude that $\gamma(\mathfrak{A}) = \mathfrak{A}$ , i.e. $\gamma \in \mathrm{Aut}\, \mathfrak{A}$.

QED.

The converse is also true

7.5.3. <u>Theorem</u>

Let $\gamma \in \Gamma(\mathcal{O}_1)$, some double cone $\mathcal{O}_1$, and set $\hat{\pi} = \pi_0 \circ \gamma$. Then $(\mathcal{H}_0, \hat{\pi} \upharpoonright \mathfrak{A}(\mathcal{O}_1^s)) \simeq (\mathcal{H}_0, \pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ and for all $\mathcal{O} \supset \mathcal{O}_1$, we have

$$\hat{\pi}(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}^s))'$$

<u>Proof</u>

As in 7.4.2, it is trivial that $(\mathcal{H}_0, \hat{\pi} \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ and $(\mathcal{H}_0, \pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}_1^s))$ are unitarily equivalent.

Let $\mathcal{O} \supset \mathcal{O}_1$, be a double cone. We claim that $\gamma(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O})$.

Indeed, we have

$$[\gamma(\mathfrak{A}(\mathcal{O})), \mathfrak{A}(\mathcal{O}^s)] = \gamma[\mathfrak{A}(\mathcal{O}), \gamma^{-1}(\mathfrak{A}(\mathcal{O}^s))]$$

$$= \gamma[\mathfrak{A}(\mathcal{O}), \mathfrak{A}(\mathcal{O}^s)]$$

since $\mathfrak{A}(\mathcal{O}^s) \subset \mathfrak{A}(\mathcal{O}_1^s)$ and $\gamma^{-1} \in \Gamma(\mathcal{O}_1)$

$$= 0 .$$

Hence $\gamma(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O}^s)' = \mathfrak{A}(\mathcal{O})$ by duality for $\pi_0$. Applying the same argument to $\gamma^{-1}$, we obtain the result

$$\gamma(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O}) \quad \text{, for all } \mathcal{O} \supset \mathcal{O}_1 .$$

Thus, for $\mathcal{O} \supset \mathcal{O}_1$, we have

$$\pi(\mathfrak{A}(\mathcal{O})) = \gamma(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$$

$$= \mathfrak{A}(\mathcal{O}^s)' \quad \text{by duality for } \pi_0$$

$$= \gamma(\mathfrak{A}(\mathcal{O}^s))' \text{ since } \gamma \in \Gamma(\mathcal{O}_1),\ \mathcal{O}^s \subset \mathcal{O}_1^s$$

$$= \pi(\mathfrak{A}(\mathcal{O}^s))' .$$

QED.

7.5.2. and 7.5.3 say that those sectors which are strongly equivalent to the vacuum sector and satisfy duality correspond to those sectors given by localized automorphisms.

It has been shown by Doplicher et al (1969a) that in the case of a non-abelian gauge group, duality cannot hold in any sector other than the vacuum sector. This means that these sectors will be given by localized morphisms, but <u>not</u> automorphisms.

**7.5.4. <u>Proposition</u>**

Let ( $\mathcal{H}$ , $\pi$ ) be a representation given by $\gamma \in$ Aut $\mathfrak{A}$.
Then $(\mathcal{H}, \pi)$ is irreducible.

<u>Proof</u>

By hypothesis, $(\mathcal{H}, \pi) \simeq (\mathcal{H}_0, \pi_0 \circ \gamma)$.
Since $\gamma \in$ Aut $\mathfrak{A}$, $\gamma(\mathfrak{A}) = \mathfrak{A}$ and so $\pi_0(\gamma(\mathfrak{A}))$ and $\pi_0(\mathfrak{A})$ are equal as sets of operators. The result follows from the irreducibility of $(\mathcal{H}_0, \pi_0)$.

QED.

If ( $\mathcal{H}$ , $\pi$ ) is given by an automorphism, then it is cleary faithful. Thus automorphisms always give rise to faithfull irreducible representations of $\mathfrak{A}$ .

To end this chapter, we will just make a few remarks concerning the results of Doplicher et al. Rather that present their construction of charge carrying fields, which is somewhat technical, we will, in the next chapter, carry out such a construction for a simple two-dimensional model.

A sector is said to be covariant if it carries a strongly continuous unitary representation of the (covering group of the) Poincaré group implementing the corresponding automorphism of $\mathfrak{A}$ . Denote by $\Gamma_c$ the set of those $\gamma \in \Gamma$ which lead to covariant sectors.

## 7.6. Some Properties of the Sectors

### 7.6.1. Definition

We recall that an inner automorphism of $\mathfrak{A}$ is one of the form $A \longrightarrow \sigma_U(A) = UAU^*$, where $U \in \mathfrak{A}$ is unitary. Denote by $\mathcal{J}$ the group of inner automorphisms of $\mathfrak{A}$ which are localized. Let us denote by $\Gamma$ the group $\bigcup_{\mathcal{O}} \Gamma(\mathcal{O})$ .

Clearly, if $\mathcal{O}_1 \subset \mathcal{O}_2$ , then $\Gamma(\mathcal{O}_1) \subset \Gamma(\mathcal{O}_2)$. We alsonote that $\mathcal{J}$ is a normal subgroup of $\Gamma$ : for if $\gamma \in \Gamma$ , $\sigma_U \in \mathcal{J}$ , then $\gamma \sigma_U \gamma^{-1} = \sigma_{\gamma(U)} \in \mathcal{J}$ .

7.6.2. <u>Proposition</u>

If $\gamma_1, \gamma_2 \in \Gamma$, then $\pi_0 \circ \gamma_1 \simeq \pi_0 \circ \gamma_2$ if and only if $\gamma_1 \gamma_2^{-1} \in \mathcal{J}$.

<u>Proof</u>

Clearly $\gamma_1 \gamma_2^{-1} \in \mathcal{J}$ implies that $\pi_0 \circ \gamma_1 \simeq \pi_0 \circ \gamma_2$. Conversely, suppose $U : \mathcal{H}_0 \to \mathcal{H}_0$ is unitary and

$$\pi_0(\gamma_2(A)) = U \, \pi_0(\gamma_1(A)) \, U^*$$

for all $A \in \mathfrak{A}$.

Let $\mathcal{O}$ be such that $\gamma_1, \gamma_2 \in \Gamma(\mathcal{O})$. Then, for any $A \in \mathfrak{A}(\mathcal{O}^s)$, $\pi_0(A) = U \pi_0(A) U^*$. That is $U \in \pi_0(\mathfrak{A}(\mathcal{O}^s))' = \pi_0(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$ by duality.

Hence,

$$\gamma_2(A) = U \gamma_1(A) U^* = \sigma_U \gamma_1(A)$$

for all $A \in \mathfrak{A}$,

i.e. $\gamma_2 = \sigma_U \gamma_1$.

QED.

<u>Remark</u> $U$ is determined up to a phase by $\sigma_U$. Indeed, if $\sigma_U = \sigma_V$, then $\sigma_U \sigma_{V^*}$ = identity, which gives $UV^*A = AUV^*$ for all $A \in \mathfrak{A}$. By irreducibility of

$\alpha$ ( on $\mathcal{H}_0$ ), we see that $U = e^{i\theta} V$, $\theta \in \mathbb{R}$ .

If $\hat{\pi}$ denotes the unitary equivalence class (i.e. sector) containing the representation $\pi$ , we can define the maps $\gamma \longrightarrow (\pi_0 \circ \gamma)^{\wedge}$ . 7.6.2 says that there is a one-one correspondence between the sectors obtained from localized automorphisms and $\Gamma/\mathcal{J}$ . Since $\mathcal{J}$ is a normal subgroup of $\Gamma$, $\Gamma/\mathcal{J}$ forms a group in the obvious way. This means that the sectors obtained from $\Gamma$ inherit this group structure.

7.6.3. Proposition

Let $\gamma_1 \in \Gamma_c(\mathcal{O}_1)$, $\gamma_2 \in \Gamma_c(\mathcal{O}_2)$ with $\mathcal{O}_1$ and $\mathcal{O}_2$ space-like with respect to each other. If $\gamma_1$ and $\gamma_2$ lead to the same sector, then they commute.

7.6.4. Proposition

Let $\gamma_1, \gamma_2 \in \Gamma_c$ correspond to the same sector. Let U satisfy $\sigma_U \gamma_1 = \gamma_2$ (as in 7.6.2). If $\gamma_1$ and $\gamma_2$ have space-like separated localizations, then

$$\gamma_1(U) = \pm U .$$

The sign depends only on the sector and not on $\gamma_1$ and $\gamma_2$ explicitly.

For proof see Doplicher et al (1969b).

<u>Remark</u> This result is not true in 2 space-time dimensions.

Those sectors corresponding to a plus sign are called Bose sectors, those corresponding to a minus sign are called Fermi sectors. It is a consequence of this proposition that the charge carrying fields fall into two classes - Bose and Fermi.

This result depends on the $\gamma$'s being automorphisms. In the general case of monomorphisms, one is led to parastatistics.

7.6.5. <u>Proposition</u>

$T_c$ is a group, and $T_c/\mathcal{J}$ is abelian.

<u>Remark</u> This result together with 7.5.2 and 7.5.3 implies that the "superselection quantum numbers" of those covariant sectors strongly locally equivalent to the vacuum representation and satisfaying duality form an abelian group.

7.6.7. <u>Proposition</u>

The energy-momentum spectrum for the representations $\pi_0 \circ \gamma$, $\gamma \in T_c$, lies in the closed forward ligth-cone.

In fact, if $S(\gamma)$ denotes this spectrum in $\pi_0 \circ \gamma$ we have

$$S(\gamma\gamma') \supset S(\gamma) + S(\gamma').$$

For the proofs of these result and further discussion we refer to Doplicher et al (1969b). The more general case of monomorphisms is treated by Doplicher et al (1971, 1974), where one finds a particle-antiparticle structure and a spin and statistics theorem (See also Haag (1970)).

# 8 - A Two-Dimensional Model

We shall construct explicitly, for a simple model, some localized automorphism and their sectors and the "charge-carrying" fields mapping one sector into another. These will turn out to satisfy Bose or Fermi commutation relations, or neither, depending on the charge they carry. The "charge" will take continuous values - the sectors can be parametrized by $\mathbb{R}^2$.

We wish to emphasize that everything is constructed from a theory of free bosons. The model is suggested by one of Skyrme (1961). We shall follow the treatment of Streater and Wilde (1970).

## 8.1 - Heuristic Construction of Fermions from Bosons

To see how one can write down fermi fields in terms of bose fields, consider a free bose field $\phi$ in one space dimension. Then it satisfies the usual time-zero commutation relations

$$\left[ \phi(x) , \pi(x') \right] = i\, \delta(x - x')$$

where $\pi$ is the conjugate momentum at time-zero.

Integrating over x' from y to $\infty$, we have

$$\left[ \phi(x) , \int_y^\infty \pi(x')\, dx' \right] = i \int_y^\infty \delta(x - x')\, dx'$$

and, exponentiating, we obtain

$$\exp i\alpha\,\phi(x)\;\exp i\beta\int_y^\infty \Pi(x')\,dx'$$
$$= \exp i\beta\int_y^\infty \Pi(x')\,dx'\;\exp i\alpha\,\phi(x)\;\exp - i\alpha\beta\int_y^\infty \delta(x-x')\,dx'$$

Define

$$\psi(x) = \exp i\alpha\,\phi(x)\;\exp i\beta\int_x^\infty \Pi(x')\,dx'$$
$$\equiv U(x)\,V(x)$$

Then using the commutation relations for the U's and V's , we have

$$\psi(x)\;\psi(y) = U(x)\;V(x)\;U(y)\;V(y)$$
$$= U(x)\;U(y)\;V(x)\;V(y)\;\exp\; i\alpha\beta\int_x^\infty \delta(y-x')\,dx'$$
$$= U(y)\;V(y)\;U(x)\;V(x)\;\exp\left(i\alpha\beta\int_x^\infty \delta(y-x')\,dx' - i\alpha\beta\int_y^\infty \delta(x-x')\,dx'\right)$$
$$= \psi(y)\;\psi(x)\;e^{i\lambda}$$

where $\lambda = \alpha\beta\int_x^\infty \delta(y-x')\,dx' - \alpha\beta\int_y^\infty \delta(x-x')\,dx'$

If $x = y$, $\lambda = 0$ and $\psi(x)$, $\psi(y)$ commute, as expected. If $x < y$, $\lambda = \alpha\beta$, and if $y < x$, $\lambda = -\alpha\beta$. So if we choose $\alpha\beta = (2n+1)\pi$, then $\psi(x)$ and $\psi(y)$ anticommute for $x \neq y$. On the other hand, if $\alpha\beta = 2n\pi$, then $\psi(x)$ and $\psi(y)$ commute.

If $\alpha\beta$ is not $2n\pi$ nor $(2n+1)\pi$, then $\psi(x)$ and $\psi(y)$ neither commute nor anticommute - there is always a phase factor which depends on which field is to the left of which, i.e. depends on whether $x < y$ or $y < x$. We shall encounter precisely this situation for our charge-carrying fields in 8.5.

Of course, this discussion so far is completely heuristic. However, using the ideas of Doplicher, Haag and Roberts, we can rigorously define these fields (suitably regularized) as charge-carrying fields operating between different sectors. The point is that $\psi$ gives rise to an automorphism which leads to an inequivalent representation of the algebra of observables.

## 8.2 The Algebra of Observables

Let us now begin the rigorous construction of our model. We must first define our algebra of observables. We do this in terms of the free bose field of mass zero in two space-time dimensions. This choice appears to be necessary in order for our automorphisms to be localized in bounded regions.

Our Hilbert space, then, is the Fock space, $\mathcal{H}$, over $L^2(\mathbb{R}, dk)$, and the time-zero field and its conjugate momentum are given by

$$\phi(x) = (4\pi)^{-1/2} \int (e^{-ikx} a^*(k) + a(k)e^{ikx}) \frac{dk}{|k|^{1/2}}$$

$$\pi(x) = i\,(4\pi)^{-1/2} \int (e^{-ikx} a^*(k) - a(k)e^{ikx})\, |k|^{1/2}\, dk$$

We note that $\phi(x)$ is <u>not</u> an operator-valued distribution - we cannot smear with an arbitrary $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ because of the factor $|k|^{1/2}$ in the denominator. However, $\phi(f)$ is a well-defined self-adjoint operator (as usual) provided $\tilde{f}(k)/|k|^{1/2}$ belongs to $L^2(\mathbb{R}, dk)$. We will, in fact, restrict our test-functions, as suggested by Schroer (1963).

To define the local algebras, we will be concerned with functions of compact support, i.e. $f \in C_0^\infty(\mathbb{R})$. (Our test-functions will always be taken to be real-valued, even if this is not explicitly stated).

We see that if $f \in C_0^\infty(\mathbb{R})$, then $\tilde{f}(k)/|k|^{1/2}$ is square-integrable if and only if $\tilde{f}(0) = 0$. It is clearly necessary. On the other hand, if $\tilde{f}(0) = 0$, then $\int f(x)\,dx=0$. Hence f is the derivative of $h(x) = \int_{-\infty}^{x} f(y)\,dy$, and $h \in C_0^\infty(\mathbb{R})$. Then $\tilde{f}(k) = -ik\,\tilde{h}(k)$, and so $\tilde{f}(k)/|k|^{1/2}$ is square-integrable.

Let $\mathcal{D}_0 = \{ f \in C_0^\infty(\mathbb{R}) \mid \tilde{f}(0) = 0 \}$.

By abuse of notation we will write $\mathcal{D}$ for $C_0^\infty(\mathbb{R})$. Then, for $(f,g) \in \mathcal{D}_0 \times \mathcal{D}$, we can define the fields $\phi(f)$ and $\pi(g)$ as self-adjoint operators in $\mathcal{H}_0$.

Let us write $\mathcal{M}$ for $\mathcal{D}_0 \times \mathcal{D}$.

Any $(f,g) \in \mathcal{M}$ uniquely defines a real solution $\xi(x,t)$ of the wave equation $\Box\xi = 0$ by the requirement that

$$\dot{\xi}(x,0) = f(x), \qquad \xi(x,0) = g(x)$$

i.e. f and g are the Cauchy-data for $\xi$.

Let us also denote by $\mathcal{M}$ the family of real solutions to $\Box\xi = 0$ with Cauchy-data $(\dot{\xi}(x,0), \xi(x,0)) \in \mathcal{M}$.

The two-dimensional Poincaré group $\mathcal{P}$, acts on $\mathcal{M}$ as $\xi \longrightarrow \xi_{a,\Lambda}$ where

$$\xi_{a,\Lambda}(x,t) = \xi(\Lambda^{-1}(x - a_1, t - a_0)).$$

8.2.1 <u>Proposition</u> $\mathcal{M}$ is invariant under Poincaré transformations.

<u>Proof</u>

If $\xi$ has Cauchy-data in $\mathcal{D} \times \mathcal{D}$ then it is well-known that $\xi_{a,\Lambda}$ has too. We only have to show that $\dot{\xi}_{a,\Lambda}(x,0) \in \mathcal{D}_0$.

Now, if $\eta$ is any solution of $\Box \eta = 0$, then the Wronskian

$$\{\eta, \xi\} = \int \left( \eta(x,0)\, \dot{\xi}(x,0) - \dot{\eta}(x,0)\, \xi(x,0) \right) dx$$

is $\mathcal{P}$ - invariant. Setting $\eta(x,t) = 1$ implies that $\int \dot{\xi}(x,0)\, dx$ is invariant.

QED.

Let $\xi \in \mathcal{M}$. Then if $\xi \longleftrightarrow (f,g)$, we see that $\phi(f) - \pi(g)$ defines a self-adjoint operator in $\mathcal{H}_0$.

We denote this operator by $\{\phi, \xi\}$ - it is the Wronskian between $\phi(x,t)$ and $\xi(x,t)$.

We set $W(\xi) = \exp i\{\phi, \xi\}$. Then it is not difficult to show that, for $\xi_1, \xi_2 \in \mathcal{M}$

$$\begin{aligned} W(\xi_1)\, W(\xi_2) &= \exp -\frac{1}{2}\{\xi_1, \xi_2\}\, W(\xi_1 + \xi_2) \\ &= \exp - i\{\xi_1, \xi_2\}\, W(\xi_2)\, W(\xi_1). \end{aligned}$$

We call these the Segal-Weyl relations.

We can give $\mathcal{M}$ a local structure as follows.

Let $\mathcal{O} \subset \mathbb{R}^2$ be an open connected bounded set.

Let $I(x_1, x_2) = \{ (x,t) \mid t = vx + t_0,\ x_1 \leq x \leq x_2,\ |v| < 1 \}$ be a space-like interval inside $\mathcal{O}$. If $\xi \in \mathcal{M}$ and $(\dot{\xi}, \xi) \restriction I(-\infty, \infty)$ is zero outside $I(x_1, x_2)$, we say that $\xi$ lives on $I(x_1,x_2)$. By varying the space-like interval $I(x_1, x_2)$ over $\mathcal{O}$, we obtain a family of solutions. The linear span of these is denoted by $\mathcal{M}(\mathcal{O})$.

It is easy to see that any $\xi$ which lives on $I(x_1, x_2)$ is the Poincaré transform $\eta_{a,\Lambda}$, some $(a, \Lambda)$, of some $\eta \in \mathcal{M}$. The image of the support of $(\dot{\eta}, \eta)$ under $(a, \Lambda)$ is simply $I(x_1, x_2)$.

Evidently, $\mathcal{M} = \bigcup_{\mathcal{O}} \mathcal{M}(\mathcal{O})$, and we also see that $(a, \Lambda) \in \mathcal{P}$ maps $\mathcal{M}(\mathcal{O})$ into $\mathcal{M}(\mathcal{O}_{a,\Lambda})$.

8.2.2 <u>Definition</u> To each region $\mathcal{O}$ in $\mathbb{R}^2$, we define the local algebra of observables $\mathfrak{A}(\mathcal{O})$ to be the von Neumann algebra generated by $\{W(\xi) \mid \xi \in \mathcal{M}(\mathcal{O})\}$. The quasilocal algebra $\mathfrak{A}$ is the C*-algebra generated by all the $\mathfrak{A}(\mathcal{O})$, $\mathcal{O}$ in $\mathbb{R}^2$.

We can interpret the condition $\int \dot{\xi}(x,0)\,dx = 0$ as follows: $\phi$ is a potential and so the observables should be given by its current $\partial^\mu \phi$. $\partial^0\phi$ is just $\pi$, and $\partial^1\phi$, in smeared form, is

$$\partial^1\phi(h) = -\phi(\partial^1 h) = -\phi(f)$$

where $f = \partial^1 h \in \mathcal{D}_0$ if $h \in \mathcal{D}$.

So the infra-red problem has forced us to consider the algebra generated by the current.

As usual, we can define a unitary representation of $\mathcal{P}$ in $\mathcal{H}_0$, and one can check that under this action $\mathfrak{A}(\mathcal{O})$ is mapped onto $\mathfrak{A}(\mathcal{O}_{a,\Lambda})$ for $(a,\Lambda) \in \mathcal{P}$.

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated, then $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ commute. This follows from the Segal-Weyl relations because $\{\xi_1, \xi_2\} = 0$ if $\xi_1 \in \mathcal{M}(\mathcal{O}_1)$ and $\xi_2 \in \mathcal{M}(\mathcal{O}_2)$.

## 8.3 Localized Automorphisms

Having defined our algebra we are now in position to define localized automorphisms.

Let $\theta : \mathbb{R} \to \mathbb{R}$ be such that $d\theta/dx \in \mathcal{D}$ and $\theta(-\infty) = 0$. That is, $\theta$ is a smooth step-function which vanishes for large negative values.

Each such $\theta$ defines a pair of real solutions to the wave equation $(\partial_t^2 - \partial_x^2)\Theta(x,t) = 0$ by setting $\Theta(x,t) = \theta(x+t)$ or $\theta(x-t)$

8.3.1 Definition Let $\mathcal{N}^{\pm}$ denote the above set of real solutions of the wave equation, and let $\mathcal{N}$ denote the real linear span of $\mathcal{N}^+$ and $\mathcal{N}^-$.

8.3.2 Lemma Let $\Theta(x,t)$ be a smooth solution of the wave equation. Then $\Theta \in \mathcal{N}$ if and only if $\dot{\Theta}(x,t) \in \mathcal{M}_0$ and $\Theta(-\infty, 0) = 0$.

Proof If $\Theta \in \mathcal{N}$ then it is clear that $\dot{\Theta} \in \mathcal{M}_0$ and $\Theta(-\infty, 0) = 0$.

Conversely, suppose $\Box\Theta = 0$, and $\dot{\Theta} \in \mathcal{M}_0$ and $\Theta(-\infty, 0) = 0$.

$\Theta$ can be written as

$\Theta(x,t) = f(x+t) + g(x-t)$

for some smooth f and g. Then

$$\dot{\Theta}(x,t) = f'(x+t) - g'(x-t)$$

$$\ddot{\Theta}(x,t) = f''(x+t) + g''(x-t)$$

$\dot{\Theta} \in \mathcal{M}$ implies that $f''(x) + g''(x) \in \mathcal{D}_0$, i.e. $f'(x) + g'(x) \in \mathcal{D}$. Also $f'(x) - g'(x) \in \mathcal{D}$ and so $f'(x)$ and $g'(x) \in \mathcal{D}$. Then

$$\Theta(x,t) = C_1 + \int_{-\infty}^{x+t} f'(y)\,dy + C_2 + \int_{-\infty}^{x-t} g'(y)\,dy$$

$\Theta(-\infty, 0) = 0$ implies that $C_1 + C_2 = 0$. If we set

$$\theta_1(x) = \int_{-\infty}^{x} f'(y)\,dy \quad \text{and} \quad \theta_2(x) = \int_{-\infty}^{x} g'(y)\,dy$$

we see that $\Theta(x,t) = \theta_1(x+t) + \theta_2(x-t) \in \mathcal{N}$

QED.

8.3.3 <u>Definition</u> Let $\mathcal{O}$ be a region in $\mathbb{R}^2$. We define $\mathcal{N}(\mathcal{O})$ to be real linear span of the sets

$$\{\Theta \in \mathcal{N}^+ \mid \dot{\Theta} \in \mathcal{M}(\mathcal{O})\},\ \{\Theta \in \mathcal{N}^- \mid \dot{\Theta} \in \mathcal{M}(\mathcal{O})\}.$$

8.3.4 <u>Lemma</u> If $\Theta \in \mathcal{N}(\mathcal{O})$ and if $\mathcal{O}_1$ is space-like with repect to $\mathcal{O}$, then $\dot{\Theta}(x,t)$ and $\partial_x \Theta(x,t)$ vanish on $\mathcal{O}_1$; i.e. $\Theta(x,t)$ is a constant on $\mathcal{O}_1$.

<u>Proof</u> $\Theta$ can be decomposed into $\Theta_\pm \in \mathcal{N}^\pm$. Since $\partial_x \Theta_\pm(x,t) = \pm \dot{\Theta}_\pm(x,t)$ we need only consider $\dot{\Theta}_\pm(x,t)$. But if $\xi \in \mathcal{M}(\mathcal{O})$, the hyperbolic propogation character of solutions to the wave

equation implies that $\xi = 0$ on any $\mathcal{O}_1$ space-like with respect to $\mathcal{O}$.

QED.

8.3.5 <u>Definition</u> For any $\Theta \in \mathcal{N}$, we define a transformation $\gamma$ on elements of $\mathfrak{A}$ of the $W(\xi)$ by

$$\gamma : W(\xi) \longrightarrow \exp i\{\Theta, \xi\}\, W(\xi)$$

8.3.6 <u>Lemma</u> For each region $\mathcal{O}$, there is a (non-unique) unitary operator $V \in \mathfrak{A}$ which effects the transformation $\gamma$ :

$$\gamma(W(\xi)) = V\, W(\xi)\, V^*$$

for all $W(\xi) \in \mathfrak{A}(\mathcal{O})$.

<u>Proof</u> Let $\Theta_1(x,t) \in \mathcal{M}$ be such that $\Theta_1(x,t) = \Theta(x,t)$ whenever $(x,t) \in \mathcal{O}$. Set $V = W(\Theta_1)^*$. Then, by the Segal-Weyl relations, for $\xi \in \mathcal{M}(\mathcal{O})$,

$$V\, W(\xi)\, V^* = W(\xi)\, e^{i\{\Theta_1, \xi\}}$$
$$= W(\xi)\, e^{i\{\Theta, \xi\}}$$

since $\Theta_1 = \Theta$ on $\mathcal{O}$.

QED.

8.3.7 <u>Proposition</u> To each $\Theta \in \mathcal{N}$, there exixts a unique automorphism of $\mathfrak{A}$ which reduces to $\gamma$ on elements of the form $W(\xi)$, $\xi \in \mathcal{M}(\mathcal{O})$, some $\mathcal{O}$.

__Proof__ Since $\mathfrak{A}(\mathcal{O})$ is generated by the $W(\xi)$, $\xi \in \mathcal{M}(\mathcal{O})$, $\gamma$ can be extended to an automorphism of $\mathfrak{A}(\mathcal{O})$, say $\gamma_{\mathcal{O}}$, implemented by V as in 8.3.6.

If $\mathcal{O}_1 \subset \mathcal{O}$ it is clear that

$$\gamma_{\mathcal{O}} \upharpoonright \mathfrak{A}(\mathcal{O}_1) = \gamma_{\mathcal{O}_1} .$$

Thus we can define an automorphism $\gamma$ of $\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ which agrees with each $\gamma_{\mathcal{O}}$. This $\gamma$ extends, by continuity, to an automorphism of $\mathfrak{A}$.

QED.

8.3.8 __Theorem__ Let $\Theta \in \mathcal{N}(\mathcal{O})$, and let $\gamma$ be the corresponding automorphism of $\mathfrak{A}$ given by 8.3.7.

Then $\gamma$ is localized in $\mathcal{O}$.

__Proof__ Let $\mathcal{O}_1$ be space-like w.r.t. $\mathcal{O}$. We want to show that

$$\gamma \upharpoonright \mathfrak{A}(\mathcal{O}_1) = i \upharpoonright \mathfrak{A}(\mathcal{O}_1)$$

It is enough if we can show that

$$\gamma(W(\xi)) = W(\xi)$$

for all $\xi \in \mathcal{M}(\mathcal{O}_1)$,

i.e. that $\{\Theta, \xi\} = 0$ for all $\xi \in \mathcal{M}(\mathcal{O}_1)$.

But, by 8.3.4, $\Theta$ is constant on $\mathcal{O}_1$, so $\{\Theta, \xi\} = 0$, for all $\xi \in \mathcal{M}(\mathcal{O}_1)$.

QED.

<u>Remark</u> It is evident that $\Theta(x,t) + c$, where $c \in \mathbb{R}$ defines the same automorphism as $\Theta$;

$$\{c, \xi\} = \int \dot{\xi}(x,0)\, c\,dx = 0,$$

since $\dot{\xi} \in \mathcal{D}_0$. Thus, the requirement that $\Theta(-\infty, 0) = 0$ is merely one of convenience, i.e. a "normalization". The important property, as far as we are concerned, is the value of the difference $\Theta(+\infty, 0) - \Theta(-\infty, 0)$.

To see more clearly what $\gamma$ is, consider $\Theta(x,t) = \theta(x + t)$. Then $\gamma$ corresponds to

$$\phi(f) \rightarrow \phi(f) + \int f(x)\, \theta(x)\, dx$$

$$\pi(g) \rightarrow \pi(g) + \int g(x) \frac{d\theta}{dx}(x)\, dx$$

or

$$\phi(x) \rightarrow \phi(x) + \theta(x)$$

and $$\pi(x) \rightarrow \pi(x) + \theta'(x)$$

If $\Theta(x,t) = \theta(x - t)$ then $\gamma$ corresponds to

$$\phi(x) \rightarrow \phi(x) + \theta(x)$$

$$\pi(x) \rightarrow \pi(x) - \theta'(x)$$

In general, since $\pi = \dot{\phi}$, we have

$$\phi(x,t) \rightarrow \phi(x,t) + \Theta(x,t).$$

We notice that if $\operatorname{supp} g \cap \operatorname{supp} \theta' = \emptyset$ then $\pi(g)$ remains unchanged. If also $\operatorname{supp} f \cap [\operatorname{supp} \theta'] = \emptyset$, where $[\operatorname{supp} \theta']$ is the smallest closed <u>interval</u> containing $\operatorname{supp} \theta'$, then $f \in \mathcal{D}_0$

implies that $\int f(x)\ \theta(x)\ dx = 0$, and so $\phi(f)$ is also unchanged.

This is why $\Theta$ is localized em terms of its derivative, and we have chosen double-cones - the relevant point is that they are convex sets in $\mathbb{R}^2$. This results in $\Theta$ being localized in terms of $[\operatorname{supp} \theta']$ rather supp $\theta'$.

If we allow the limiting procedure $\theta_1 \longrightarrow H(x-x_o)$ where H is the Heavyside step-function, and $\Theta_1(x,t) = \theta_1(x+t)$, the unitary operator V in lemma 8.3.6 becomes essentially the fermion operator of Skyrme which we constructed in 8.1.

## 8.4 The "Charged" Sectors

With the air of our localized automorphisms we can construct a family of sectors.

8.4.1 Definition Let $\Gamma, \Gamma(\mathcal{O}), \Gamma^{\pm}$ denote the groups of automorphism given by $\Theta$ in $\mathcal{N}, \mathcal{N}(\mathcal{O})$ and $\mathcal{N}^{\pm}$ respectively.

We denote by $\pi_o$ the representation of $\mathcal{O}\!\mathcal{L}$ by itself on $\mathcal{H}_o$.

8.4.2. Theorem

(i) If $\gamma_1 \in \Gamma^{+}$ and $\gamma_2 \in \Gamma^{-}$, then $\pi_o \cdot \gamma_1 \simeq \pi_o \cdot \gamma_2$ if and only if $\theta_1(\infty) = \theta_2(\infty)$ where $\theta_1$ and $\theta_2$ define $\gamma_1$ and $\gamma_2$, respectively.

(ii) If $\gamma^{\pm} \in \Gamma^{\pm}$ are defined by $\theta(x \pm t)$, resp., then $\pi^{\pm} = \pi_o \cdot \gamma^{\pm}$ are unitarily inequivalent unless $\theta(\infty) = 0$.

**Proof** (i) Suppose first, that $\theta_1(\infty) = \theta_2(\infty)$ .Let $\theta(x) = \Theta_1(x) - \Theta_2(x)$. Then $\Theta(x,t) = \theta(x+t) \in \mathcal{M}$.
Hence, if $\Theta_j(x,t) = \theta_j(x+t)$, $j = 1,2$,

$$W(\Theta)\ \gamma_1(W(\xi))\ W(\Theta)^* = W(\Theta)\,W(\xi)\,W(\Theta)^*\,e^{i\{\Theta_1,\xi\}}$$
$$= W(\xi)e^{-i\{\Theta,\xi\}}\ e^{i\{\Theta_1,\xi\}}$$
$$= W(\xi)e^{i\{\Theta_1-\Theta,\xi\}} \qquad = W(\xi)\,e^{i\{\Theta_2,\xi\}}$$
$$= \gamma_2(W(\xi)).$$

Since $\mathfrak{A}$ is generated by $\{W(\xi) \mid \xi \in \mathcal{M}\}$, we have

$$W(\Theta)\ \gamma_1(A)\ W(\Theta)^* = \gamma_2(A)$$

for all $A \in \mathfrak{A}$.

Thus $W(\Theta)$ provides the unitary equivalence.

Now suppose $\theta_1(\infty) \neq \theta_2(\infty)$.

$\pi_0 \circ \gamma_1 \simeq \pi_0 \circ \gamma_2$ if and only if $\pi_0 \circ \gamma_1\gamma_2^{-1} \simeq \pi_0$, so we need only prove that $\pi_0 \circ \gamma$ is inequivalent to $\pi_0$ whenever $\gamma$ is given by $\theta$ with $\theta(\infty) \neq 0$.

Suppose $\pi_0 \circ \gamma \simeq \pi_0$ . Then, in particular, we have

$$UW(\xi)\,U^* = W(\xi)\,e^{i\{\Theta,\xi\}}$$

for all $\xi \in \mathcal{M}$, for some unitary U.

Picking $\xi \in \mathcal{M}$ to be of the form $(o,g)$, $g \in \mathcal{D}$, we obtain

$$U\,e^{-i\,\pi(g)}\,U^* = e^{-i\,\pi(g)}\,e^{-i\int g(x)\,\theta'(x)dx}$$

We will obtain a contradiction (as in 3.3) if we can find a sequence $g_n \in \mathcal{D}$ such that

$$\int |k|\,|\tilde{g}_n(k)|^2\,dk \longrightarrow 0 \text{ but } \int g_n(x)\,\theta'(x)dx \longrightarrow \pi$$

( - because then $e^{i\pi(g_n)} \longrightarrow \mathbb{1}$ , but the r.h.s. $\longrightarrow -\mathbb{1}$ )

This can be done if we can show that the functional

$$\chi : \tilde{g} \longrightarrow \int \tilde{g}(k)\ \tilde{\theta}'(-k)\, dk$$

is unbounded on $\tilde{\mathcal{D}}_{\mathbb{R}}$ w.r.t the norm

$$|||\tilde{g}|||^2 = \int |k|\ |\tilde{g}(k)|^2\, dk$$

First, let us note that $\theta'(x) \in \mathcal{D}$ , but because $\theta(\infty) \neq 0$, $\theta'(x) \notin \mathcal{D}_0$ , i.e. $\tilde{\theta}'(0) \neq 0$. Since $\tilde{\theta}'(k)$ is continuons, there is $\delta > 0$ and $b > 0$ such that, for $|k| < \delta$ , we have $|\operatorname{Re}\ \tilde{\theta}'(k)| > b$.

( $\tilde{\theta}'(0)$ is real because $\theta'(x)$ is real).

Now suppose $\chi$ is bounded. Then it has a continuons extension, say $\hat{\chi}$ , to the completion $\tilde{\mathcal{D}}^c_{\mathbb{R}}$ , of $\tilde{\mathcal{E}}_{\mathbb{R}}$ w.r.t. $|||\cdot|||$ . Since $\mathcal{D}_{\mathbb{R}}$ is dense in $\mathcal{S}_{\mathbb{R}}$ w.r.t. the topology of $\mathcal{S}$ , we see that $\tilde{\mathcal{S}}_{\mathbb{R}} \subset \tilde{\mathcal{D}}^c_{\mathbb{R}}$ ;

$$\tilde{\mathcal{S}}_{\mathbb{R}} = \{ f \in \tilde{\mathcal{S}} \mid \overline{f(k)} = f(-k) \}.$$

Moreover, the functional $f \longrightarrow \int f(k)\ \tilde{\theta}'(-k)dk$ is continuons w.r.t the $\mathcal{S}$ - topology.

Let $f \in \tilde{\mathcal{S}}_{\mathbb{R}}$ . Then there is $\tilde{g}_n \in \tilde{\mathcal{D}}_{\mathbb{R}}$ such that $\tilde{g}_n \longrightarrow f$ in $\mathcal{S}$.

Therefore

$$\int f(k)\ \tilde{\theta}'(-k)dk = \lim_n \int \tilde{g}_n(k)\ \tilde{\theta}'(-k)dk$$

$$= \lim_n \chi(\tilde{g}_n) = \hat{\chi}(f),$$

i.e. $\hat{\chi}$ is given on $\tilde{\mathcal{S}}_{\mathbb{R}}$ by $\hat{\chi}(f) = \int f(k)\ \tilde{\theta}'(-k)dk$, and, by definition, is continuons w.r.t. the $|||\cdot|||$ -norm.

We obtain a contradiction to this continuity of $\hat{\chi}$ by considering smooth approximations to the functions

$$h_n(k) = \begin{cases} |k|^{-1+1/n} & ; \ |k| \leq \delta \\ 0 & ; \ \text{otherwise} \end{cases}$$

since $\left| \int h_n(k) \ \tilde{\theta}'(-k)dk \right| = \int_{-\delta}^{\delta} |k|^{-1+1/n} \ |\text{Re } \tilde{\theta}'(-k)| \ dk$

since Re $\tilde{\theta}'(-k)$ dois not change sign in (- , .))

$$> b \int_{-\delta}^{\delta} |k|^{-1+1/n} dk = 2\,b\,n\,\delta^{1/n}$$

But $\int |h_n(k)|^2 \ |k| \ dk = n\,\delta^{2/n}$

Therefore

$$\left| \int h_n(k) \ \tilde{\theta}'(-k)dk \right| > 2b\sqrt{n} \ \| h_n \|$$

This completes the proof of (i). The proof of (ii) is analogous.

QED.

8.4.3 <u>Theorem</u> Let $\gamma \in \Gamma^{\pm}$; then the restricted Poincaré group is implemented in $\Pi_0 \circ \gamma$.

<u>Proof</u> This is not difficult (See streater and Wilde (1970)).

QED.

<u>Remark</u> Space and time inversions are <u>not</u> implemented except in $\Pi_0$, because, by 8.4.2,

$\Theta(-x,t)$ and $\Theta(x,-t)$ lead to representations inequivalent to that defined by $\Theta(x,t)$.

8.4.4 <u>Theorem</u> Let $\gamma \in \Gamma^{\pm}$. Then $\pi_0$ and $\pi_0 \circ \gamma$ are strongly locally equivalent.

<u>Proof</u> Let $\mathcal{O}$ be given. If $\gamma \in \Gamma^{\pm}(\mathcal{O})$, by 8.3.8, $\gamma \restriction \mathfrak{A}(\mathcal{O}^s) = i \restriction \mathfrak{A}(\mathcal{O}^s)$ and there is nothing to prove.

Let $\gamma \in \Gamma^{\pm}(\mathcal{O}_1)$, and suppose $\gamma$ is defined by $\Theta \in \mathcal{N}^{\pm}(\mathcal{O}_1)$. Let $\Theta_2 \in \mathcal{N}^{\pm}(\mathcal{O})$ with $\Theta_2(\infty, 0) = \Theta_1(\infty, 0)$, and let $\gamma_2$ be the automorphism corresponding to $\Theta_2$.

By 8.4.2, $\gamma \cdot \gamma_2^{-1}$ is implemented, and so $\pi_0 \simeq \pi_0 \circ \gamma \cdot \gamma_2^{-1}$.

But $\gamma_2^{-1} \in \Gamma^{\pm}(\mathcal{O})$ and so $\gamma_2^{-1} \restriction \mathfrak{A}(\mathcal{O}^s) = i \restriction \mathfrak{A}(\mathcal{O}^s)$.

Therefore $\pi_0 \restriction \mathfrak{A}(\mathcal{O}^s) \simeq \pi_0 \cdot \gamma \restriction \mathfrak{A}(\mathcal{O}^s)$

QED.

8.4.5 <u>Theorem</u> If $\gamma \in \Gamma^{\pm}$, then the operators implementing space-time translation in $\pi_0 \circ \gamma$ may be chosen to have their joint-spectrum in the closed forward light-cone.

<u>Proof</u> One shows that the generators of space-time translations are given by

$$P_\gamma = - \int k\, b^*(k)\, b(k)\, dk$$

$$H_\gamma = \int |k|\, b^*(k)\, b(k)\, dk$$

where $b^*(k) = a^*(k) + \frac{1}{\sqrt{2}}\, i\, \frac{\tilde{\Theta}'(k)}{|k|^{1/2}}\, \frac{(|k| - k)}{k}$

This is precisely the displacement corresponding to that experienced by the fields under $\gamma$.

QED.

<u>Remark</u> The vacuum in $\Pi_0$, $\Omega_0 = (1,0,0,\ldots\ldots) \in \mathcal{H}_0$, the Fock space, defines a vector state in $\Pi = \Pi_0 \circ \gamma$ but it no longer has zero energy,

$$(\Omega, H_\gamma \Omega_0) = \frac{1}{2} \int \left\{ \dot{\Theta}(x,0)^2 + \frac{d\Theta}{dx}(x,0)^2 \right\} dx$$

This is the classical energy of the solution $\Theta(x,t)$ of the wave equation.

## 8.5 <u>The Construction of Charge Carrying Fields</u>

We have seen that $\Theta_1$ and $\Theta_2 \in \mathcal{N}^+$ (or $\mathcal{N}^-$) lead to equivalent representations if and only if $\Theta_1(\infty,0) = \Theta_2(\infty,0)$. For simplicity, let us consider only $\mathcal{N}^+$. Then the sectors are labelled by the values $\Theta_1(\infty,0)$, i.e. by $\mathbb{R}$ - the "charge". (In general, when considering $\mathcal{N}^\pm$, we would be able to label the sectors by $\mathbb{R}^2$ ).

In the set-up of 7.6.4, we have, with

$\gamma_2(\cdot) = U \gamma_1(\cdot) U^*$, that

$\gamma_1(U) = e^{\pm i\alpha^2} U$

where $\gamma_1$ corresponds to $\Theta$ and has charge $\alpha$, i.e. $\Theta(\infty,0) = \alpha$. So we expect fields that are of neither Bose nor Fermi type (-as also predicted in the first paragraph 8.1). This is a special feature of two dimensions.

We select an arbitrary, but fixed, $\Theta \in \mathcal{N}^+$, with $\Theta(\infty, 0) = \theta(\infty) = 1$. For each $\alpha \in \mathbb{R}$, we shall write $\Theta_\alpha$ for $\alpha\Theta \in \mathcal{N}^+$. Such a $\Theta_\alpha$, and its corresponding automorphism, $\gamma_\alpha$, will be called standard.

We want to consider the various representations as taking place on different Hilbert spaces. For each $\alpha \in \mathbb{R} \setminus \{0\}$. let $\mathcal{H}_\alpha$ be a Hilbert space isomorphic to $\mathcal{H}_0$. Let $\psi_\alpha^*$ be isometric from $\mathcal{H}_0$ onto $\mathcal{H}_\alpha$, and set $\psi_0^* = 1\!\!1_{\mathcal{H}_0}$. We define the representation $(\mathcal{H}_\alpha, \pi_\alpha)$ of $\mathfrak{A}$ by

$$\pi_\alpha(A) = \psi_\alpha^* \; \pi_0 \circ \gamma_\alpha(A) \, \psi_\alpha, \quad A \in \mathfrak{A},$$

where $\psi_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_0$ is the inverse of $\psi_\alpha^*$.

By 8.4.2, $(\mathcal{H}_\alpha, \pi_\alpha)$ and $(\mathcal{H}_\beta, \pi_\beta)$ are inequivalent unless $\alpha = \beta$.

Define $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$, $\pi = \bigoplus_\alpha \pi_\alpha$

The field algebra will be defined in $\mathcal{H}$. If $U_\alpha(a, \Lambda)$ represents $\mathcal{P}_+^\uparrow$ in $(\mathcal{H}_\alpha, \pi_\alpha)$, then $\mathcal{H}$ carries the representation $\bigoplus_\alpha U_\alpha$ which also has energy-momentum spectrum in the closed forward light-cone.

We define $\psi_\alpha^*$ on $\mathcal{H}$ by linear extension of

$$\psi_\alpha^* \upharpoonright \mathcal{H}_\beta = \psi_{\alpha+\beta}^* \, \psi_\beta \upharpoonright \mathcal{H}_\beta, \quad \text{all } \beta \in \mathbb{R}$$

Cleary $\psi_\alpha^*$ is unitary on $\mathcal{H}$. It is the charged field corresponding to the standard $\Theta_\alpha$.

We can extend this to the general case.

8.5.1 <u>Definition</u> Let $M(x,t) = \mu(x+t) \in \mathcal{N}^+$, $\mu(\infty) = \alpha$. Let $\gamma_\mu$ be the corresponding automorphism. The field $\psi^*(\mu)$, with charge $\alpha = \mu(\infty)$ is defined by linear extension of

$$\psi^*(\mu) \restriction \mathcal{H}_\beta = \pi_{\alpha+\beta}(W(M - \Theta_\alpha))\, \psi_\alpha^* \restriction \mathcal{H}_\beta$$

for each $\beta \in \mathbb{R}$, where $\pi_{\alpha+\beta}$ is a standard representation.

We understand this definition as follows:

$\psi_\alpha^*$ acting on $\mathcal{H}_\beta$ creates a standard charge in $\mathcal{H}_{\alpha+\beta}$. We can change this standard to the required state, determined by M, by an element of $\mathfrak{A}$, namely $W(M - \Theta_\alpha)$. However, this must be done in the representation $\pi_{\alpha+\beta}$.

Evidently, $\psi^*(\mu)$ is unitary on $\mathcal{H}$.

8.5.2 <u>Lemma</u> Let $\mathcal{O}$ be any region, and let $M \in \Gamma^+(\mathcal{O})$. Then $\psi^*(\mu)$ commutes with $\pi(\mathfrak{A}(\mathcal{O}^s))$.

<u>Proof</u> It suffices to prove this on each $\mathcal{H}_\beta$, since, by construction, $\pi$ is reduced by $\bigoplus_\beta \mathcal{H}_\beta$.

Let $A \in \mathfrak{A}(\mathcal{O}^s)$, and suppose $\mu(\infty) = \alpha$.

Then

$$\psi^*(\mu)\pi(A) \restriction \mathcal{H}_\beta = \psi^*(\mu)\, \pi_\beta(A)$$
$$= \pi_{\alpha+\beta}(W)\, \psi_\alpha^* \psi_\beta^*\, \pi_0(\gamma_\beta(A))\, \psi_\beta,$$

where $W = W(M - \Theta)$

$$= \psi_{\alpha+\beta}^*\, \pi_0(\gamma_{\alpha+\beta}(W))\, \pi_0(\gamma_\beta(A))\, \psi_\beta$$
$$= \psi_{\alpha+\beta}^*\, \pi_0(\gamma_\beta(\gamma_\alpha(W)A))\, \psi_\beta$$
$$= \psi_{\alpha+\beta}^*\, \pi_0 \circ \gamma_\beta(\gamma_\alpha(W)\gamma_\mu(A))\, \psi_\beta$$

by 8.3.8, since $\gamma_\mu \in \Gamma^+(\mathcal{O})$,

$$= \psi_{\alpha+\beta}^*\, \pi_0 \circ \gamma_\beta(\gamma_\alpha(W)\gamma_\mu(A)\gamma_\alpha(W)^*\gamma_\alpha(W))\, \psi_\beta$$

Now, $\gamma_\alpha(W) = e^{i\vartheta} W$, some $\vartheta \in \mathbb{R}$, so

$$\gamma_\alpha(W)\,\gamma_\mu(A)\,\gamma_\alpha(W)^* = W\,\gamma_\mu(A)\,W^*$$
$$= W(M - \Theta_\alpha)\,\gamma_\mu(A)\,W(M - \Theta_\alpha)^*$$
$$= \gamma_\alpha(A) \qquad \text{by 8.4.2 (i)}$$

Hence

$$\psi^*(\mu)\,\pi(A) \restriction \mathcal{H}_\beta = \psi^*_{\alpha+\beta}\,\pi_0 \circ \gamma_\beta(\gamma_\alpha(A)\gamma_\alpha(W))\,\psi_\beta$$
$$= \psi^*_{\alpha+\beta}\,\pi_0 \circ \gamma_{\alpha+\beta}(AW)\,\psi_\beta$$
$$= \psi^*_{\alpha+\beta}(\pi_0 \circ \gamma_{\alpha+\beta}(A))(\pi_0 \circ \gamma_{\alpha+\beta}(W))\,\psi_\beta$$
$$= \pi(A)\,\psi^*(\mu) \restriction \mathcal{H}_\beta$$

QED.

8.5.3 <u>Definition</u> We define the local field algebra $\mathcal{F}^+(\mathcal{O})$ to be the von Neumann algebra generated by the set

$$\{\psi^*(\mu) \mid \mu \longleftrightarrow M \in \mathcal{N}^+(\mathcal{O})\} \cup \{\pi(A) \mid A \in \mathfrak{A}(\mathcal{O})\}$$

By 8.5.2, we see that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated regions, then

$$[\mathcal{F}^+(\mathcal{O}_1), \pi(\mathfrak{A}(\mathcal{O}_2))] = 0$$

8.5.4 <u>Theorem</u>

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be space-like separated regions (double-cones). if $M_1 \in \mathcal{N}^+(\mathcal{O}_1)$ and $M_2 \in \mathcal{N}^+(\mathcal{O}_2)$ then

$$\psi^*(\mu_1)\,\psi^*(\mu_2) = \psi^*(\mu_2)\,\psi^*(\mu_1)\,e^{i\vartheta}$$

where $M_j\ (x,t) = \mu_j\ (x+t)$, $j = 1,2$ and

$\nu = \pm \mu_1(\infty)\,\mu_2(\infty)$ according as to whether $O_1$ is to the left or right of $O_2$.

<u>Proof</u> Let $\mu_1(\infty) = \alpha$, $\mu_2(\infty) = \beta$, and consider $\psi^*(\mu_1)\,\psi^*(\mu_2)$ on any subspace $\mathcal{H}_\varepsilon$ of $\mathcal{H}$.

$$\psi^*(\mu_1)\,\psi^*(\mu_2) \restriction \mathcal{H}_\varepsilon = \Pi_{\alpha+\beta+\varepsilon}(W_1)\,\psi^*_\alpha\,\Pi_{\varepsilon+\beta}(W_2)\,\psi^*_\beta$$

where $W_1 = W(M_1 - \Theta_\alpha)$, $W_2 = W(M_2 - \Theta_\beta)$

$$= \psi^*_{\alpha+\beta+\varepsilon}\,\Pi_0\left\{\gamma_{\alpha+\beta+\varepsilon}(W_1)\,\gamma_{\beta+\varepsilon}(W_2)\right\}\psi_\varepsilon$$

$$= \psi^*_{\alpha+\beta+\varepsilon}\,\Pi_0\left\{W(M_1 - \Theta_\alpha)\,W(M_2 - \Theta_\beta)\,e^{iX}\right\}\psi_\varepsilon$$

where $X = \{\Theta_{\alpha+\beta+\varepsilon}, M_1 - \Theta_\alpha\} + \{\Theta_{\beta+\varepsilon}, M_2 - \Theta_\beta\}$.

But $\{\Theta_i, \Theta_j\} = 0$ for any standards because they are proportional.

Thus

$$X = \{\Theta_{\alpha+\beta+\varepsilon}, M_1\} + \{\Theta_{\beta+\varepsilon}, M_2\}.$$

By the Segal-Weyl relations,

$$W(M_1 - \Theta_\alpha)\,W(M_2 - \Theta_\beta)\,e^{iX} = e^{iY}\,W(M_1 + M_2 - \Theta_\alpha - \Theta_\beta),$$

where $Y = -\frac{1}{2}\{M_1 - \Theta_\alpha, M_2 - \Theta_\beta\} + \{\Theta_{\alpha+\beta+\varepsilon}, M_1\} + \{\Theta_{\beta+\varepsilon}, M_2\}$

Interchanging $\alpha$ and $\beta$, and $M_1$ and $M_2$, we obtain

$$\psi^*(\mu_2)\,\psi^*(\mu_1) = \psi^*_{\alpha+\beta+\varepsilon}\,\Pi_0\left\{W(M_1 + M_2 - \Theta_\alpha - \Theta_\beta)\right\}e^{iZ}$$

where $Z = \frac{1}{2}\{\Theta_\alpha - M_1, \Theta_\beta - M_2\} + \{\Theta_{\alpha+\beta+\varepsilon}, M_2\} + \{\Theta_{\alpha+\varepsilon}, M_1\}$.

Hence

$$\psi^*(\mu_1)\,\psi^*(\mu_2) = \psi^*(\mu_2)\,\psi^*(\mu_1)\,e^{i\nu}$$

where $$\begin{aligned}\nu &= X - Y\\ &= \{\Theta_\alpha - M_1, \Theta_\beta - M_2\} + \{\Theta_\alpha, M_2\} - \{\Theta_\beta, M_1\}\\ &= \{M_1, M_2\}\\ &= \int \left(\mu_1(x)\mu_2'(x) - \mu_1'(x)\mu_2(x)\right) dx\\ &= \pm\, \mu_1(\infty)\mu_2(\infty) \qquad \text{as required.}\end{aligned}$$

QED.

For futher remarks, and the definition of the gauge group, we refer to Streater and Wilde (1970).

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