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A NEW SOLUTION TO THE U(n) THIRRING MODEL ?

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ABSTRACT

The Thirring model with internal U(n) symmetry is discussed both in perturbation theory and outside of it. In the latter case we propose a mechanism whereby the quantized theory has more conserved currents than its classical counter part. In this second case we obtain a partial solution for the value π/n of one of the coupling constants. We also discuss related attempts of solving the present model which have appeared in the literature.

I. INTRODUCTION

One of the most important reasons for studying two dimensional field theoretical models consists in their relevance as a laboratory for new ideas. Operator product expansions and anomalous dimensions (1,2) are ideas that were primarily realized in the Thirring model (3) (massless spinor field with quadrilinear interaction); another two dimensional model, the Schwinger model (4), can be referred to as exhibiting a very simple mechanism for dynamical generation of mass. These discoveries were made possible, mainly because of the solubility of various models that occur in the two dimensional world. The basic reason for this solubility, is the fact that vector and exial vector currents are not independent but are related because

$$\gamma^\mu \gamma^5 = \varepsilon^{\mu\nu} \gamma_\nu \,, \quad \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \varepsilon_{\mu\nu} \varepsilon^{\nu\rho} = g_\mu^\rho$$

in two dimensions. Thus the currents are explicitly known if they are all conserved (as it happens in the Thirring model) or if the anomaly is mild in the sense that it still permits the integration of the equations of motion (the Schwinger model).

The appearence of anomalies, which one generally expects in any theory needing renormalization, usually may destroy the conservation of currents, which is valid on the classical level. One may on the other hand take advantage of this feature and consider models, which on the classical level

have nonconserved currents, i.e.

$$\partial^\mu j^{cl}_{\mu,i}(x) = \Delta_i(x)$$

One may now try to use the anomalies to cancel exactly the term $\Delta_i(x)$ and thus produce currents, which are conserved on the quantum level:

$$\partial^\mu j^{q}_{\mu,i}(x) = 0$$

The quantized version of the model is thus more symmetric than the classical counter part.

It is immediately obvious that this mechanism can only operate outside of perturbation theory. For example, the classical equations are valid in the tree aproximation, whereas anomalies only appear once one considers closed loops. The above cancellation can only happen for particular values of the coupling constants. Off these values the theory does not satisfy current conservation, but should exhibit all the general properties of order by order perturbation theory, like unitarity and analyticity and they should still be true for some particular values of the coupling constants.

In this paper we investigate such a possibility in the context of a Thirring model with U_n symmetry. There are four types of currents in this model which are important for its solubility. These are caracterised by their transformation properties under the U_n and Lorentz groups and classically it is always possible to arrange things so that only the axial isovector current $\bar\psi \gamma^\mu \gamma^5 \lambda^a \psi$ is not conserved. To get conservation in the quantized version of the model it is necessary for the anomaly of this current to have the same operator form as its classical divergence. To investigate this possibility we use initially perturbation theory for the massive case. We find that asymptotic conservation of the various currents is possible along a curve in the space of the three coupling constants g_1, g_2, g_3, that includes the origin. As expected from the above argument, this curve corresponds to a trivial model, in the sense that the interaction does not mix the n components of the basic spinor. One has n selfinteracting U(1) Thirring models and all the currents are already conserved classically. This result is the same as that obtained by Mitter and Weisz (5), although they use a parametrization different from ours. What one learns from perturbation theory is that the anomalies have the same form as the classical divergencies $\Delta_i(x)$ and makes a nontrivial cancellation mechanism highly plausible.

In section II we study the zero mass limit in perturbation theory, conveniently using a generalized Taylor subtraction scheme instead of the Bogoliubov-Parasiuk-Hepp-Zimmermann (6) renormalization procedure. It becomes clear however that the only assymptotically symmetric theory that we can get using perturbative methods consists of n non interacting Thirring fields.

The important problem of the existence of other, non perturbative, solutions is discussed in section III. Previously there were some attempts (7, 8) to accomplish this goal but to our knowledge all proposed solutions are trivial in the sense that they are or can by a reparametrization be reduced to the case of n independent Thirring fields, i.e. a solution ψ, which is the product $\prod_{i=1}^{n} \psi_i(x)$ of n fields ψ_i, each one interacting via $g_1(\bar\psi_i\gamma_\mu\psi_i)(\bar\psi_i\gamma^\mu\psi_i)$, i not summed over. The latter possibility arises due the existence of an algebraic relation for the free field which assumes the form of an interacting equation of motion if a different representation for the Dirac matrices is used. Instead of this we use a construction of Johnson's type (9) to obtain an explicit form for the two and four point Green func-

tions. Higher order point functions are not explicitly known yet, but it seems clear that our result is intrinsically non trivial. Furthermore following arguments that parallel those of ref. (7) we are able to determine the value π/n for the iso-vector coupling constant. Finally, section IV contains a discussion of related work by Dashen and Frishman (7) and a comment on the complete solution of the present model. In the Appendix we present a different form of the four-point function, which may be useful for a complete solution of the model.

II. PERTURBATIVE DISCUSSION

In this section methods of renormalized perturbation theory will be applied to the Thirring model with internal U(n) symmetry, aiming at a classification of various situations according to the number of conserved currents. We also discuss an equation of motion for ψ, written in a limit form, adequate for the comparison with the extactly soluble model of section III.

II.1) Asymptotic scale invariance

The perturbative treatment can be done in a sistematic way through normal product methods. We consider the model described by the following effective Lagrangian density

$$\mathcal{L}_{eff} = \frac{i}{2}\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi - m\bar\psi\psi + \frac{g_1-c_1}{2}(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi) +$$

$$+ \frac{g_2-c_2}{2}(\bar\psi\gamma_\mu\vec\lambda\,\psi)(\bar\psi\gamma^\mu\vec\lambda\,\psi) + \frac{g_3-c_3}{2}(\bar\psi\psi)(\bar\psi\psi) \qquad \text{(II.1)}$$

where $\vec\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ are matrices of the fundamental representation of SU_n, satisfying

$$\lambda_a \lambda_b = \frac{2}{n}\,\delta_{ab} + (d_{abc} + i f_{abc})\,\lambda_c$$

The finite constants c_1, c_2 and c_3 are mass independent counter terms fixed by normalization conditions to be specified later. The effective Lagrangian (II.1) is the most general one satisfying the requirements of renormalizability, Lorentz covariance, parity, charge conjugation and U(n) symmetry. The Green functions are given by the usual finite part of the Gell-Mann Low formula, with a subtraction scheme which uses the forest formula with modified "Taylor" operators (10):

Logarithmically divergent integrands require one subtraction:

$$\tau^{(0)} F(p,m) = F(0,\mu) \qquad \text{(II.2)}$$

whereas linearly divergent integrands should be subtracted with $\tau^{(1)}$:

$$\tau^{(1)} F(p,m) = F(0,0) + p^\mu \frac{\partial F}{\partial p^\mu}\bigg|_{\substack{p=0\\ m=\mu}} + m\frac{\partial F}{\partial m}\bigg|_{\substack{p=0\\ m=\mu}} \qquad \text{(II.3)}$$

The subtraction scheme (II.2-3) is convenient for the derivation of homogeneous parametric differential equations for the N point functions (10).

Defining the 2N-point vertex function $\Gamma^{(2N)}$ by

$$(2\pi)^2\,\delta\Big(\sum_{i=1}^{N} p_i + \sum_{i=1}^{N} q_i\Big)\,\Gamma^{(2N)}(p_1,p_2,\dots p_N, q_1,q_2,\dots q_N) =$$

$$= \int \prod_{i=1}^{N} dx_i\,dy_i\,\exp\Big\{i\sum_{i=1}^{N}(p_i x_i + q_i y_i)\Big\}\,\langle 0|\,T\prod_{i=1}^{N}\psi(x_i)\,\bar\psi(y_i)\,|0\rangle^{prop}$$

where the superscript Prop indicates that only proper (amputated, one particle irreducible) diagrams are included, we obtain from (II.2) and (II.3) the results

$$\frac{\partial\Gamma^{(2)}}{\partial p^\mu}\bigg|_{\substack{p=0\\ m=\mu}} = i\gamma_\mu\ ;\ \frac{\partial\Gamma^{(2)}}{\partial m}\bigg|_{\substack{p=0\\ m=\mu}} = -i\ ;\ \Gamma^{(2)}\bigg|_{\substack{p=0\\ m=0}} = 0 \qquad \text{(II.4)}$$

Using methods similar to those of reference (10) the following differential equation can be established

$$\Big[\mu\frac{\partial}{\partial\mu} + \delta m\frac{\partial}{\partial m} + \sum_{i=1}^{3}\beta_i\frac{\partial}{\partial g_i} - 2N\gamma\Big]\Gamma^{(2N)} = 0 \qquad \text{(II.5)}$$

where δ, β_i (i=1,2,3) and γ are mass independent functions of g_i, i=1,2,3 which can be obtained from the equation above by application of the normalization conditions satisfied by $\Gamma^{(2N)}$.

Normal products are introduced in the usual way. In particular for proper functions containing only one special normal product vertex the following notation will be used:

NORMAL PRODUCT	NOTATION
$N_1[\bar\psi\gamma_\mu\psi]$	$\Gamma_{1,\mu}^{(2N)}$
$N_1[\bar\psi\gamma^5\psi]$	$\Gamma_2^{(2N)}$
$N_1[\bar\psi\gamma_\mu\lambda^a\psi]$	$\Gamma_{3,\mu a}^{(2N)}$
$N_1[\bar\psi\gamma^5\lambda^a\psi]$	$\Gamma_{4,a}^{(2N)}$
$N_2[(\bar\psi\gamma^5\psi)(\bar\psi\psi)]$	$\Gamma_5^{(2N)}$
$N_2[f_{abc}(\bar\psi\gamma^\mu\gamma^5\lambda^b\psi)(\bar\psi\gamma_\mu\lambda^c\psi)]$	$\Gamma_{6,a}^{(2N)}$
$N_2[(\bar\psi\gamma^5\lambda^a\psi)(\bar\psi\psi)]$	$\Gamma_{7,a}^{(2N)}$

(II.6)

These normal product vertex functions satisfy differential equations analogous to (II.5)

$$[D-(2N-2)\gamma]\begin{Bmatrix}\Gamma^{(2N)}_{1,\mu}\\ \Gamma^{(2N)}_{2}\\ \Gamma^{(2N)}_{3,\mu a}\\ \Gamma^{(2N)}_{4,a}\end{Bmatrix} = \begin{Bmatrix} t_1\,\Gamma^{(2N)}_{1,\mu}\\ t_2\,\Gamma^{(2N)}_{2}\\ t_3\,\Gamma^{(2N)}_{3,\mu a}\\ t_4\,\Gamma^{(2N)}_{4,a}\end{Bmatrix}(\delta-1)$$

(III.7)

$$[D-(2N-4)]\begin{Bmatrix}\Gamma^{(2N)}_{5}\\ \Gamma^{(2N)}_{6,a}\\ \Gamma^{(2N)}_{7,a}\end{Bmatrix} = \begin{Bmatrix}\delta_1 p^\mu \varepsilon_\mu{}^\lambda \Gamma^{(2N)}_{1,\lambda} + \delta_2 \Gamma^{(2N)}_{5}\\ \delta_3 p^\mu \varepsilon_\mu{}^\lambda \Gamma^{(2N)}_{3,\lambda a} + \delta_4 \Gamma^{(2N)}_{6,a} + \delta_5 \Gamma^{(2N)}_{7,a}\\ \delta_6 p^\mu \varepsilon_\mu{}^\lambda \Gamma^{(2N)}_{3,\lambda a} + \delta_7 \Gamma^{(2N)}_{6,a} + \delta_8 \Gamma^{(2N)}_{7,a}\end{Bmatrix}(\delta-1)$$

where $\delta_1,\dots,\delta_8$ and $t_1,\dots t_4$ are functions of g_i, known in perturbation theory and

$$D = \delta m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} + \sum_{i=1}^{3}\beta_i\frac{\partial}{\partial g_i}$$

The model (II.1) has two conserved vector currents

$$p^\lambda \Gamma^{(2N)}_{1,\lambda} = i\sum_{k=1}^{N}\left[\Gamma^{(2N)}(\dots,q_k+p,\dots) - \Gamma^{(2N)}(\dots,p_k+p,\dots)\right] \quad \text{(II.8a)}$$

$$p^\lambda \Gamma^{(2N)}_{3,\lambda a} = i\sum_{k=1}^{N}\left[\lambda^a\Gamma^{(2N)}(\dots,q_k+p,\dots) - \lambda^{aT}\Gamma^{(2N)}(\dots,p_k+p,\dots)\right] \quad \text{(II.8b)}$$

whereas for the axial currents we have

$$(\alpha-e_1)p^\lambda\varepsilon_\lambda{}^\mu\Gamma^{(2N)}_{1,\mu} = -2m\,\Gamma^{(2N)}_{2} + \alpha_1\Gamma^{(2N)}_{5} + $$

$$+ i\alpha\sum_{k=1}^{N}\left[\Gamma^{(2N)}(\dots,q_k+p,\dots)\gamma_5^{q_k} + \gamma_5^{p_k}\Gamma^{(2N)}(\dots,p_k+p,\dots)\right] \quad \text{(II.8c)}$$

$$(\eta-e_2)p^\lambda\varepsilon_\lambda{}^\mu\Gamma^{(2N)}_{3,\mu a} = -2m\,\Gamma^{(2N)}_{4,a} + \alpha_2\Gamma^{(2N)}_{6,a} + \alpha_3\Gamma^{(2N)}_{7,a} + $$

$$+ i\eta\sum_{k=1}^{N}\left[\Gamma^{(2N)}(\dots,q_k+p,\dots)\gamma_5^{q_k}\lambda^a + \gamma_5^{p_k}\lambda^{aT}\Gamma^{(2N)}(\dots,p_k+p,\dots)\right] \quad \text{(II.8d)}$$

The anomalies in (II.8c) and (II.8d) come from the two identities

$$2\alpha N_2[m\bar\psi\gamma^5\psi] = 2m N_1[\bar\psi\gamma^5\psi] + e_1 N_2[\partial^\mu(\bar\psi\gamma_\mu\gamma^5\psi)] + $$
$$+\lambda_1 N_2[(\bar\psi\gamma^5\psi)(\bar\psi\psi)]$$

$$2\eta N_2[m\bar\psi\gamma^5\lambda^a\psi] = 2m N_1[\bar\psi\gamma^5\lambda^a\psi] + e_2 N_2[\partial^\mu(\bar\psi\gamma_\mu\gamma^5\lambda^a\psi)] + $$
$$+\lambda_2 f_{abc} N_2[(\bar\psi\gamma^\mu\gamma^5\lambda^b\psi)(\bar\psi\gamma_\mu\lambda^c\psi)] + \lambda_3 N_2[(\bar\psi\gamma^5\lambda_a\psi)(\bar\psi\psi)]$$

(II.9)

$$\alpha_1 = 2(g_3-c_3)\alpha - \lambda_1$$
$$\alpha_2 = 2i(g_2-c_2)\eta - \lambda_2$$
$$\alpha_3 = 2(g_3-c_3)\eta - \lambda_3$$

where

$$\gamma_5\alpha = 2\frac{\partial}{\partial m}\langle 0|T\, m N_1[\bar\psi\gamma^5\psi](0)\,\tilde{\bar\psi}(0)\,\tilde\psi(0)|0\rangle^{prop}\Big|_{m=\mu}$$

$$e_1\gamma_\mu\gamma_5 = \frac{2\mu}{i}\frac{\partial}{\partial q^\mu}\langle 0|T N_1[\bar\psi\gamma_5\psi](0)\,\tilde{\bar\psi}(q/2)\,\tilde\psi(q/2)|0\rangle^{prop}\Big|_{\substack{q=0\\ m=\mu}}$$

$$2n(n-1)\lambda_1 = -\mu\langle 0|T N_1[\bar\psi\gamma_5\psi](0)\,(\tilde{\bar\psi}(0)\gamma^5\tilde\psi(0))\,\tilde{\bar\psi}(0)\tilde\psi(0)|0\rangle^{prop}\Big|_{m=\mu}$$

and analogous formulae for η, e_2, λ_1 and λ_3.

One interesting feature of the renormalized theory is the possibility of asymptotic conservation of the axial currents, though classically this is impossible. Formulae (II.7) and (II.8) can be used to obtain a number of interesting results:

i) By applying D to both sides of (II.8a) and (II.8b) and using (II.7) we get

$$\delta t_1 = \delta t_3 = 2\gamma$$

So that at an eigenvalue $\bar g_0 = (g_{10}, g_{20}, g_{30})$ defined by

$$\beta_1 = \beta_2 = \beta_3 = 0$$

all currents will scale canonically in the asymptotic region. Such a line of fixed points has been investigated by Mitter and Weisz (5) in the neighborhood of the origin $g_1=g_2=g_3=0$.

ii) There are two surfaces containing the origin $g_1=g_2=g_3=0$ on which either β_1 or β_2 is zero.

Applying the differential operator

$$\tilde{D} = m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} + \sum_{i=1}^{3} \beta_i \frac{\partial}{\partial g_i} \tag{II.10}$$

to both sides of (II.8c) and using

$$\frac{\partial \Gamma^{(2N)}}{\partial m} = -\Delta_o \Gamma^{(2N)}$$

$$(\alpha - e_1)(-\tilde{p}^\lambda \Delta_o \Gamma^{(2N)}_{1,\lambda}) = i\alpha \sum_{k=1}^{N} [\Delta_o \Gamma^{(2N)}(\ldots, q_k+p, \ldots)\gamma_5^{q_k} +$$
$$+ \gamma_5^{p_k} \Delta_o \Gamma^{(2N)}(\ldots, p_k+p, \ldots)] - 2[-\alpha + m(\Delta_o - t_2)]\Gamma_2^{(2N)} + \alpha_1 \Delta_o \Gamma_5^{(2N)}$$

where Δ_o is the soft mass vertex insertion, we obtain the equations

$$\begin{aligned} &\alpha \tilde{D}\left(1 - \frac{e_1}{\alpha}\right) - (\delta - 1)\alpha_1 \delta_1 = 0 \\ &1 - 2\gamma - \frac{\tilde{D}\alpha}{\alpha} - (\delta - 1)\alpha = 0 \\ &\alpha_1 \frac{\tilde{D}\alpha}{\alpha} - \tilde{D}\alpha_1 + 4\gamma\alpha_1 - (\delta - 1)\alpha_1 \delta_2 = 0 \end{aligned} \tag{II.11a}$$

We now choose g_3-c_3 so that $\alpha_1=0$ and furthermore c_1 so that e_1/α depends only on g_1 (this is always possible in perturbation theory, since $e_1/\alpha=A(g_1-c_1)+$ higher order terms, with $A\neq 0$). Thus from the first of the equations (II.10) we get

$$\tilde{D}\frac{e_1}{\alpha} = 0 \quad \text{implying} \quad \beta_1 = 0$$

Analogolously, if we apply $\tilde{D}$ to both sides of (II.8d) we will get

$$\begin{aligned} &\eta \tilde{D}\left(1 - \frac{e_2}{\eta}\right) - (\delta - 1)(\alpha_2\delta_3 + \alpha_3\delta_6) = 0 \\ &1 - 2\gamma - \frac{\tilde{D}\eta}{\eta} - (\delta - 1)\eta = 0 \\ &\alpha_2 \frac{\tilde{D}\eta}{\eta} - \tilde{D}\alpha_2 + 4\gamma\alpha_2 - (\delta - 1)(\alpha_2\delta_4 + \alpha_3\delta_7) = 0 \\ &\alpha_3 \frac{\tilde{D}\eta}{\eta} - \tilde{D}\alpha_3 + 4\gamma\alpha_3 - (\delta - 1)(\alpha_2\delta_5 + \alpha_3\delta_8) = 0 \end{aligned} \tag{II.11b}$$

Thus if c_2 is chosen so that e_2/η depends only on g_2 and g_2 is fixed so that $\alpha_2\delta_3 + \alpha_3\delta_6 = 0$ it follows that $\beta_2=0$.

From the equations above we also have that at the curve $\alpha_1=0$, $\alpha_2\delta_3+\alpha_3\delta_6=0$, all β's vanish. But only in the case of SU(2), when $\Gamma_{7,a}$ and $\Gamma_{6,a}$ are linearly dependent, does this mean conservation of all the currents.

II.2) Solubility conditions

In this section we want to investigate the conditions of exact solubility of the $m \to 0$ limit theory. In the present context this means an exact conservation of all vector and axial vector currents, for then we would be able to execute a Johnson type construction (8,9), as will be done in the next section.

For this purpose it is usefull to start with an equation of motion written in terms of currents, such that one can easily read off the consequences of their conservation instead of using the parametrization provided by L_{eff} of equation (II.9). We will thus express the currents as limits of fields, when the separating distance goes to zero.

Besides the usual definition (9)

$$\left\{ \begin{matrix} j_\mu(x) \\ j_\mu^a(x) \end{matrix} \right\} \equiv \tag{II.12}$$

$$\equiv \lim_{\varepsilon \to 0} \frac{1}{4} \left\{ \left[\bar{\psi}(x+\varepsilon) \left\{ \begin{matrix} \gamma_\mu \\ \gamma_\mu \lambda^a \end{matrix} \right\} \psi(x) - \left\{ \begin{matrix} \gamma_\mu \\ \gamma_\mu \lambda^a \end{matrix} \right\} \psi(x+\varepsilon) \bar{\psi}(x) \right] + [\varepsilon \to \tilde{\varepsilon}] \right\} \left\{ \begin{matrix} Z_s \\ Z_{\cdot} \end{matrix} \right.$$

we may also define the following currents

$$\left\{\begin{matrix} J_\mu(x) \\ J_\mu^a(x) \end{matrix}\right\} \equiv \lim_{\varepsilon\to 0} \frac{1}{4}\Big[\Big(\bar\psi(x+\varepsilon)\left\{\begin{matrix}\gamma_\alpha \\ \gamma_\alpha\lambda^a\end{matrix}\right\}\psi(x) - \left\{\begin{matrix}\gamma_\alpha \\ \gamma_\alpha\lambda^a\end{matrix}\right\}\psi(x+\varepsilon)\bar\psi(x)\Big)\cdot$$

$$\cdot\frac{\varepsilon_\mu\varepsilon^\alpha+\tilde\varepsilon_\mu\tilde\varepsilon^\alpha}{\varepsilon^2} + (\varepsilon\to\tilde\varepsilon)\Big]\left\{\begin{matrix} Z_s'(\varepsilon) \\ Z_v'(\varepsilon)\end{matrix}\right\} \tag{II.13}$$

where

From the Wilson expansions:

$$\bar\psi(x+\varepsilon)\gamma_\mu\psi(x) - \gamma_\mu\psi(x+\varepsilon)\bar\psi(x) = 2a_1^s N_1[\bar\psi\gamma_\mu\psi](x) + 4a_2^s\frac{\varepsilon_\mu\varepsilon_\nu}{\varepsilon^2}N_1[\bar\psi\gamma^\nu\psi](x) + \mathcal{O}(\varepsilon)$$

$$\bar\psi(x+\varepsilon)\lambda^a\gamma_\mu\psi(x) - \lambda^a\gamma_\mu\psi(x+\varepsilon)\bar\psi(x) = 2a_1^v N_1[\bar\psi\gamma_\mu\lambda^a\psi](x) + 4a_2^v\frac{\varepsilon_\mu\varepsilon_\nu}{\varepsilon^2}N_1[\bar\psi\gamma^\nu\lambda^a\psi](x) + \mathcal{O}(\varepsilon) \tag{II.14}$$

we obtain

$$j_\mu(x) = J_\mu(x) = N_1[\bar\psi\gamma_\mu\psi](x)$$

$$j_\mu^a(x) = J_\mu^a(x) = N_1[\bar\psi\gamma_\mu\lambda^a\psi](x) \tag{II.15}$$

where $a_i^{s,v}$ depend logarithmically on $(\varepsilon^2\mu^2)$. We also have $(Z_{s,v})^{-1} = a_1^{s,v} + a_2^{s,v}$, $(Z'_{s,v})^{-1} = a_2^{s,v}$.

The renormalization factors Z_i convert the renormalized field ψ into the unrenormalized field ψ_u.

The two currents j_μ and J_μ are thus identical, since $N_1[\bar\psi\gamma_\mu\psi]$ is the only conserved current one can construct in perturbation theory. We will obtain in Sect. III a suggestion that outside of perturbation a new isovector current (11) can be constructed due to the fact that the logarithms present in a_i^v may sum up to a positive power of ε, i.e. $(\varepsilon^2\mu^2)^\alpha$, $\alpha > 0$, so that equ. (II.14) is not any more valid as a power series in the coupling constants g_2 and g_3.

The field ψ described by the effective Lagrangian (II.1) satisfies

$$\langle 0|T\big[(i\partial\!\!\!/-m)\psi(x) - (g_1-c_1)N_{3/2}[\gamma_\mu\psi(\bar\psi\gamma^\mu\psi)](x) - (g_2-c_2)N_{3/2}[\lambda^a\gamma_\mu\psi(\bar\psi\gamma^\mu\lambda^a\psi)](x) - (g_3-c_3)N_{3/2}[\psi(\bar\psi\psi)](x)\big]\prod_{i=1}^N\psi(x_i)\prod_{j=1}^N\psi(y_j)|0\rangle = \tag{II.16}$$

$$= \text{delta-terms}$$

We will now show the existence of the zero mass limit of the above equation of motion (II.16), around the point $g_1=g_2=g_3=0$ along the curve found in the previous section, where all β_i and all anomalies are zero.

We rewrite eq. (II.16) using the Wilson expansions (12)

$$A = E_1N_{3/2}[(\bar\psi\gamma_\mu\psi)\gamma^\mu\psi](x) + E_2N_{3/2}[(\bar\psi\gamma_\mu\lambda^a\psi)\gamma^\mu\lambda^a\psi](x) + E_3N_{3/2}[(\bar\psi\psi)\psi](x) + iE_4\partial\!\!\!/\psi(x) + mE_5\psi(x) + \mathcal{O}(\eta) \tag{II.17}$$

$$B = F_1N_{3/2}[(\bar\psi\gamma_\mu\psi)\gamma^\mu\psi](x) + F_2N_{3/2}[(\bar\psi\gamma_\mu\lambda^a\psi)\gamma^\mu\lambda^a\psi](x) + F_3N_{3/2}[(\bar\psi\psi)\psi](x) + iF_4\partial\!\!\!/\psi(x) + mF_5\psi(x) + \mathcal{O}(\eta)$$

where we introduced the notation

$$A(x) = \lim_{\eta\to 0}\{N_1[\bar\psi\gamma_\mu\psi](x+\eta)\gamma^\mu\psi(x) - \gamma^\mu\psi(x+\eta)N_1[\bar\psi\gamma_\mu\psi](x)\}$$

$$B(x) = \lim_{\eta\to 0}\{N_1[\bar\psi\gamma_\mu\lambda^a\psi](x+\eta)\gamma^\mu\lambda^a\psi(x) - \gamma^\mu\lambda_a\psi(x+\eta)N_1[\bar\psi\gamma_\mu\lambda^a\psi](x)\} \tag{II.18}$$

$$\Delta_5(x) = N_{3/2}[(\bar\psi\psi)\psi](x)$$

Note that the coefficients E_i and F_i are m-independent due to our subtraction scheme (II.2) and (II.3). The absence of direction dependent terms of the form $\varepsilon_\alpha\varepsilon_\beta/\varepsilon^2$, $\varepsilon_\alpha\partial^\alpha/\varepsilon^2$ follows from the conservation of all vector and axialvector currents in the $m\to 0$ limit. Using (II.17) the $m\to 0$ limit of eq. (II.16) can be written:

$$[E_1F_2 - E_2F_1 - (g_1-c_1)(E_2F_4 - F_2E_4) - (g_2-c_2)[F_1E_4 - F_4E_1]]\,i\partial\!\!\!/\psi(x) =$$

$$= [(g_1-c_1)F_2 - (g_2-c_2)F_1]A + [(g_2-c_2)E_1 - (g_1-c_1)E_2]B + \tag{II.19}$$

$$+ [(g_3-c_3)(E_1F_2 - E_2F_1) + (g_1-c_1)(E_2F_3 - F_2E_3) + (g_2-c_2)(F_1E_3 - F_3E_1)]\Delta_5$$

i.e. $\quad i\not\partial\,\psi(x) = d_1 A(x) + d_2 B(x) + d_3 \Delta_5(x) \qquad$ (II.20)

To show the finiteness of d_i, i=1,2,3 we use (II.20) to get:

$$-i\partial^\mu\{(a_1^s+a_2^s)^{-1}:\bar\psi(x+\epsilon)\gamma_\mu\gamma^5\psi(x):\} = \frac{d_1}{a_1^s+a_2^s}\{\bar\psi(x+\epsilon)\gamma^5 A(x) + \bar A(x+\epsilon)\gamma^5\psi(x)\} + \frac{d_2}{a_1^s+a_2^s}\{\bar\psi(x+\epsilon)\gamma^5 B(x) + \bar B(x+\epsilon)\gamma^5\psi(x)\} + \frac{d_3}{a_1^s+a_2^s}\{\bar\psi(x+\epsilon)\gamma^5\Delta_5(x) + \bar\Delta_5(x+\epsilon)\gamma^5\psi(x)\} \qquad \text{(II.21)}$$

Now from:

$$(a_1^s+a_2^s)^{-1}\{\bar\psi(x+\epsilon)\gamma^5 A(x) + \bar A(x+\epsilon)\gamma^5\psi(x)\} = i r_1^s N_2[\partial^\mu(\bar\psi\gamma_\mu\gamma^5\psi)](x) + O(\epsilon)$$

$$(a_1^s+a_2^s)^{-1}\{\bar\psi(x+\epsilon)\gamma^5 B(x) + \bar B(x+\epsilon)\gamma^5\psi(x)\} = i r_2^s N_2[\partial^\mu(\bar\psi\gamma_\mu\gamma^5\psi)](x) + O(\epsilon)$$

$$(a_1^s+a_2^s)^{-1}\{\bar\psi(x+\epsilon)\gamma^5 \Delta_5(x) + \bar\Delta_5(x+\epsilon)\gamma^5\psi(x)\} = i r_3^s N_2[\partial^\mu(\bar\psi\gamma_\mu\gamma^5\psi)](x) + r_4^s N_2[(\bar\psi\gamma^5\psi)(\bar\psi\psi)](x) + O(\epsilon)$$

We obtain

$$-i\partial^\mu\{(a_1^s+a_2^s)^{-1}:\bar\psi(x+\epsilon)\gamma_\mu\gamma^5\psi(x):\} = i(d_1 r_1^s + d_2 r_2^s + d_3 r_3^s)\cdot\partial^\mu[N_1(\bar\psi\gamma_\mu\gamma^5\psi)](x) + d_3 r_4^s N_2[(\bar\psi\gamma^5\psi)(\bar\psi\psi)](x) \qquad \text{(II.22)}$$

In eq. (II.22) we have dropped direction dependent terms, since they are absent in the exactly soluble case.

Had we used the iso-vector axial current $:\bar\psi(x+\epsilon)\gamma_\mu\lambda^a\psi(x):$ we would get a similar equation

$$-i\partial^\mu\{(a_1^v+a_2^v)^{-1}:\bar\psi(x+\epsilon)\lambda^a\gamma_\mu\gamma^5\psi(x):\} = i(d_1 r_1^v + d_2 r_2^v + d_3 r_3^v)\,\partial^\mu N_1[\bar\psi\gamma_\mu\gamma^5\lambda^a\psi](x) + d_2 r_4^v f_{abc} N_2[(\bar\psi\gamma^\mu\lambda^b\psi)(\bar\psi\gamma_\mu\gamma^5\lambda^c\psi)](x) + d_3 r_5^v N_2[(\bar\psi\gamma^5\lambda^a\psi)(\bar\psi\psi)](x) \qquad \text{(II.23)}$$

For solubility we thus require $d_3 r_4^S = d_2 r_4^V = d_3 r_5^V = 0$. Since in zeroth order r_4^S, r_4^V and r_5^V are nonzero, we conclude that $d_2 = d_3 = 0$ in perturbation theory. The finiteness of d_1 is obtained applying $\tilde D$ given by eq. (II.10) to both sides of the field equation with $d_2 = d_3 = 0$, together with the knowledge that the current scales canonically. One obtains $\tilde D d_1 = 0$, which implies $\partial/\partial\mu, (d_1) = 0$, i.e. d_1 is an ϵ-independent constant.

III) NON-PERTURBATIVE SOLUTION

In this section we want to discuss non-perturbative solutions to the Thirring model with U(n) symmetry, which are not equivalent to the trivial case. One way to do this systematically is to construct Green functions by integrating Ward identities (W.I.) as was first done by Johnson (9) for the Thirring model without internal symmetry. This is possible if one knows $\partial_\mu j^\mu$ and $\partial_\mu \tilde{j}^\mu = \partial_\mu \varepsilon^{\mu\nu} j_\nu$. It is thus immediately obvious, that one has to get rid of the anomalies in equ. (II.8c) and (II.8d), since they do not only change the normalization of the current. As we know from previous sections, this will in general only be possible for particular values of some coupling constants. In particular, this happens at $g_2 = g_3 = 0$, but we do not consider this case, since it in no way differs from the usual U(1) Thirring model.

We thus consider the massless case with the following equation of motion:

$$i\not{\partial}_x \langle 0|T\psi(x)\,\Sigma\,|0\rangle = Z\sum_{j=1}^{M}\delta(x-y_j)(-1)^{j-1+N}\langle 0|T\,\Sigma_j\,|0\rangle + \tag{III.1}$$
$$+\langle 0|T[g_1 :j^\mu\gamma_\mu\psi: + g_2 :j^\mu_a\gamma_\mu\lambda^a\psi:](x)\,\Sigma\,|0\rangle$$

$$\Sigma = \psi(x_1)\ldots\psi(x_N)\bar\psi(y_1)\ldots\bar\psi(y_M)$$
$$\Sigma_j = \psi(x_1)\ldots\psi(x_N)\bar\psi(y_1)\ldots\bar\psi(y_{j-1})\bar\psi(y_{j+1})\ldots\bar\psi(y_M) \tag{III.2}$$

Z is some finite constant.

In the following we will present a partial solution of this model, which is however sufficient to fix all parameters ocurring in the solution.

In the following of this section we will denote

$\langle 0|TO(x)|0\rangle$ by $\langle O(x)\rangle_T$ for simplicity.

III.1) The two and four point functions

The equal time commutators of the isoscalar, isovector (12), vector and axial vector charges with $\psi(x)$ suggest the validity of the following W.I.:

$$\partial_\mu\langle j^\mu(x)\psi(x_1)\bar\psi(y_1)\rangle_T = \tag{III.3a}$$
$$= a_1[-\delta(x-x_1)+\delta(x-y_1)]\langle\psi(x_1)\bar\psi(y_1)\rangle_T$$

$$\partial_\mu\langle\tilde j^\mu(x)\psi(x_1)\bar\psi(y_1)\rangle_T =$$
$$= a_2[-\delta(x-x_1)\gamma^5_{x_1}-\delta(x-y_1)\gamma^5_{y_1}]\langle\psi(x_1)\bar\psi(y_1)\rangle_T \tag{III.3b}$$

$$\partial_\mu\langle j^\mu_a(x)\psi(x_1)\bar\psi(y_1)\rangle_T = \tag{III.3c}$$
$$= b_1[-\delta(x-x_1)\lambda^a_{x_1}+\delta(x-y_1)\lambda^a_{y_1}]\langle\psi(x_1)\bar\psi(y_1)\rangle_T$$

$$\partial_\mu\langle\tilde j^\mu_a(x)\psi(x_1)\bar\psi(y_1)\rangle_T = \tag{III.3d}$$
$$= b_2[-\delta(x-x_1)\lambda^a_{x_1}\gamma^5_{x_1}-\delta(x-y_1)\lambda^a_{y_1}\gamma^5_{y_1}]\langle\psi(x_1)\bar\psi(y_1)\rangle_T$$

where a_i, b_i are functions of g_i, to be determined. Equs. (III.3) are easily integrated yielding

$$\langle j^\mu(x)\psi(x_1)\bar\psi(y_1)\rangle_T = \tag{III.4a}$$
$$= (a_1 g^{\mu\nu}+a_2\varepsilon^{\mu\nu}\gamma^5_{x_1})\,\partial_\nu[D(x-x_1)-D(x-y_1)]\langle\psi(x_1)\bar\psi(y_1)\rangle_T$$

$$\langle j^\mu_a(x)\psi(x_1)\bar\psi(y_1)\rangle_T = \tag{III.4b}$$
$$= (b_1 g^{\mu\nu}+b_2\varepsilon^{\mu\nu}\gamma^5_{x_1})\lambda^a_{x_1}\partial_\nu[D(x-x_1)-D(x-y_1)]\langle\psi(x_1)\bar\psi(y_1)\rangle_T$$

where $D(x) = i\pi/(2\pi)^2\log(x^2-i\varepsilon)\mu^2$ is the causal solution of $\Box D(x) = -\delta^{(2)}(x)$ and we allways mean $\partial_\mu D(x-y) \equiv \frac{\partial}{\partial x^\mu}D(x-y)$.

With equ. (III.4) and the definition (III.2) the equation of motion (III.1) becomes

$$i\not{\partial}_{x_1}\langle\psi(x_1)\bar\psi(y_1)\rangle_T = iZ\,\delta(x_1-y_1) -$$
$$-[g_1(a_1-a_2)+g_2C(b_1-b_2)]\cdot\not{\partial}D(x_1-y_1)\langle\psi(x_1)\bar\psi(y_1)\rangle_T$$

whose solution is

$$\langle\psi(x)\,\bar\psi(y)\rangle_T =$$

$$= \exp\{i[g_1(a_1-a_2)+g_2C(b_1-b_2)]D(x-y)\}\langle\psi(x)\bar\psi(y)\rangle_T^{(0)} \qquad \text{(III.5)}$$

where the free field two-point function is given by

$\langle\psi(x)\bar\psi(y)\rangle_T^{(0)} = \not\partial D(x-y)$ and $C \equiv \lambda^a\lambda^a = 2(n-1/n)$.

Thus Z in equ. (III.1) is actually zero.

For four fields the isovector Ward identities are

$$\partial_\mu\langle j^\mu_a(x)\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T = b_1[-\delta(x-x_1)\lambda^a_{x_1} - \qquad \text{(III.6a)}$$

$$-\delta(x-x_2)\lambda^a_{x_2}+\delta(x-y_1)\lambda^a_{y_1}+\delta(x-y_2)\lambda^a_{y_2}]\langle\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T$$

$$\partial_\mu\langle\tilde j^\mu_a(x)\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T = b_2[-\delta(x-x_1)\lambda^a_{x_1}\gamma^5_{x_1}-\delta(x-x_2)\lambda^a_{x_2}\gamma^5_{x_2}-$$

$$-\delta(x-y_1)\lambda^a_{y_1}\gamma^5_{y_1}-\delta(x-y_2)\lambda^a_{y_2}\gamma^5_{y_2}]\langle\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T \qquad \text{(III.6b)}$$

Integrating gives

$$\langle j^\mu_a(x)\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T = \{\lambda^a_{x_1}(b_1\partial^\mu+b_2\gamma^5_{x_1}\tilde\partial^\mu)D(x-x_1)+$$

$$+\lambda^a_{x_2}(b_1\partial^\mu+b_2\gamma^5_{x_2}\tilde\partial^\mu)D(x-x_2)+\lambda^a_{y_1}(-b_1\partial^\mu+b_2\gamma^5_{y_1}\tilde\partial^\mu)D(x-y_1)+$$

$$+\lambda^a_{y_2}(-b_1\partial^\mu+b_2\gamma^5_{y_2}\tilde\partial^\mu)D(x-y_2)\}\langle\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T \qquad \text{(III.7)}$$

For $\langle j^\mu(x)\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T$ just replace λ^a by one and b_i by a_i in equ. (III.7).

The equation of motion now becomes

$$i\not\partial_{x_1}\langle\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T = \{g_1[(a_1+a_2\gamma^5_{x_1}\gamma^5_{x_2})\not\partial D(x_1-x_2)-$$

$$-(a_1-a_2\gamma^5_{x_1}\gamma^5_{y_1})\not\partial D(x_1-y_1)-(a_1-a_2\gamma^5_{x_1}\gamma^5_{y_2})\not\partial D(x_1-y_2)]+$$

$$+g_2[\lambda^a_{x_1}\lambda^a_{x_2}(b_1+b_2\gamma^5_{x_1}\gamma^5_{x_2})\not\partial D(x_1-x_2)-\lambda^a_{x_1}\lambda^a_{y_1}(b_1-b_2\gamma^5_{x_1}\gamma^5_{y_1})\not\partial D(x_1-y_1)-$$

$$-\lambda^a_{x_1}\lambda^a_{y_2}(b_1-b_2\gamma^5_{x_1}\gamma^5_{y_2})\not\partial D(x_1-y_2)]\}\,\times$$

$$\times\langle\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T \qquad \text{(III.8)}$$

whose solution is

$$\langle\psi(x_1)\psi(x_2)\bar\psi(y_1)\bar\psi(y_2)\rangle_T = \exp\{-i[g_1(a_1-a_2\gamma^5_{x_1}\gamma^5_{x_2})+g_2(b_1-b_2\gamma^5_{x_1}\gamma^5_{x_2})\lambda^a_{x_1}\lambda^a_{x_2}]\cdot$$

$$\cdot[D(x_1-x_2)+D(y_2-y_1)-D(x_1-y_1)-D(y_2-x_2)]\}\langle\psi(x_1)\bar\psi(y_2)\rangle_T\langle\psi(x_2)\bar\psi(y_1)\rangle_T -$$

$$-\exp\{-i[g_1(a_1-a_2\gamma^5_{x_1}\gamma^5_{x_2})+g_2(b_1-b_2\gamma^5_{x_1}\gamma^5_{x_2})\lambda^a_{x_1}\lambda^a_{x_2}]\cdot[D(x_1-x_2)+$$

$$+D(y_1-y_2)-D(y_1-x_2)-D(x_1-y_2)]\}\langle\psi(x_1)\bar\psi(y_1)\rangle_T\langle\psi(x_2)\bar\psi(y_2)\rangle_T \qquad \text{(III.9)}$$

Up to the four-point function no problems due to the noncommutativity of the λ-matrices arise . In particular one easily verifies that $[\lambda^a\otimes\lambda^a, \exp\lambda^b\otimes\lambda^b] = 0$. (See also the appendix).

III.2) Definition of currents

In order to determine the coeficients a_i and b_i, one has to exhibit the currents as limits of products of the field $\psi(x)$, manufacture from an N-point function some N'-point function of the current and compare the result with the W.I..

For a_1 and a_2 the procedure is the same as given by Johnson (9) or Dell'Antonio et al (13), the two differing only by a renaming of the coupling constant g_1.

One defines the isoscalar current as:

$$\langle j_\mu(x)\underline{X}\rangle_T \equiv$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4}\langle\{[\bar\psi(x+\varepsilon)\gamma_\mu\psi(x)-\gamma_\mu\psi(x+\varepsilon)\bar\psi(x)]+[\varepsilon\to\tilde\varepsilon]\}\underline{X}\rangle_T Z_s(\varepsilon) \qquad \text{(III.10)}$$

The normalization can be fixed, requireing the isoscalar Schwinger term to have some value.

The satisfaction of the isovector Ward identities with a suitably defined isovector current, turns out to be a nontrivial endeavour. We have found the following definition to do the job in the 2-and 4-point sectors (plus any number of currents):

$$\langle j_\mu^a(x)\underline{X}\rangle_T \equiv \lim_{\varepsilon\to 0}\frac{\mathcal{N}}{4}\Big\langle\Big\{\Big[\bar\psi(x+\varepsilon)\gamma_\alpha\lambda^a\psi(x) -$$

$$-\gamma_\alpha\lambda^a\psi(x+\varepsilon)\bar\psi(x)\Big]\frac{\varepsilon_\mu\varepsilon^\alpha+\tilde\varepsilon_\mu\tilde\varepsilon^\alpha}{\varepsilon^2}+(\varepsilon\to-\varepsilon)\Big\}\underline{X}\Big\rangle_T\; Z(\varepsilon) \qquad \text{(III.11)}$$

where $Z(\varepsilon) = \exp i\, g_2(b_1-b_2)C\, D(\varepsilon)$

Because of the non abelian structure, the normalization factor N is not arbitrary and will be fixed by manufacturing one current out of the 4-point function $\langle\psi\psi\bar\psi\bar\psi\rangle_T$ and comparing the result with the one current W.I. equ. (III.4b). As opposed to the isoscalar case only the term proportional to $\langle\psi(x_1)\bar\psi(y_1)\rangle_T\langle\psi(x_2)\bar\psi(y_2)\rangle_T$ will contribute to $\langle j_\mu^a(x_1)\psi(x_2)\bar\psi(y_2)\rangle_T$, when $x_1 \to y_1$, since appearently the term $\sim\langle\psi(x_1)\bar\psi(y_2)\rangle_T * \langle\psi(x_2)\bar\psi(y_1)\rangle_T$ does not allow the extraction of some finite part with a definition like equ. (III.10). In order for one term alone to satisfy the W.I. equ. (III.4b) one needs to flip the sign of the $\gamma^5\otimes\gamma^5$ part in $(b_1-b_2\gamma_5\otimes\gamma_5)$ of the four-point function in order to obtain a nontrivial solution with $b_1\neq b_2$. This is effected by $(\varepsilon_\mu\varepsilon_\alpha+\tilde\varepsilon_\mu\tilde\varepsilon_\alpha)/\varepsilon^2$.

To determine $\mathcal{N}$ we take the limit $y_1=x_1+\varepsilon$, $\varepsilon\to 0$ in equ. (III.9):

$$\langle\psi(x_1)\psi(x_2)\bar\psi(x_1+\varepsilon)\bar\psi(y_2)\rangle_T = \Big(\exp\Big\{-i\big[g_1(a_1-a_2\gamma^5_{x_1}\gamma^5_{x_2})+g_2(b_1-b_2\gamma^5_{x_1}\gamma^5_{x_2})\lambda^a_{x_1}\lambda^a_{x_2}\big] *$$

$$*\big[D(x_1-x_2)+D(y_2-x_1-\varepsilon)-D(\varepsilon)-D(y_2-x_2)\big]\Big\}\exp i\Big\{\big[g_1(a_1-a_2)+g_2C(b_1-b_2)\big]\big[D(x_1-y_2)+$$

$$+D(x_2-x_1-\varepsilon)\big]\Big\}\langle\psi(x_1)\bar\psi(y_2)\rangle^{(0)}\langle\psi(x_2)\bar\psi(x_1+\varepsilon)\rangle^{(0)} - \exp\Big\{-i\big[g_1(a_1-a_2\gamma^5_{x_1}\gamma^5_{x_2})+$$

$$+g_2(b_1-b_2\gamma^5_{x_1}\gamma^5_{x_2})\lambda^a_{x_1}\lambda^a_{x_2}\big]\big[D(x_1-x_2)-D(x_1-x_2-\varepsilon)-D(x_1-y_2)+D(x_1-y_2-\varepsilon)\big]\Big\}\cdot$$

$$\cdot\langle\psi(x_2)\bar\psi(y_2)\rangle_T\frac{i}{2\pi}\frac{\cancel{\varepsilon}}{\varepsilon^2}\Big)\; Z(\varepsilon) \qquad \text{(III.12)}$$

multiplying by $(\gamma_\alpha\lambda_a)_{x_1}$ we obtain for the first term above:

$$\exp i\big\{\big[g_1(a_1-a_2)+g_2C(b_1-b_2)\big]D(y_2-x_2)\big\}\,*$$

$$*\exp\Big\{i g_2(b_1-b_2)\Big(\frac{2}{n}+C\Big)\big[D(x_1-x_2)-D(y_2-x_2)+D(y_2-x_1)-$$

$$-D(\varepsilon)\big]\Big\}\langle\psi(x_1)\bar\psi(y_2)\rangle_T^{(0)}\langle\psi(x_2)\bar\psi(y_1)\rangle_T^{(0)}$$

$$\text{(III.13)}$$

where we used $\lambda^b\lambda^a\lambda^b = -\frac{2}{n}\lambda^a$, $\langle\gamma^5\psi\bar\psi\rangle_T = -\langle\psi\bar\psi\gamma^5\rangle_T$.

The first factor gives $\langle\psi(x_2)\bar\psi(y_2)\rangle_T$ and the second goes to zero for $g_2(b_1-b_2)(-2/n-C)=g_2(b_1-b_2) * (-2n) < 0$, i.e.

$$g_2(b_1-b_2) > 0 \qquad \text{(III.14)}$$

Our solution does thus not have a perturbative expansion, since we cannot take $g_2\to 0$.

The remaining term in equ. (III.12) now gives

$$\langle j^a_\mu(x)\psi(x_1)\bar\psi(y_1)\rangle_T = \frac{2g_2}{\pi}\mathcal{N}\Big\{\lambda^a_{x_1}(b_1\partial_\mu+b_2\gamma^5_{x_1}\tilde\partial_\mu)*$$

$$*\big(D(x-x_1)-D(x-y_1)\big)\Big\}\langle\psi(x_1)\bar\psi(y_1)\rangle_T$$

Comparing with equ. (III.4b) we find

$$\frac{2g_2}{\pi}\mathcal{N}=1 \quad\text{or}\quad \mathcal{N}=\frac{\pi}{2g_2} \qquad \text{(III.15)}$$

To determine b_1 and b_2 we integrate the two current W.I. given by the non abelian structure:

$$\langle j^a_\mu(x)j^b_\nu(y)\psi(x_2)\bar\psi(y_2)\rangle_T = \big[\lambda^a_{x_2}(b_1\partial_\mu+b_2\gamma^5_{x_2}\tilde\partial_\mu)D(x-x_2)+$$

$$+\lambda^a_{y_2}(-b_1\partial_\mu+b_2\gamma^5_{y_2}\tilde\partial_\mu)D(x-y_2)\big]\langle j^b_\nu(y)\psi(x_2)\bar\psi(y_2)\rangle_T -$$

$$-i f_{abc}\big[\partial_\mu D(x-y)\langle j^b_\nu(y)\psi(x_2)\bar\psi(y_2)\rangle + \qquad \text{(III.16)}$$

$$+\tilde\partial_\mu D(x-y)\langle\tilde j^b_\nu(y)\psi(x_2)\bar\psi(y_2)\rangle_T\big] -$$

$$-i\, g_\nu\,\delta_{ab}(\partial_\mu\partial_\nu+\tilde\partial_\mu\tilde\partial_\nu)D(x-y)\langle\psi(x_2)\bar\psi(y_2)\rangle_T$$

where $\mathcal{S}_V$ is the isovector Schwinger term. It is easy to see that equ. (III.16) gives the correct equal time commutation relations for the vector and axialvector charges among themselves and $\psi(x)$. The interested reader may also want to verify symmetry under the exchange $(x,a,\mu) \leftrightarrow (y,b,\nu)$.

Now start from the one current W.I. and take limits:

$$\langle j^a_\mu(x)(\gamma_\alpha\lambda^b\psi)(x_1)\psi(x_2)\bar\psi(x_1+\varepsilon)\bar\psi(y_1)\rangle_T = \Big[\lambda^a_{x_1}(b_1\partial_\mu + b_2\gamma^5_{x_1}\tilde\partial_\mu)D(x-x_1) +$$

$$+ \lambda^a_{x_2}(b_1\partial_\mu + b_2\gamma^5_{x_2}\tilde\partial_\mu)D(x-x_2) + \lambda^a_{x_1+\varepsilon}(-b_1\partial_\mu + b_2\gamma^5_{x_1+\varepsilon}\tilde\partial_\mu)D(x-x_1-\varepsilon) +$$

$$+ \lambda^a_{y_2}(-b_1\partial_\mu + b_2\gamma^5_{y_2}\tilde\partial_\mu)D(x-y_2)\Big]\Big(\exp\{-i[g_1(a_1 - a_2\gamma^5_{x_1}\gamma^5_{x_2}) +$$

$$+ g_2\lambda^c_{x_1}\lambda^c_{x_2}(b_1 - b_2\gamma^5_{x_1}\gamma^5_{x_2})][D(x_1-y_2) - D(\varepsilon) - D(y_2-x_2) + D(y_2-x_2-\varepsilon)]\} \cdot$$

$$\cdot \exp\{i[g_1(a_1-a_2) + Cg_2(b_1-b_2)][D(x_1-y_2) - D(x_2-x_1-\varepsilon)]\} \cdot$$

$$\cdot \langle\psi(x_1)\bar\psi(y_2)\rangle^{(0)}_T \langle\psi(x_2)\bar\psi(x_1+\varepsilon)\rangle^{(0)}_T - \exp\{-i[g_1(a_1 - a_2\gamma^5_{x_1}\gamma^5_{x_2}) +$$

$$+ g_2\lambda^c_{x_1}\lambda^c_{x_2}(b_1 - b_2\gamma^5_{x_1}\gamma^5_{x_2})][D(x_1-x_2) - D(x_1-y_2) - D(x_1-x_2+\varepsilon) +$$

$$+ D(y_2-x_1-\varepsilon)]\}\langle\psi(x_2)\bar\psi(y_2)\rangle_T\langle\psi(x_1)\bar\psi(x_1+\varepsilon)\rangle_T\Big) \qquad \text{(III.18)}$$

One verifies that again under condition (III.14) the first term goes to zero, whereas the second term yields:

$$\langle j^a_\mu(x) j^b_\nu(y)\psi(x_2)\bar\psi(y_2)\rangle_T = -i\delta_{ab}\frac{2\mathcal{N}}{\pi}(b_1\partial_\mu\partial_\nu - b_2\tilde\partial_\mu\tilde\partial_\nu)\cdot$$

$$\cdot D(x-y)\langle\psi(x_2)\bar\psi(y_2)\rangle_T + \frac{2i\mathcal{N}}{\pi} f_{abc}\lambda^c\partial^\alpha D(x-x_1)\, g_2 \cdot$$

$$\cdot \{g_{\mu\alpha}(b_1^2\partial_\nu + b_1 b_2\gamma^5_{x_2}\tilde\partial_\nu) - \varepsilon_{\mu\alpha}(b_2^2\partial_\nu\gamma^5_{x_2} - b_1 b_2\tilde\partial_\nu)\}\cdot$$

$$\cdot [D(x_1-y_2) - D(x_1-x_2)]\langle\psi(x_2)\bar\psi(y_2)\rangle_T - 2g_2\partial^\alpha[(\lambda^a\lambda^b)D(x-x_2) -$$

$$- (\lambda^b\lambda^a)D(x-y_2)](b_1 g_{\mu\alpha} + b_2\gamma^5_{x_2}\varepsilon_{\mu\alpha})(b_1\partial_\nu + b_2\gamma^5_{x_2}\tilde\partial_\nu)\cdot$$

$$\cdot [D(x_1-y_2) - D(x_1-x_2)]\langle\psi(x_2)\bar\psi(y_2)\rangle_T \qquad \text{(III.19)}$$

Comparing with equ. (III.16) we get

$$b_1 = -b_2 = \frac{1}{2}\,, \quad \mathcal{S}_V = \frac{1}{2g_2} \qquad \text{(III.19)}$$

III.3) Determination of g_2 via the energy-momentum tensor

By solving our model in the sector

$$\langle\psi\bar\psi j_{\mu_1}\cdots j_{\mu_n} j^{a_1}_{\nu_1}\cdots j^{a_m}_{\nu_m}\rangle_T\,, \langle\psi\psi\bar\psi\bar\psi j_{\mu_1}\cdots j_{\mu_n} j^{a_1}_{\nu_1}\cdots j^{a_m}_{\nu_m}\rangle_T$$

we have been able to fix all our constants, except g_1 and g_2. On the other hand we do not expect g_2 to be arbitrary. Presumably g_2 will be fixed by looking at higher point Green function or at a possible operator solution, which we have not been able to do.

We thus take a route via the energymomentum tensor, much in the spirit of Ref. (7). We first notice, that our W.I. equ. (III.4b) and (III.16) differ from the free field W.I. only by changing

$$\mathcal{S}_V \to \frac{1}{4\pi}\,, \quad \gamma^5 \to -\gamma^5 \qquad \text{(III.20)}$$

Our current Green functions are thus equivalent to the free ones, except for the value of the Schwinger term. We know then that our energy-momentum tensor has to be of the Sugawara form

$$\langle\theta_{\mu\nu}(x)\,\overline{X}\rangle_T = \langle\{E_1[:j^a_\mu j^a_\nu:(x) - \tfrac{1}{2}g_{\mu\nu}:j^a_\alpha j^{a\alpha}:(x)] +$$

$$+ E_2[:j_\mu j_\nu:(x) - \tfrac{1}{2}g_{\mu\nu}:j_\alpha j^\alpha:(x)]\}\,\overline{X}\rangle_T$$

$$\text{(III.21)}$$

The constants are now fixed by requireing the correct commutation relations with j_μ and j^a_μ say. One gets for example (see also ref. 7).

$$i\left[\partial^\mu \theta_{\mu\nu}(x), j^a_\rho(y)\right]_{E.T} = E_1\left(\frac{n}{2\pi}+S_\nu\right)\partial_\nu j^a_\rho(y)\,\delta(x'-y')$$

$$\Rightarrow \quad g_2\left(\frac{n}{2\pi}+\frac{1}{2g_2}\right) = 1 \qquad \text{(III.22)}$$

since comparing with the equation of motion one gets

$E_2 = g_2$ and $E_1 = \dfrac{2g_1}{a_1-a_2}$

We finally obtain

$$g_2 = \frac{\pi}{n} \qquad \text{(III.23)}$$

a_1 and a_2 are given in the parametrization of Ref. 13 by

$$a_1-a_2=g_1S_s\ ,\ a_1a_2= S_s$$

where S_s is the isoscalar Schwinger term.

Our partial solution is thus sufficient to fix all parameters, except g_1, which is expected to remain arbitrary. We hope to come back in a future publication to deal with a complete operator solution.

IV) DISCUSSION

Guided by the simplicity of the usual Thirring model we tried in this paper to investigate the existence of possible solutions for the model with U_n symmetry. Thus after some perturbative motivation, we looked at the class of massless spinor field theories generated by the (formal) interacting Lagrangian density

$$\mathcal{L}_I = g_1(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi) + g_2(\bar\psi\gamma_\mu\lambda^a\psi)(\bar\psi\gamma^\mu\lambda^a\psi) \qquad \text{(IV.1)}$$

We have found explicit solutions in the case $g_2=0$. These are in a certain sense trivial, since they correspond to n

independent self-interacting Thirring fields. The operator solution can be written in Klaiber's form

$$\psi(x) = e^{i\chi^+(x)}\,\varphi(x)\,e^{i\chi^-(x)} \qquad \text{(IV.2)}$$

where φ is a canonical free spinor field,

$$\chi = \alpha j + \beta\gamma^5\tilde{j} + q + \gamma^5\tilde{q}\ ;\ \alpha-\beta = \frac{g_1}{\sqrt{\pi}} \qquad \text{(IV.3)}$$

and we are using the same notation employed by Klaiber. The N point function has Klaiber's form (see formula IV.25 of ref. 14) with

$$a = n\alpha^2 - 2\sqrt{\pi}\,\alpha \quad ,\ b = n\beta^2 - 2\sqrt{\pi}\,\beta$$

$$\lambda = \alpha\beta n - (\alpha+\beta)\sqrt{\pi}$$

The dimension of the field is

$$d = \frac{1}{4\pi}\left[n(\alpha-\beta)^2 + 2\lambda\right] + \frac{1}{2}$$

and the spin has the value $s = \dfrac{1}{2} + \dfrac{\lambda}{2\pi}$

We have also discussed the existence of other solutions, associated with $g_2\neq 0$. Following a Johnson type construction we obtained the two and the four point Green functions associated with the value $g_2=\frac{\pi}{n}$. The dimension of the spin 1/2 field is (15)

$$d = \frac{1}{2}\left[1 + n\frac{(g_1/2\pi)^2}{1-(g_1/2\pi)^2} + \frac{n^2-1}{4n^2}\right] \qquad \text{(IV.4)}$$

The existence of a second non trivial solution was already conjectured in a paper by R.Dashen and Y. Frishman (7). These authors, following an algebraic construction similar to that of ref. 13 claim to have solutions for $g_2=0$ and also for $g_2=4\pi/n+1$. However, as Schroer has pointed out (15), Dashen and Frishman's second solution is only a different parametrization of the first one (IV.2) and not a really new solution. More precisely

$$\psi_{D.F.} = \gamma_0 \psi_K \tag{IV.5}$$

where ψ_K is a solution of the type (IV.2) with $\lambda = -2\pi$. It satisfies

$$i\not{\partial}\psi_{D.F.} = -g_s :\partial_\mu \gamma^\mu \psi_{D.F.}: - g_v :j^a_\mu \gamma^\mu \frac{\lambda^a}{2} \psi_{D.F.}: \tag{IV.6}$$

with $g_v = \frac{4\pi}{n+1}$ and $g_s = (\alpha+\beta)\sqrt{\pi} - \frac{2\pi}{n}$

This can be verified by using

$$\begin{aligned} J_\mu &= j_\mu - \frac{n\alpha}{\sqrt{\pi}} \partial_\mu \Delta Q - \frac{n\beta}{\sqrt{\pi}} \partial_\mu \tilde{\Delta} \tilde{Q} \\ J^a_\mu &= :\varphi \gamma_\mu \lambda^a \varphi: \\ :J_\mu \psi: &= \lim_{\varepsilon \to 0} \left\{ J^{(+)}_\mu (x+\varepsilon)\psi(x) + \psi(x) J^{(-)}_\mu (x-\varepsilon) \right\} \end{aligned} \tag{IV.7}$$

and the fact that if φ is a free spinor field then $\varphi' = \gamma_0 \varphi$ satisfies

$$i\not{\partial}\varphi' = \frac{2\pi}{n} :\partial^\mu \gamma_\mu \varphi': + \frac{2}{n+1} \lambda^a :j^a_\mu \gamma^\mu \varphi': \tag{IV.8}$$

which results from

$$\lambda^i_{ab} \otimes \lambda^i_{cd} = -2\frac{n+1}{n} \delta_{ab}\delta_{cd} + 2(n+1)\delta_{ac}\delta_{bd}$$

The field $\psi_{D.F.}$ has spin 1/2 and dimension

$$d = \frac{1}{2} + n(g_s/2\pi)^2 + \frac{n-1}{n}$$

According to the arguments of ref. 7, there is only one solution corresponding to the value $g_v = 4\pi/n+1$ and having the same short distance behaviour as a free field. We conclude therefore that $\gamma_0 \psi_K$ and Dashen and Frishman's solution are identical. There is another point with relation reference 7, which deserves some more comment. If one wants to introduce mass in theories that are not free one should start considering composite objects $N[\bar{\psi}\psi]$, where N indicates some normal product prescription (in the case of the Thirring model we could

use Lowenstein's normal products for example). The mass is then introduced by modifying the Lagrangian to

$$\mathcal{L} + m\, N[\bar{\psi}\psi] \tag{IV.9}$$

where $\mathcal{L}$ is the original (not free) Lagrangian and the mass insertion operator $N[\bar{\psi}\psi]$ can have an anomalous dimension. In this scheme the equation of motion for ψ should be derived and a priori there is no reason at all to restrict the field spin to the value 1/2.

Finally we want to comment on the removal of the question mark in the title. A complete solution is most probably not obtainable by applying an exponential depending on the currents to a free spinor field, because of the noncommutativity of the λ-matrices. We expect the 2N-point Green function of a nontrivial ($g_2 \neq 0$) solution to consist of a sum of terms, corresponding to the number of independent tensors made out of the λ's and γ_5's, each one multiplyied by a combination of exponentials. Since the number of these terms increases with N, an operator solution is probably very difficult to obtain.

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FOOTNOTES AND REFERENCES

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11) By isoscalar and isovector we actually mean the U(1) and SU(n) parts of U(N).

12) In this section all operator equations should be understood as time-ordered matrix elements with suitable δ-terms added.

13) G.F.Dell'Antonio, Y.Frishman and D.Zwanziger; Phys. Rev. 6, 988 (1972)

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15) Continuous spin $\tilde{\lambda}$ can easily be incorporated into this scheme and leads to a one-parameter family of solutions depending on $\tilde{\lambda}$.

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APPENDIX

In this appendix we want to show how the SU_2 algebra can be used to rewrite the four point function given by equ. (III.9) in a form where the λ-matrices in the exponent have disappeared. Consider the expansion

$$\exp i\alpha\, \sigma_a \otimes \sigma_a = 1 + i\alpha\, \sigma_a \otimes \sigma_a + \frac{(i\alpha)^2}{2!} \sigma_b \sigma_a \otimes \sigma_b \sigma_a + \dots$$

$$= 1 + \sum_n \frac{(i\alpha)^n}{n!} \left(a_n \sigma_a \otimes \sigma_a + b_n I \otimes I \right)$$

where the relation $\sigma_a \sigma_b = i \varepsilon_{abc} \sigma_c + \delta_{ab}$

was used. The numbers a_n and b_n satisfy

$$a_n = -b_n + 1$$

$$b_n = -3 b_{n-1} + 3$$

and so are given by

$$a_n = -(-3)^n - \frac{1}{4}\left(-1 + (-3)^{n+1}\right)$$

$$b_n = (-3)^n + \frac{1}{4}\left(3 + (-3)^{n+1}\right)$$

Thus

$$\exp i\alpha\, \sigma \otimes \sigma = e^{-3i\alpha} + \frac{3}{4} e^{3i\alpha} - \frac{3}{4} e^{-3i\alpha} +$$

$$+ \left[-e^{-3i\alpha} + \frac{1}{4} e^{i\alpha} + \frac{3}{4} e^{-3i\alpha} \right] \sigma \otimes \sigma$$

Besides that the relation

$$\exp\left[a + i b\, \gamma^5 \otimes \gamma^5\right] = e^a \left(\cos b + i \sin b \cdot \gamma^5 \otimes \gamma^5\right)$$

is easily verified. Therefore we get

$$\exp -i \left\{ g_1 (a_1 - a_2 \gamma^5 \otimes \gamma^5) + g_2\, \sigma \otimes \sigma\, (b_1 - b_2 \gamma^5 \otimes \gamma^5 \right\} C =$$

$$= \frac{1}{4} e^{-i g_1 C a_1} \left[\cos(g_1 a_2 C) + i \sin(g_1 a_2 C) \gamma^5 \otimes \gamma^5 \right] \otimes$$

$$\otimes \left\{ I \otimes I \left[e^{3i g_2 b_1 C} \left(\cos(3 g_2 b_2 C) - i \sin(3 g_2 b_2 C) \gamma^5 \otimes \gamma^5 \right) + \right. \right.$$

$$\left. + 3 e^{-i g_2 b_1 C} \left(\cos(g_2 b_2 C) + i \sin(g_2 b_2 C) \gamma^5 \otimes \gamma^5 \right) \right] +$$

$$+ \sigma \otimes \sigma \left[- e^{-3i g_2 b_1 C} \left(\cos(3 g_2 b_2 C) - i \sin(3 g_2 b_2 C) \gamma^5 \otimes \gamma^5 + \right. \right.$$

$$\left. \left. + e^{-i g_2 b_1 C} \left(\cos(b_2 g_2 C) + i \sin(b_2 g_2 C) \gamma^5 \otimes \gamma^5 \right) \right] \right\}$$

which can be used to remove the σ-matrices from the exponent in a straightforward way.