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ABSTRACT:

Rigorous lower bounds to the critical exponents γ, ν in ferromagnetic ising systems are derived, using an idea of Glimm and Jaffe [1] and some ideas and results of Fisher [5].

I) INTRODUCTION AND SUMMARY

Glimm and Jaffe ([1]) have recently proved that the critical indices η, ν and γ (3) are bounded below by their classical (Goldstone) values in the $(\phi^4)_2$ quantum field theory, namely

$$\eta \geq 0 \quad (I.1)$$

$$\nu \geq 1/2 \quad (I.2)$$

$$\gamma \geq 1 \quad (I.3)$$

In their paper they also derived, among others, the inequalities

$$2\nu \geq \gamma \quad (I.3)$$

$$2\nu \geq \alpha \quad (I.4)$$

Probably all the above field theory results hold also in statistical mechanics. In fact, one of them ((3)) is in field theory, a corollary of an inequality which holds for ferromagnetic ising systems (Lebowitz's inequality (8)), together with the so-called lattice approximation (2). However, most of the proofs given in field theory for (I.1-I.5) are specific to field theory: for instance, the simple inequality (I.1) is, in field theory, a simple consequence of the Kallēn-Lehmann representation in space-time dimensions higher than two ([1]), while in two dimensions it follows from a result of Simon ([2]). A proof of (I.1) in statistical mechanics exists, however (and is trivial) only in two dimensions, if we define η by*

$$\Gamma_2(r) - \Gamma_1^2 \underset{r\to\infty}{\sim} \frac{1}{r^{d-2+\eta}} \quad (I.6)$$

where Γ_r is the (infinite-volume) r-spin correlation function.

The proof of (I.3) in statistical mechanics following the method of ([1]) also requires some additional argument, which we provide in sect.II. In particular, two assumptions

* The notation $f(x) \underset{x\to b}{\sim} x^a$ means $\lim_{x\to b} \frac{\log f(x)}{\log x} = a$ for $x \in R_+ \to f(x) \in R_+$, as is standard ([3]).

((A) and (B) of section II), some applications of Griffiths' inequalities and a result of Fisher ([5]) are required. The r.h.s. of (I.3) is just the mean-field exponent (see, e.g. ([11])), and we feel that the importance and interest of this result makes isolation of the necessary assumptions and a complete proof desirable, although the central idea is, of course, provided by ([1]). As a corollary of (I.3), we have, by Fisher's inequality (5)

$$(2-\eta)\nu \geq \gamma \quad , \; 0 \leq 2-\eta \leq d \tag{I.7}$$

the bound

$$\nu \geq 1/d \tag{I.8}$$

(d is the dimension of the system), which is weaker than (I.2) and coincides with it only for d=2. Of course, (I.2) would follow from (I.1), (I.3) and (I.7), but a proof of (I.1) is at present missing, as remarked previously.

II) RESULT

We consider a ferromagnetic ising system. If Ω_N is a finite lattice region of N spins in Z^d, we define its Hamiltonian by

$$H_N(\{S_i\}_{i\in[1,N]}) = - \sum_{p,q\in\Omega_N} J(p-q)S_pS_q \tag{II.9}$$

where $\{S_i\}_{i\in[1,N]}$ are spin variables ($S_i=\pm1$), $J(p)\geq0$, and

$$\sum_{p\in Z^d} |J(p)| < \infty \tag{II.10}$$

(II.10) is just the usual stability condition. We shall be concerned with the thermodynamic limit along a sequence $\{\Omega_N\}$ of lattice regions such that $\Omega_N \supset \Omega_{N'}$, if N>N', of the susceptibility per spin χ_N, defined by (h>0)

$$\chi_N(\beta,h) \equiv \frac{\partial M_N(\beta,h)}{\partial h} \tag{II.11a}$$

where

$$M_N(\beta,h) \equiv \frac{\beta^{-1}}{N} \frac{\partial \log Z_N(\beta,h)}{\partial h} \qquad \text{(II.11b)}$$

is the magnetization per spin, and

$$Z_N(\beta,h) \equiv \sum_{\substack{S_i=\pm 1 \\ i\in[1,N]}} e^{-\beta\left[H_N(\{S_i\}) - h \sum_{k=1}^{N} S_K\right]} \qquad \text{(II.11c)}$$

By the GKS inequalities ([4]), $\lim_{N\to\infty} M_N(\beta,h) \equiv M(\beta,h)$ exists and is an analytic function of h for Reh>0 by the Lee-Yang theorems ([6]). The (infinite-volume) susceptibility is defined by

$$\chi(\beta,h) = \frac{\partial m(\beta,h)}{\partial h}$$

By the Lee-Yang theorems ([6]), this definition coincides with the following one:

$$\chi(\beta,h) \equiv \lim_{N\to\infty} \chi_N(\beta,h) \qquad \text{(II.12)}$$

(h>0), where the existence of the r.h.s. of (II.12) is also guaranteed by ([6]). However- there is at present no proof that

(A) $\chi(\beta,0) \equiv \lim_{h\to 0_+} \chi(\beta,h) = \lim_{N\to\infty} \chi_N(\beta,0)$ if $0<\beta<\beta_c$

which we assume as a reasonable conjecture ([5]) (or, alternatively ([5]), define the susceptibility in terms of which the critical exponent γ is defined, by $\chi(\beta)\equiv\lim_{N\to\infty} \chi_N(\beta,0)$). We assume further (see ([5]) for a discussion)

(B) $\chi(\beta,0) = \lim_{h\to 0_+} \chi(\beta,h)$ is analytic in the open disk of

the complex β-plane with center at the origin and radius β_c
(B) has been proved only for $0<|\beta|<\beta_c'$ with $\beta_c' << \beta_c$, ([7]).

The critical exponent γ is now defined as usual ([3]) by

$$\chi(\beta,0) \sim (\beta_c-\beta)^{-\gamma} \tag{II.13}$$

<u>Theorem</u>: Assumptions (A) and (B) imply (I.3).
<u>Proof</u>: The idea is in ([1]). As in (5), we define the "reduced" susceptibility $\hat{\chi}_N(\beta,h) \equiv \beta^{-1}\chi_N(\beta,h)$, $\hat{\chi}(\beta,h)\equiv\beta^{-1}\chi(\beta,h)$, $\hat{\chi}(\beta,0)=\lim_{h\to 0_+} \hat{\chi}(\beta,h)=\lim_{N\to\infty} \hat{\chi}_N(\beta,0)$ by (A) $(0<\beta<\beta_c)$. Of course $\hat{\chi}(\beta,0) \sim (\beta_c-\beta)^{-\gamma}$. We have

$$\hat{\chi}_N(\beta,0) = \frac{1}{N} \sum_{i,j\in\Omega_N} \left[\Gamma_{2,N}(i,j) - \Gamma_{1,N}(i)\,\Gamma_{1,N}(j)\right] \tag{II.14}$$

where $\Gamma_{r,N}\,(i_1,\ldots,i_r)$ is the r-spin correlation function for the region Ω_N. Now.

$$\frac{d\,\hat{\chi}_N(\beta,0)}{d\beta} = \frac{1}{N} \sum_{i,k,p,q\in\Omega_N} |J(p-q)|\,\{\Gamma_{4,N}(p,q,k,i) - \\ -\Gamma_{2,N}(p,q)\Gamma_{2,N}(k,i)\} \tag{II.15}$$

As in ([1]), by the Lebowitz inequality ([8]) and (II.15), we have

$$\frac{d\hat{\chi}_N(\beta,0)}{d\beta} \leq \frac{2}{N} \sum_{i,k,p,q\in\Omega_N} |J(p-q)|\;\Gamma_{2,N}(p,k)\;\Gamma_{2,N}(q,i) \tag{II.16}$$

By the GKS inequalities (4), $0\leq\Gamma_{r,N}\,(i_1,\ldots,i_r)\leq\Gamma_r(i_1,\ldots,i_r)$ and $\Gamma_{r,N}(i_1,\ldots,i_r) \underset{N\to\infty}{\to} \Gamma_r(i_1,\ldots,i_r)$ where $\Gamma_r(i_1,\ldots,i_r)$ are the infinite-volume correlation functions. Hence, from (II.16)

$$\frac{d\hat{\chi}_N(\beta,0)}{d\beta} \leq \frac{2}{N} \sum_{i,k,p,q\in\Omega_N} |J(p-q)|\;\Gamma_2(p,k)\;\Gamma_2(q,i) = \\ = \frac{2}{N} \sum_{i,k,p,q\in\Omega_N} |J(p-q)|\;\Gamma_2(p-k)\Gamma_2(q-i) \tag{II.17}$$

Let R_N be the diameter of Ω_N. Using now the GKS inequalities one more in the form $\Gamma_{r,N}(i_1,\dots,i_r) \le \Gamma_{r,N'}(i_1,\dots,i_r)$ for $N \le N'$, we have from (I.17)

$$\frac{d\hat{\chi}_N(\beta,0)}{d\beta} \le \frac{2}{N} \sum_{i \in \Omega_N} \sum_{q;|q-i| \le R_N} \sum_{p;|p-q| \le R_N} \sum_{k;|k-p| \le R_N}$$

$$|J(p-q)| \Gamma_2(p-k) \Gamma_2(q-i) = 2 \sum_{|r| \le R_N} |J(r)| \left(\sum_{|r| \le R_N} \Gamma_2(r) \right)^2 \le$$

$$\le 2 \sum_{r \in Z^d} |J(r)| \quad \chi(\beta,0)^2 \tag{II.18}$$

using Fisher's result [5] that $\chi(\beta,0) = \sum_{r \in Z^d} \Gamma_2(r)$. By (B) and the bound (([5]),(A14)) and Vitali's convergence theorem we have that $\{\chi_N(\beta,0)\}$ converges uniformly in any compact subset of the open disk with center at the origin and radius β_c to $\chi(\beta,0)$, hence

$$\frac{d\hat{\chi}_N(\beta,0)}{d\beta} \underset{N \to \infty}{\longrightarrow} \frac{d\hat{\chi}(\beta,0)}{d\beta} \quad (0<\beta<\beta_c)$$

Hence, by (II.18),

$$\frac{d\hat{\chi}(\beta,0)}{d\beta} \le c\, \hat{\chi}(\beta,0)^2$$

with $c = 2 \sum_{r \in Z^d} |J(r)|$, whence $\dfrac{d\hat{\chi}(\beta,0)^{-1}}{d\beta} \le c$ (II.19)

Integrating (II.19) from β to β_c, and assuming (II.13), we obtain (I.3). ∎

<u>Remark 1</u> - For ferromagnetic ising models characterized by

$$J(r) = r^{-s} \qquad d+1>s>d \tag{II.20}$$

one has Suzuki's inequalities ([9])

$$\nu(s) \ge \nu(t)$$
$$\gamma(s) \ge \gamma(t)$$

if $d+1 > s \geq t > d$. If the somewhat looser statement that "γ,ν decrease when the interaction range increases" holds, one has:

$$\left.\begin{array}{c} 1 \leq \gamma \leq 7/4 \\ 1/2 \leq \nu \leq 1 \end{array}\right. \quad d=2$$

and

$$\left.\begin{array}{c} 1 \leq \gamma \leq 5/4 \\ 1/3 \leq \nu \leq 5/8 \end{array}\right. \quad d=3$$

for all models with long-range interactions in particular the ones described by (II.20). Here we have used the well-known exponents for the nearest-neighbour two and three dimensional Ising models. In this respect, the fact that the mean field exponents should be lower bounds to these exponents is also quite intuitive, since mean field theory corresponds, in a sense, to the infinite-range limit $\gamma \to 0_+$ of infinite range interactions of type $J(r) = \gamma^d v(\gamma r)$ (see ([10])). A direct proof of (I.2,I.3) using this fact is an interesting open problem.

Remark 2 - If the assumptions of Suzuki's paper ([9]) are true, then the exponents γ_I, ν_I of the Heisenberg-Ising model also satisfy the inequalities

$$\gamma_I \geq 1 \qquad \nu_I \geq 1/d$$

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