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GOLDSTONE'S THEOREM FOR QUANTUM SPIN SYSTEMS OF FINITE RANGE

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<u>ABSTRACT</u>:

Goldstone's theorem is proved for quantum spin systems of finite range with rotationally invariant Hamiltonian under the assumptions that the ground state of the infinite system is unique, invariant under a subset of the translation group (which depends on the system) and has long range order.

In this paper we prove that the spectrum of the physical Hamiltonian of a quantum spin system of finite range with rotationally invariant (finite-region) Hamiltonian has no gap, under three assumptions on the ground state of the infinite system. This result, which is a form of Goldstone's theorem (whose analog in quantum field theory is (6), to which we also refer for additional and original references), was proved by Streater in (2) for ferromagnetic Heisenberg systems of finite range. As Streater, we use the results of (6), but have to generalize his Theorem 7, which relies upon structure specific to the ferromagnet, in particular upon the fact that the physical Hilbert space is the direct sum of dynamically independent "n-magnon sectors", and the restriction of the physical Hamiltonian to each of them is a bounded operator (see specially ((2),(23))).

In a recent paper, Reeh ((1)) proved by explicit construction the existence of zero energy ("Goldstone") states in the spectrum of any rotationally invariant antiferromagnetic Hamiltonian, under a number of assumptions on the ground state. Although he did not prove Goldstone's theorem, we shall see that just three of his four assumptions (unicity, translation invariance and long range order) suffice to provide a proof: his assumption of invariance of the ground state under rotations around the z-axis will not be needed. Furthermore, his proof, as given, seems to be restricted to one dimension[(1)] while ours holds for any number of dimensions.

For clarity, we divide the forthcoming proofs into two parts: the results of part 1 are independent of the above mentioned assumptions on the ground state (which will be more precisely formulated in part 2), while part 2 contains the results which explicity depend on them.

(1) Reeh used the fact that the number of terms in the commutator $\left[H_{\Lambda_1}, \underline{S}_\Lambda \right]$, which is considered in part 1, tends to a finite constant as $\Lambda\to\infty$ (see part 1 for this notation), which is true only in one dimension.

1) RESULTS INDEPENDENT OF ASSUMPTIONS ON THE GROUND STATE

We consider a quantum spin system in Z^ν in the sense of ([2],[3]). Specifically, to each finite region $\Lambda \subset Z^\nu$ we associate a Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$, where $\mathcal{H}_x$ are Hilbert spaces of dimensions $(2S_x+1)$, S_x being the magnitude of the total spin associated to the site x, and a matrix algebra $\mathfrak{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$ generated by the matrices $\{S_x^{(1)}, S_x^{(2)}, S_x^{(3)}\}_{x\in\Lambda}$ satisfying $[S_x^{(1)}, S_{x'}^{(2)}] = 2i\, S_x^{(3)} \delta_{x,x'}$ plus cyclic permutations. For further use we let also $\underline{S}_\Lambda = \sum_{x\in\Lambda} \underline{S}_x \quad \underline{S}_x \equiv (S_x^{(1)}, S_x^{(2)}, S_x^{(3)})$.
For A, B, C finite sets, let

$$S_{ABC} = \left\{ \prod_{x\in A} S_x^{(1)} \prod_{x\in B} S_x^{(2)} \prod_{x\in C} S_x^{(3)} \right.$$

if none of A,B or C is empty, otherwise $S_{ABC} = 1$.
To each $\Lambda \subset Z^\nu$ we associate a rotationally invariant Hamiltonian

$$H_\Lambda = \sum_{A\cup B\cup C \subset \Lambda} c(A,B,C)\; S_{ABC} \tag{1}$$

where c(A,B,C) are real functionals of A,B,C such that c(A,B,C) is zero unless A,B and C are mutually disjoint and c(A,B,C) = = c(A',B',C') whebever there exists a translation in the lattice carrying simultaneously A into A', B into B' and C into C' (translation invariance). We also assume c has finite range Δ ([3]), i.e., the set

$$\Delta = -\Delta = \{ X \subset \mathbb{Z}^\nu \text{ such that } X \ni \{0\} \text{ and } X = A\cup B\cup C \quad \text{,with} \quad c(A,B,C) \neq 0 \} \tag{2}$$

is finite. We also assume that the functionals c(A,B,C) occurring in H_Λ have a bound independent of A,B and C:

$$|c(A,B,C)| \leq \text{const. (independent of A,B,C)} \tag{3}$$

Rotational invariance of the Hamiltonian means that for all $\Lambda \subset \mathbb{Z}^\nu$, $[H_\Lambda, \underline{S}_\Lambda] = \underline{0}$. An example of (1) is the rotationally invariant antiferromagnetic Hamiltonian ("ferrimagnetic" if there are some unequal spins, ferromagnetic if one of A or B is empty):

$$H_\Lambda = -\sum_{x,y\in A\subset\Lambda} J_1(x-y)\, \underset{=}{S}_x\cdot \underset{=}{S}_y - \sum_{x,y\in B\subset\Lambda} J_2(x-y)\, \underset{=}{S}_x\cdot \underset{=}{S}_y + \sum_{x\in A\subset\Lambda,\, y\in B\subset\Lambda} J_3(x-y)\, \underset{=}{S}_x\cdot \underset{=}{S}_y \tag{4}$$

where $J_i(x) \geq 0$ and $J_i(x) = 0$ if $|x| >$ diam (Δ) (diameter of the range Δ of the potential), $i=1,2,3$, and A and B are "sublattices" of Λ, with $A \cup B = \Lambda$ (not necessarily uniquely defined, see ((10))). Let $\mathfrak{A}_\Gamma = \bigcup_{\Lambda\subset\mathbb{Z}^\nu} \mathfrak{A}_\Lambda$ be the local algebra and $\mathfrak{A}$ be the quasi-local algebra (norm closure of $\mathfrak{A}_\Gamma$). It is shown in ((3)) that

$$\exists\ n\text{-}\lim_{\Lambda\to\infty} e^{itH_\Lambda} A e^{-itH_\Lambda} \equiv \tau^t(A) \quad \forall A\in\mathfrak{A}_\Gamma \tag{5}$$

where the "$\lim_{\Lambda\to\infty}$" is in the sense of Van Hove (see, e.g., ((11)), and that τ^t may be extended to an automorphism of $\mathfrak{A}$. Let Ω_Λ be a ground state of H_Λ in $\mathcal{H}_\Lambda$. We construct a (ground) state φ on $\mathfrak{A}$ by taking the limit along some subsequence of $(\Omega_\Lambda, \cdot\, \Omega_\Lambda)$. It defines by the G.N.S. construction a representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ with cyclic vector Ω (the physical ground state), such that $\varphi(A) = (\Omega, \pi(A)\Omega)\ \forall A\in\mathfrak{A}$. From the time translation automorphism τ and the space translation automorphism (see, e.g., (2)) one constructs the unitary groups on $\mathcal{H}$, $t\in\mathbb{R}\to U_t$, $x\in\mathbb{Z}^\nu\to W(x)$, such that $W_x\, S_0^{(1,2,3)}\, W(x)^{-1} = S_x^{(1,2,3)}$ and the generator of U is the physical Hamiltonian, a densely defined selfadjoint operator on $\mathcal{H}$ satisfying $H\Omega = 0$, $H \geq 0$ (see (9), which is also the standard reference for ground state representations).

The main result of this part is the following theorem, which is the precise analogue of Theorem 7 of ((2)) for the ferromagnet, and may be considered as a generalization of that result. For conciseness, we always omit explicit reference to the representation π, which is however implicit whenever operators on $\mathcal{H}$ are considered.

THEOREM 1 - Let $f \in \mathcal{S}(\mathbb{R})$, $A \in \mathfrak{A}(\Lambda_0)$ and τ^t be the automorphism corresponding to a rotationally invariant Hamiltonian of finite range of the form (1). Further, let $A(f) \equiv \int dt\, f(t)\, \tau^t(A)$

(6)

Then,

$$\lim_{\Lambda \to \infty} (\Omega, [S_\Lambda, A(f)]\Omega) = (\Omega, [S_{\Lambda_0}, A]\Omega) \int_{-\infty}^{\infty} dt\, f(t) \tag{7}$$

Remark: The integral (6) is a norm integral on $\mathfrak{A}$, which exists since the sutomorphism τ is strongly continuous ([3]). In the proof of the above theorem, a crucial role will be played by the following proposition, which is a trivial consequence of the results of Lieb and Robinson ([4]).

PROPOSITON: For all $A \in \mathfrak{A}(\Lambda_0)$ there exist constants a and b (depending on A and Λ_0) such that for all i=1,2,3:

$$\| [\tau^t(S_x^{(i)}), A] \| \leq b \exp[-\tfrac{1}{2}\, \mathrm{dist}(x, \Lambda_0)] \quad \forall t \in \mathbb{R}$$

such that $|t| \leq a|x|$ (8)

Proof of Theorem 1: since $H\Omega = 0$, we have

$$(\Omega, [S_\Lambda, \tau^t(A)]\Omega) = (\Omega, [\tau^{-t}(S_\Lambda), A]\Omega) \tag{9}$$

Now, we have ([3])

$$\tau^t(S_\Lambda) = S_\Lambda + i \int_0^t ds\, \tau^s([H_{\Lambda_1}, S_\Lambda]) \quad \forall \Lambda_1 \text{ s.t. } \Lambda_1 \supset (\Lambda + \Delta) \tag{10}$$

as an equation on $\mathfrak{A}$, the above integral, as well as the following ones being norm integrals on $\mathfrak{A}$ (they are applications of $\mathfrak{A}_r$ into $\mathfrak{A}$). Now,

$$[\tau^t(S_\Lambda), A] = [S_\Lambda, A] + i \int_0^t ds\, [\tau^s([H_{\Lambda_1}, S_\Lambda]), A] =$$

$$= [S_{\Lambda_0}, A] + i \int_0^t ds\, [\tau^s([H_{\Lambda_1}, S_\Lambda]), A]$$

$$\forall \Lambda_1 \text{ st } \Lambda_1 \supset (\Lambda + \Delta) \tag{11}$$

We now prove that, $\forall f \in \mathcal{S}(\mathbb{R})$, $\forall A \in \mathfrak{A}(\Lambda_c)$, and H_Λ rotationally invariant

$$\exists \lim_{\Lambda \to \infty} \left\| \int_{-\infty}^{\infty} dt\, f(t) \int_0^t ds\, [\tau^s([H_{\Lambda_1}, S_\Lambda]), A] \right\| = 0 \tag{12}$$

(12) and (11) yield

$$\lim_{\Lambda \to \infty} \| [\tau^{-t}(S_\Lambda), A] - [S_{\Lambda_0}, A] \| = 0 \tag{13}$$

whence, in particular, (7) results.

To prove (12), we use the proposition. By rotational invariance of H_Λ, $[H_{\Lambda'}, S_\Lambda] = 0$ if $\Lambda' \subseteq \Lambda$. Hence, $[H_{\Lambda_1}, S_\Lambda]$ for $\Lambda_1 \supset (\Lambda + \Delta)$, is a sum of a number, $N(\Lambda)$ say, of operator monomials of the form $d(A,B,C)$ S_{ABC} concentrated in a region $\partial(\Lambda,\Delta)$ within Δ of the boundary of Λ (i.e., $A \cup B \cup C \subset \partial(\Lambda,\Delta)$ in the above monomials). The coefficients $d(A,B,C)$ depend on the $c(A,B,C)$ occurring in the Hamiltonian (1) and are, by (3), uniformly bounded by a constant independent of A,B,C. Let $r(\Lambda)$ be the diameter of Λ. It follows then easily that there exists a constant e independent of Λ such that

$$N(\Lambda) \le e\, (r(\Lambda))^{\nu-1} \tag{14}$$

Let $\quad I_\Lambda^t \equiv \int_0^t ds\, [\tau^s([H_{\Lambda_1}, S_\Lambda]), A]$

and $\quad p(\Lambda^\circ) = \mathrm{dist}\, (\partial(\Lambda,\Delta), \Lambda_c)$

Clearly, one may find a constant $g > 0$ independent of Λ such that

$$p(\Lambda) \ge g\, r(\Lambda) \tag{15}$$

We now write

$$\int_{-\infty}^{\infty} dt\, f(t)\, I_\Lambda^t = I_\Lambda^{(1)} + I_\Lambda^{(2)} . \tag{16}$$

where $$I_\Lambda^{(1)} \equiv \int_{|t| \le a p(\Lambda)} dt\, f(t)\, I_\Lambda^t \tag{16a}$$

and $$I_\Lambda^{(2)} \equiv \int_{|t| \ge a p(\Lambda)} dt\, f(t)\, I_\Lambda^t \tag{16b}$$

where a is the constant (depending on A and Λ_0) occurring in the proposition. Using now the proposition and eqs. (14), (15), we find that there exists a constant h independent of Λ such that

$$\| I_\Lambda^{(1)} \| \leq h \, (r(\Lambda))^{\nu-1} \exp[-\tfrac{1}{2} g\, r(\Lambda)] \int_{-\infty}^{\infty} dt\, |t|\, |f(t)| \tag{17}$$

Further, by (14) and (15) there exists a constant s and a positive constant m independent of Λ such that

$$\| I_\Lambda^{(2)} \| \leq s \, (r(\Lambda))^{\nu-1} \int_{|t| \geq m\, r(\Lambda)} dt\, |t|\, |f(t)| \tag{18}$$

By (17), $\lim_{\Lambda\to\infty} I_\Lambda^{(1)} = 0$ and by (18) and the property that $f \in \mathcal{S}(\mathbb{R})$. $\lim_{\Lambda\to\infty} I_\Lambda^{(2)} = 0$ Those two facts and (16) imply (12). ∎

2) RESULTS DEPENDING ON ASSUMPTIONS ON GROUND STATE

In this section, we state more precisely the assumptions on φ (or Ω) mentioned in the introduction. To do that we restrict ourselves to the model described by the Hamiltonian (4) which includes a large number of cases of physical interest. The same proof is applicable to several Hamiltonians with more complicated interactions and similar structure. We assume that:

i) φ is pure, i.e., Ω is the unique state in $\mathcal{H}$ such that $H\Omega = 0$;

ii) Ω is invariant under translations within A or B, namely: $W(x-y)\Omega = \Omega$ for all $x,y \in A$ or $x,y \in B$;

iii) Ω has long range order, i.e.,

$$\lim_{|x-y|\to\infty\,;\; x,y \in A \text{ or } B} (\Omega,\, S_x^{(3)} S_y^{(3)} \Omega) = \gamma > 0 \tag{19}$$

From these assumptions and Theorem 1 we have as a corollary "Goldstone's theorem":

THEOREM 2 - For the model (4) and under assumptions (1)-(3), there is no gap in the spectrum of the physical Hamiltonian H.

Proof: Using assumption (1), we may, as in ((2), pg. 245) follow the proof of ((6)) to prove ((2), Lemma 3) holds as long as the spectrum of H has a gap. This lemma, in conjunction with Theorem 1, implies that, as long as the spectrum of H has a gap, for all $A \in \mathfrak{A}(\Lambda_0)$, that

$$\lim_{\Lambda \to \infty} (\Omega, [\underline{S}_\Lambda, A]\Omega) = 0 \tag{20}$$

Specializing in (20) A to $S_0^{(2)}$, and taking the 1-component of $\underline{S}_\Lambda$, we have

$$0 = \lim_{\Lambda \to \infty} (\Omega, [S_\Lambda^{(1)}, S_0^{(2)}]\Omega) = 2i\,(\Omega, S_0^{(3)}\Omega) \tag{21}$$

On the other hand, using assumptions (2) and (3) it is a corollary from asymptotic abelianness ((1)) that

$$(\Omega, S_0^{(3)}\Omega) = \pm\sqrt{\gamma} \neq 0 \tag{22}$$

Eq. (22) contradicts (21) and hence the fact that the spectrum of H has a gap. ∎

Remarks: a) Assumption (1) was not known for the ferromagnet at the time of writing of ((2)) and was only proved later ((8)). Hence it was replaced by ((2), Lemma 1), which was also sufficient. b) Unicity and long range order are open problems even for the isotropic one-dimensional antiferromagnetic Heisenberg chain, where much is known about the grond state for finite volume ((7)). If they were proved, the conjectured ((5)) absence of an energy gap would follow from Theorem 2. Finally, we mention a related work of Swieca ((12)) which proves Goldstone's theorem for many-body systems, under an assumption of falloff of the commutator similar but more general than the one in the proposition, which so far, however, has not been rigorously proved.

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