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A CLASS OF GHOST-FREE NON-ABELIAN GAUGE THEORIES

by

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ABSTRACT

We discuss a class of non-abelian gauge theories characterized by the gauge condition $\Lambda_\mu W_\mu^a = h^a$, where Λ_μ is an arbitrary four-vector and h^a denotes a set of arbitrary functions. We show that this class of theories is ghost free. Using the method of gauge variation of proper vertices, we prove the gauge independence and unitarity of the S-matrix elements.

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I) INTRODUCTION

Non-abelian gauge theories provide the framework for unifying the weak and electromagnetic interactions (1). Furthermore, they exhibit the phenomenon of asymptotic freedom which represents a good approximation of scaling observed in strong interactions (2). As is well known, in order to quantize gauge theories, it is necessary to choose a definite gauge. In many classes of gauges, such as the covariant Lorentz gauge, there will appear the so-called Faddeev-Popov ghosts (3). The renormalizability of gauge theories in these gauges has been discussed by many authors (4), one of the most careful analysis being contained in the recent review by G.Costa and M.Tonin (5). However serious complications due to the presence of ghosts arise principally in the context of the renormalization of gauge-invariant Wilson operators, which becomes in this case a highly non-trivial problem (6). On the other hand, calculation involving Wilson-expansions of operators are much more simple and direct in ghost free gauges (7). Furthermore these theories may provide a way of investigating the behaviour of renormalization group parameters beyond perturbation theory (8).

For these reasons, among others, it is interesting to study gauge conditions which do not lead to the appearence of ghosts (9). Consider, for example, a massless Yang-Mills theory (10) containing a set of vector boson fields W_μ^a described by the Lagrangian:

$$L_{YM} = -\frac{1}{4} F_{\mu\nu}^{a} F_{\mu\nu}^{a} \qquad (1a)$$

where the covariant curl $F_{\mu\nu}^a$ is given by:

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu \tag{1b}$$

In the above expression a is an isotopic index, g is the coupling constant and f^{abc} are the antisymmetric structure constants of the gauge group.

As is well known, this Lagrangian is invariant under the infinitesimal gauge transformations given by:

$$W^a_\mu \to W^a_\mu + (\partial_\mu \delta_{ab} + g f^{bac} W^c_\mu)\omega^b \equiv W^a_\mu + D^{ab}_\mu \omega^b \tag{2}$$

In order to define the vector boson propagator we will introduce the gauge fixing Lagrangian:

$$L_g = -\frac{1}{2}(\Lambda_\mu W^a_\mu)^2 \tag{3}$$

where Λ_μ is an arbitrary four vector with dimension one and such that $\Lambda^2 \neq 0$. Then, the free propagator for the vector boson fields becomes:

$$D^{ab}_{\mu\nu} = \delta_{ab}\frac{1}{k^2}\left[\delta_{\mu\nu} - \frac{k_\mu\Lambda_\nu + k_\nu\Lambda_\mu}{k.\Lambda} + \frac{k_\mu k_\nu \Lambda^2}{(k.\Lambda)^2}\left(1 - \frac{k^2}{\Lambda^2}\right)\right] \tag{4}$$

Notice that the second term in the round brackets is a function of degree minus two in the gauge parameter and does not lead, by superficial power counting, to a renormalizable theory. On the other hand, all other terms are functions of zero degree in Λ and have a good high momentum behaviour.

Writing $\Lambda_\mu \equiv \frac{n_\mu}{\sqrt{\beta}}$ and letting $\beta \to 0$ we note that the last term disappears while all other maintain the same form. Furthermore it has been observed that in this case no ghost fields will be present(11). The renormalization of this theory has been extensively discussed by W.Kummer, being later extended by W.Konetschny and W.Kummer(12) to a class of gauge theories including matter fields to allow for spontaneous symmetry breaking** .

On the other hand, due to the underlying gauge symmetry, one would expect that the above gauge dependent, non-renormalizable terms to drop off for observable quantities. It is the aim of this work to show that the S-matrix elements are indeed gauge independent quantities. The proof is facilitated by the fact, to be shown in the next section, that all ghost fields decouple for any values of the gauge parameter Λ . We shall make use in this paper of the methods developed by G.Costa and M.Tonin , who derived a set of identities for the gauge variation of proper vertices(13) and which provide a simple way of proving the gauge independence of the S-matrix. Although , due to the high momentum behaviour mentioned above, our theory is not a special case of the class discussed by these authors, our analysis is similar in spirit. Furthemore, due to the absence of ghosts, we are able to derive in a more direct way a set of rather simple identities for the gauge variation of proper vertices.

In order to avoid the infrared difficulties, we shall consider a theory consisting of a set of Yang-Mills fields interacting with fermions and scalar fields to allow for spontaneous symmetry breaking. We shall assume that, after the occurence of the symmetry breaking, only one massless vector boson field remains in the theory , so that the infrared problem can be treated like in QED. We remark parenthetically that off the mass-shell, the structure of this theory is in many respects similar to that of a pure Yang-Mills theory. We have used this fact in order to check explicitly in the one-loop approximation, many of the results stated in this paper.

** I thank these authors for kindly sending to me a copy of their work.

In section II, using the Faddeev Popov ansatz for the generating functional of Green functions, we show that no ghost fields are present in our theory. Using this fact we derive a set of simple identities satisfied by the Green functions, which reflect the underlying gauge symmetry of the theory. Section III is devoted to the study of the Lee's identities[(14)] for the one particle irreducible Green functions, which are very suitable for the discussion of the structure of the counterterms. In section IV we derive the identities for the gauge variation of proper vertices. Using this identity, we discuss in section V the two point vertex function. After defining the meaning of physical states, we show, among other things, the gauge independence of the physical masses. Using these results, we show in the last section that the S-matrix elements between physical states are gauge independent and unitary.

II) IDENTITIES FOR GREEN FUNCTIONS IN THE ABSENCE OF GHOSTS

We consider a theory of Yang-Mills fields, described by the Lagrangian given by equation (1), interacting with matter fields. We shall use a compact notation where all fields are collectively denoted by A_i, i standing for all attributes of the fields. Unless otherwise stated, repeated indices imply summation and integration. We assume that the Lagrangian describing our theory is invariant under the infinitesimal gauge transformations:

$$A_i \to A_i + (\partial_i^\alpha + g\, t_{ij}^\alpha A_j) \, \omega^\alpha \qquad (5)$$

where ∂_i is simply the gradient and is present only for vector boson fields (see eq. 2). In terms of a real basis of fields A_i, the matrices t^α will be real and antisymmetric. We may write our invariant lagrangian as follows:

$$L^o_{inv} = \sum_{n=2}^{4} \frac{1}{n!} \Gamma^o_{i_1 \ldots i_n} A_{i_1} \ldots A_{i_n} \tag{6}$$

The gauge fixing Lagrangian (3) may be rewritten compactly as:

$$L_g = - \frac{1}{2} \left(\Lambda_i^{\ \alpha} A_i \right)^2 \tag{7}$$

where Λ_i^α equals $\Lambda_\mu \delta_{ab}$ for vector boson fields and is zero otherwise.

We shall include spontaneous symmetry breaking by assuming that the potential of the scalar fields has a minimum at a non-zero value ε_i^c. In higher orders the vacuum expectation values of the fields A_i (which can be present only for the scalar fields) will be denoted by ε_i. In what follows, unless otherwise specified, we shall rewrite everything in terms of the shifted fields:

$$A_i' = A_i - \varepsilon_i \tag{8}$$

In terms of the primed fields, the gauge transformation (5) becomes:

$$A_i' \longrightarrow A_i + \left(\Delta_i^\alpha + g\, t^\alpha_{ij} A_j' \right) \omega^\alpha \tag{9a}$$

where

$$\Delta_i^\alpha = \partial_i^\alpha + g\, t^\alpha_{ij} \varepsilon_j \tag{9b}$$

In equation (9b) the index i of the matrix t^α reffers to the Goldstone fields which provide the vector fields with mass.

Notice that the gauge fixing Lagrangian is not affected by the shifting given by eq. (8). (In what follows we shall for simplicity drop the prime indices).

In order to derive the Ward identities we shall consider the generating functional for the Green functions in the presence of external sources J_i :

$$W(J_i) = \int [dA] \det(\Lambda_i^a D_i^a) \exp i [L^0_{inv} - \frac{1}{2}(\Lambda_i^a A_i)^2 + J_i A_i] \tag{10}$$

Usually the determinant is written in terms of the so-called Faddeev-Popov ghosts fields:

$$\det(\Lambda_i^a D_i^a) = \int [d\bar{c}][dc] \exp i \bar{c} \Lambda_i^a D_i^a c \tag{10a}$$

However, in our case, this is not necessary since, as we will now show, the ghosts fields decouple from all other fields. To see this we write:

$$\det(\Lambda_i^a D_i^a) = \det(\Lambda_\mu D_\mu) = \det \Lambda.\partial \det(1 - g \mathcal{G} f^a W_\mu^a \Lambda_\mu) \tag{11a}$$

where we have defined the function $\mathcal{G}$ by the relation:

$$(\partial.\Lambda)\, \mathcal{G}(x-y) = \delta^4(x-y) \tag{11b}$$

We will now expand the second determinant on the right hand side of equation (11a) as follows:

$$\det(1 - g \mathcal{G} f^a W_\mu^a \Lambda_\mu) = \exp \operatorname{Tr} \ln (1 - g \mathcal{G} f^a W_\mu^a \Lambda_\mu) =$$

$$= \exp\left\{ - \operatorname{Tr} \sum_\ell \frac{g^\ell}{\ell} \Lambda_{\mu_1} \mathcal{G}(x_1 - x_2) f^a W^a_{\mu_2}(x_2) \ldots \mathcal{G}(x_\ell - x_1) f^b W^b_{\mu_1}(x_1) \right. \tag{12}$$

A typical term under the trace sign in the above equation can be represented graphically by the Feynman diagram shown in Figure 1.

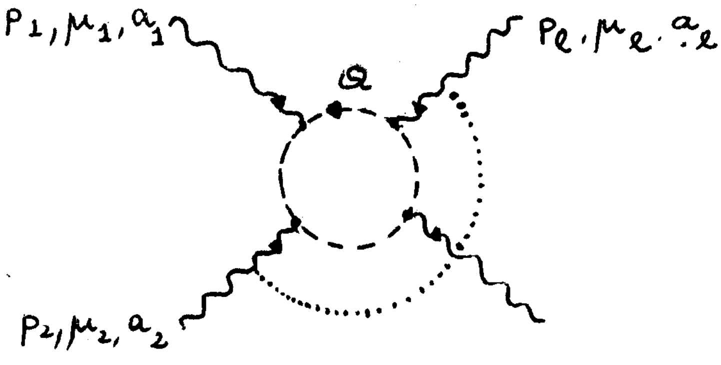


FIG. 1

In this figure all momenta are ingoing and we have $\sum p_i = 0$ The dashed line represents the ghost propagator which in momentum space is given by:

$$\mathcal{G}^{ab} = \delta_{ab}\,\mathcal{G} = \delta_{ab}\,\frac{1}{\Lambda \cdot Q} \tag{13}$$

We shall regularize our theory using the dimensional regularization scheme[15] which preserves the gauge symmetry of the theory. In this scheme the Feynman integrals are performed in a n-dimensional complex space. Using the Feynman parametrization, the above diagram can be expressed as follows:

$$\begin{aligned} I^{a_1 \dots a_\ell}_{\mu_1 \dots \mu_\ell} &= g^\ell \Lambda_{\mu_1} \cdots \Lambda_{\mu_\ell} \operatorname{Tr} f^{a_1} \dots f^{a_\ell} (\ell - 1)! \,\cdot \\ &\cdot \int_0^1 dx_1 \dots \int_0^{x_{\ell-1}} dx_{\ell-2} \int \frac{d^n Q}{[\Lambda \cdot Q + \dots - \Lambda \cdot p_2\, x_{\ell-1}]^\ell} \end{aligned} \tag{14}$$

Performing a shift of variables and integrating over the variables x we obtain:

$$I^{a_1 \dots a_\ell}_{\mu_1 \dots \mu_\ell} = g^\ell \Lambda_{\mu_1} \dots \Lambda_{\mu_\ell} \operatorname{Tr} f^{a_1} \dots f^{a_\ell}\, I \tag{15a}$$

with

$$I = \int \frac{d^n Q}{(\Lambda \cdot Q)^\ell} \tag{15b}$$

Exploiting the Lorentz invariance of this integral we find:

$$I = C(n, \ell)\, \frac{1}{\Lambda^\ell} \int \frac{d^n Q}{Q^\ell} \tag{16}$$

where C is a constant independent of Λ which depends, in general, on ℓ and n.

Now, in the dimensional regularization scheme the last integral vanishes[16]. This implies the decoupling of the ghost sector in the theory.

There is still another way of understanding this result which emerges when one considers the "derivation" of eq. (10). We start from the generating functional in the absence of external sources where the gauge condition is ensured by the presence of a delta functions:

$$W(0) = N\int[dA]\det(\Lambda_i^a D_i^a)\,\delta(\Lambda_i^a A_i - h^a)\exp i L^0_{inv} \qquad (17)$$

As is well known $W(0)$ is independent of the path h^a. Due to this fact, usually (17) is multiplied by $G(h) = e^{-\frac{i}{2}h^a h^a}$ and then an integration over h is performed. This does not modify $W(0)$ except up to an overall normalization factor which is included in N. After introducing the external sources we then find (10).

However, since $W(0)$ is path independent one could as well have multiplied eq. (17) by $\tilde{G}(h) = G(h)\det^{-1}(\Lambda\cdot\partial - gf^{a}h^{a})$ Then because of the presence of the delta function, we see that the determinants cancel and hence the ghost fields disappear.

Therefore the generating functional can simply be written as:

$$W(J_i) = \det(\Lambda\,\partial)\int[dA]\exp i\left[L^0_{inv} - \tfrac{1}{2}(\Lambda_i^a A_i)^2 + J_i A_i\right] \qquad (18)$$

where the constant term $\det(\Lambda\,\partial)$ has been maintained in order to ensure a convenient normalization condition.

We now perform a gauge transformation given by (5) in the path integral. The integration measure and L^0_{inv} remain unchanged under such a transformation. The only changes occur in L_g and in the source terms. From the condition that W is stable under a change of variables of integration, we find:

$$[\Delta_e^a + g\, t_{ej}^a \left(\frac{1}{i}\frac{\delta}{\delta J_j}\right)] \Lambda_e^b \Lambda_k^b \left(\frac{1}{i}\frac{\delta}{\delta J_k}\right) W =$$

$$= [\Delta_e^a + g\, t_{ej}^a \left(\frac{1}{i}\frac{\delta}{\delta J_j}\right)] J_e W \tag{19}$$

This equation can be simplified using the antisymmetry of the matrices t^a. Furthermore, using the fact that Λ_i^a is connected only to vector boson fields and that the matrix t_{ij}^a does not mix vector fields with other fields, (19) reduces to:

$$\Lambda_j^a \left(\frac{1}{i}\frac{\delta}{\delta J_j}\right) W = \mathcal{G}^{ab} D_j^b \left(\frac{\delta}{\delta J}\right) J_j W \tag{20a}$$

where

$$D_j^b \left(\frac{\delta}{\delta J}\right) = \Delta_j^b + g\, t_{jk}^b \left(\frac{1}{i}\frac{\delta}{\delta J_k}\right) \tag{20b}$$

Successive functional derivatives of eq. (20a) yield the following set of Ward identities:

$$\Lambda_j^a W_{j i_1 \dots i_n} = \mathcal{G}^{ab} \{ D_j^b J_j W_{i_1 \dots i_n} + \sum_p D_{i_1}^b W_{i_2 \dots i_n} \} \tag{21a}$$

with

$$W_{i_1 \dots i_n} = \frac{1}{i\delta J_{i_1}} \cdots \frac{1}{i\delta J_{i_n}} W \tag{21b}$$

and where the sum $\sum_p$ are taken over all non trivial permutations of the indices $i_1 \dots i_n$. For the case when $n = 1$ we get:

$$\Lambda_j^a W_{j i_1} = \mathcal{G}^{ab} [D_j^b J_j W_{i_1} + D_{i_1} W] \tag{22}$$

This relation will be very useful in the study of the gauge dependence of the Green functions.

III) LEE IDENTITIES AND COUNTERTERMS

In order to discuss the structure of the counterterms in our theory it is useful to derive an identify for the proper vertices. In terms of the generating functional of connected Green functions $Z = \frac{1}{i} \ln W$, the generating functional of the proper vertices is a Legendre transform of Z:

$$\Gamma(A_i^c) = Z(J_i) - J_i A_i^c \tag{23}$$

where the classical fields A_i^c are defined by:

$$A_i^c \equiv \frac{\delta Z}{\delta J_i} \tag{24a}$$

so that from (23) we have:

$$J_i = -\frac{\delta \Gamma}{\delta A_i^c} \equiv -\Gamma_i \tag{24b}$$

Using these relations, together with the antisymmetry properties of the matrices t^a, we obtain from (19) the identity:

$$D_i^a(A^c) \frac{\delta}{\delta A_i^c}\left[\Gamma + \tfrac{1}{2}(\Lambda_j^a A_j^c)^2\right] \equiv D_i^a(A_c)\, \tilde{\Gamma}_i = 0 \tag{25}$$

Following reference (5), in order to study the structure of counterterms, it is more convenient to express our quantities in terms of the fields $\tilde{A}_i$, which are related to the original unshifted fields by the relation:

$$\tilde{A}_i = A_i - \varepsilon_i^0 \tag{26}$$

which coincide with the fields A_i' only in the tree approximation.

Note that in deriving eq. (25) we did not make use of the fact that $\Gamma_i(A^c = 0) = 0$.

Therefore, in terms of the shifed fields (26) we will get a similar relation, namely:

$$\hat{D}_i^a(\tilde{A}_i^c)\,\hat{\Gamma}_i(\tilde{A}^c) = 0 \qquad (27a)$$

where

$$\hat{D}_i^a(\tilde{A}_i^c) = \partial_i^a + g\,t^a_{ij}\,\varepsilon_j^o + g\,t^a_{ij}\,\tilde{A}_j^c \equiv \hat{\Delta}_i^a + g\,t^a_{ij}\,\tilde{A}_j^c \qquad (27b)$$

although now, of course, $\hat{\Gamma}_i(\tilde{A}^c = 0)$ vanishes only in the tree approximation.

By taking successive differentiations with respect to $\tilde{A}^c$ at $\tilde{A}^c = 0$ we obtain a set of identities of which we list only but a few:

$$H^a = \hat{\Delta}_i^a\,\hat{\Gamma}_i = 0 \qquad (28a)$$

$$H^a_{i_1} = \hat{\Delta}_i^a\,\hat{\tilde{\Gamma}}_{ii_1} + g\,t^a_{ii_1}\,\hat{\Gamma}_i = 0 \qquad (28b)$$

$$H^a_{i_1 i_2} = \hat{\Delta}_i^a\,\hat{\Gamma}_{ii_1i_2} + g\,t^a_{ii_1}\,\hat{\tilde{\Gamma}}_{ii_2} + g\,t^a_{ii_2}\,\hat{\tilde{\Gamma}}_{ii_1} = 0 \qquad (28c)$$

$$H^a_{i_1 i_2 i_3} = \hat{\Delta}_i^a\,\hat{\Gamma}_{ii_1i_2i_3} + \frac{1}{2}g\sum_P \delta_{i_1j_1}\,\delta_{i_2j_2}\,\delta_{i_3j_3}\,t^a_{ij_1}\,\hat{\Gamma}_{ij_2j_3} = 0 \qquad (28d)$$

etc, where in general the above vertices depend on the (dimensional) gauge parameter Λ. We observe that, due to the arbitrariness of the gauge parameter, the above identies must be satisfied independently for each set of terms with a definite degree in Λ. In particular, the terms of zero degree in Λ satisfy separately the identities (28).

In the dimensional regularization scheme, the vertices in eq. (28) develop poles at $\varepsilon = n-4=0$. Introducing a loop parameter η, we obtain for $\hat{\Gamma}_{i_1 \ldots i_n}$:

$$\hat{\Gamma}_{i_1 \ldots i_n} = \hat{\Gamma}^{o}_{i_1 \ldots i_n} + \frac{\eta}{\varepsilon} \hat{\Gamma}^{1}_{i_1 \ldots i_n} \tag{29}$$

where the quantities $\hat{\Gamma}^{o}_{i_1 \ldots i_n}$ with $n \leq 4$ are precisely the factors appearing in L^{o}_{inv} (see eq. 6).

From power counting, we know that the part of zero degree in Λ of $\hat{\Gamma}^{1}_{i_1 \ldots i_n}$ for $n>4$ contains no poles at $\varepsilon = 0$. On the other hand, due to the asymtotic behaviour of the part of the vector boson propagator which is not of zero degree in Λ, the corresponding part of $\hat{\Gamma}^{1}_{i_1 \ldots i_n}$ will contain poles for all n.

Remark however that all these divergent parts are related by virtue of the set of identities (28). We shall show in the next sections that all these gauge dependent terms vanish on mass shell, between physical sources. Hence, for the purpose of calculating gauge invariant quantities these terms are not relevant. For certain other purposes however, it will be convenient in what follows to add a set of counterterms which render finite the Green functions. We have:

$$L_{ct} = - \frac{\eta}{\varepsilon} \sum_k \frac{1}{k!} \hat{\Gamma}^{1}_{i_1 \ldots i_k} \tilde{A}_{i_1} \ldots \tilde{A}_{i_k} = \tag{30}$$

$$= - \frac{\eta}{\varepsilon} \left[\sum_{n=1}^{4} \frac{1}{n!} \hat{\Gamma}^{1}_{o, i_1 \ldots i_n} \tilde{A}_{i_1} \ldots \tilde{A}_{i_n} + \sum_m \frac{1}{m!} \hat{\Gamma}^{1}_{\neq o, i_1 \ldots i_m} \tilde{A}_{i_1} \ldots \tilde{A}_{i_m} \right]$$

where we have explicitly separated the part of zero degree in Λ, denoted by $\hat{\Gamma}^{1}_{o, i_1 \ldots i_n}$.

Due to the structure of the vector boson propagator, $\hat{\Gamma}^{1}_{\neq o, i_1 \ldots i_m}$ must be functions of negative degree in Λ (Of course they must be invariant under the reflection $\Lambda \to -\Lambda$). Therefore, in eq. (30) no counterterms with the structure of L_g (see eq. 7) are allowed. We then conclude that the gauge fixing Lagrangian is not renormalized.

We will now show that $L_{inv} = L^{o}_{inv} + L_{ct}$ is invariant under the gauge transformation (9) (where of course we must use $\hat{\Delta}$ since the fields are shifted according to eq. 26). By assumption L^{o}_{inv} is invariant under such a transformation. We then obtain:

$$\frac{\delta L_{inv}}{\delta \omega^{\alpha}} = \frac{\delta L_{ct}}{\delta \hat{A}_i}\left(\hat{\Delta}^{\alpha}_{i} + g\, t^{\alpha}_{ij} \hat{A}_j\right) = \tag{31}$$

$$= -\frac{\eta}{\zeta}\left[H^{\alpha P} + H^{\alpha P}_{i_1} \tilde{A}_{i_1} + \frac{1}{2} H^{\alpha P}_{i_1 i_2} \tilde{A}_{i_1} \tilde{A}_{i_2} + \frac{1}{6} H^{\alpha P}_{i_1 i_2 i_3} \tilde{A}_{i_1} \tilde{A}_{i_2} \tilde{A}_{i_3} + \ldots\right]$$

where the functions $H^{\alpha P}$ are the residues at the poles of the functions appearing the Lee identities(28), and hence vanish.

The invariance of the Lagrangian with counterterms added can be similarly established inductively to all orders, due to the fact that L_g is not renormalized.

In the renormalized theory, we will now reexpress all quantities in terms of the fields shifted by the full vacuum expectation value ε . Then, the renormalized Lagrangian, expressed in terms of the fields A_i (see eq. 8) will be invariant under the set of gauge transformation expressed by (9). Due to this fact all formulae derived thus far will be valid in all orders of perturbation theory provided we replace in the generating functional L^{o}_{inv} by the full Lagrangian $L_{inv} = L^{o}_{inv} + L_{ct}$.

Of course, the requirement that the Green functions be regular at $\zeta = 0$ does not fix uniquely the counterterms. What is still necessary is another finite renormalization yielding propagators with residue 1 at the physical poles. We will return to this question in section 5.

IV) GAUGE-DEPENDENCE OF PROPER VERTICES

In this section we will derive an identity for the gauge variation of proper vertices which will be very useful for establishing the gauge independence of the S-matrix. Consider the generating functional for the renormalized Green functions.

$$W(J) = \det(\partial.\Lambda)\int [dA] \exp i\left[L_{inv} - \tfrac{1}{2}(\Lambda_i^a A_i)^2 + J_i A_i\right] \tag{32}$$

Let us vary the gauge fixing Lagrangian by making an infinitesimal transformation in the gauge parameter $\Lambda_i^a \to \Lambda_i^a + \delta\Lambda_i^a$. Under this transformation W also changes and we obtain (the determinant is also being varied):

$$\delta W(J) = W^{\Lambda+\delta\Lambda}(J) - W^{\Lambda}(J) = \tag{33}$$

$$= \delta\Lambda_j^a\left[\mathcal{G}^{ab}\Delta_j^b - \Lambda_\ell^a\left(\frac{1}{i}\frac{\delta}{\delta J_\ell}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_j}\right)\right]W$$

This expression can be put into a more convenient form with the help of eq. (22). Using the antisymmetry of the matrix t^a, together with the fact that Λ_i^a is connected only to the vector boson fields, we obtain:

$$\delta W(J) = -\delta\Lambda_j^a\,\mathcal{G}^{ab}D_k^b\left(\frac{\delta}{\delta J}\right)J_k W_j \tag{34}$$

We now reexpress (34) in terms of the generating functional of the connected Green functions. We get:

$$\delta Z(J) = Z^{\Lambda+\delta\Lambda}(J) - Z^{\Lambda}(J) = \tag{35}$$

$$= -\delta\Lambda_j^a\,\mathcal{G}^{ab}\left(D_k^b A_j + g\,t_{k\ell}^b Z_{\ell j}\right)J_k$$

An identical expression can be obtained **for** the gauge variation of the generating functional for the 1PI Green functions.
We recall that (see eq. 23):

$$\Gamma^{\Lambda}(A^c) = Z^{\Lambda}(J^{\Lambda}) - J^{\Lambda} A_i^c \tag{36}$$

where J^{Λ} is obtained by inverting eq. (24a). A similar relation is obtained for $\Gamma^{\Lambda+\delta\Lambda}(A^c)$ by replacing on the right hand side of (36) Λ by $\Lambda + \delta\Lambda$. We obtain:

$$\delta\Gamma(A^c) = \Gamma^{\Lambda+\delta\Lambda}(A^c) - \Gamma^{\Lambda}(A^c) = \tag{37}$$

$$= \delta Z = -\delta\Lambda_j^a \Lambda_k^a A_j^c A_k^c + g\,\delta\Lambda_j^a \mathcal{G}^{ab} t_{k\ell}^b Z_{\ell_j} \Gamma_k$$

where use has been made of the Lee identity (25). Note that the first term represents the contribution which results in the tree approximation by making an infinitesimal change in the gauge fixing Lagrangian.

Defining

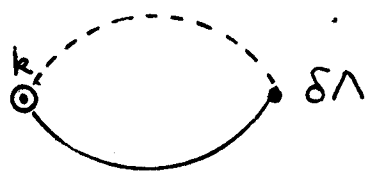
$$M_k = g\,\delta\Lambda_j^a \mathcal{G}^{ab} t_{k\ell}^b Z_{\ell j} \tag{38}$$

we can finally express eq. (37) as follows:

$$\delta\Gamma(A^c) = -\delta\Lambda_j^a \Lambda_k^a A_j^c A_k^c + M_k \Gamma_k \tag{39}$$

The quantity M_k is represented graphically in figure 2.

FIG. 2



In this figure the vertex denoted by $\odot$ is proportional to $g t^k$. The dashed line represents the "ghost" propagator $\mathcal{G}$. The full line stands for $Z_{\ell_j}(J)$ which at $J=0$, gives the full propagator.

Note that the fields A^c are determined by the condition $\Gamma_i^\Lambda(A^c=0)=0$. Let us now define δA^c so that $\Gamma_i^{\Lambda+\delta\Lambda}(\delta A^c)=0$. Then by definition, the quantity:

$$\bar{\Delta}\Gamma(A^c) \equiv \Gamma^{\Lambda+\delta\Lambda}(A^c+\delta A^c) - \Gamma^\Lambda(A^c) \tag{40}$$

is such that $\bar{\Delta}\Gamma_i(A^c=0)=0$. From (37) we see that:

$$\bar{\Delta}\Gamma(A^c) = \delta\Gamma(A^c) + \delta A_j^c \Gamma_j^\Lambda(A^c) \tag{41}$$

We now remark that, due to the antisymmetry of the matrix t^k and since $\mathcal{G}^{ab}$ is symmetric in its indices, we have .

$$M_k(A^c=0)=0 \tag{42}$$

Therefore we see that $\delta\Gamma_i(A^c)$ also vanishes at $A^c=0$. It then follows that $\delta A^c=0$. This fact implies that in the modified gauge the vacuum expectation values (V.E.V.) of the fields are the same as in the original gauge, i.e. that they are independent of the gauge parameter Λ. We can understand more directly the above result at least for the part of zero degree in Λ of the V.E.V. of the scalar fields as follows. The only way that an expression can be a non trivial function of zero degree in Λ is through scalar products with the external momenta. As the V.E.V. are independent of momenta, they must, at least to this degree, be independent of the gauge parameters Λ.

By taking the functional derivatives of eq. (39) with respect to A^c, at $A^c=0$ we then obtain a set of identities for the gauge variation of proper vertices. We will close this section by making a final remark concerning this variation. (See also ref. 5).

Note that, in order to obtain equation (39) for the gauge variation of proper vertices, we have varied only $\mathcal{L}_g$ in the exponential of the generating functional. However, as we have discussed in the previous chapter, $\mathcal{L}_{inv}$ also depends, through $\mathcal{L}_{ct}$, on the gauge parameter Λ. We remember that these counterterms were explicitly constructed so that the Green functions corresponding to $\mathcal{L}_g$ be finite. Therefore, in the modified gauge the Green functions will have in general, poles at $\varepsilon = 0$. In fact, the divergent terms will be given by the corresponding parts of:

$$\Delta \Gamma_{div} \equiv - \Delta \mathcal{L}_{ct} = - \frac{\delta \mathcal{L}_{ct}}{\delta \Lambda_i^a} \delta \Lambda_i^a \tag{43}$$

We will show that on mass shell, between physical states, these Green functions give the same gauge independent contribution. This implies in particular, that all of the above divergent, gauge dependent terms must vanish in this case.

V) TWO POINT VERTICES AND PHYSICAL STATES

Let us now consider the gauge variation of two point vertices. Using the fact that $M_k(A^c = 0) = 0$, we obtain from (39):

$$\delta \Gamma_{ij}(0) = -(\delta \Lambda_i^a \Lambda_j^a + \delta \Lambda_j^a \Lambda_i^a) + M_{k,i} \Gamma_{kj} + M_{k,j} \Gamma_{ki} \tag{44}$$

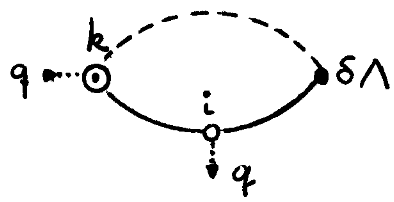
where $M_{k,i}$ is given by:

$$M_{k,i} = \delta \Lambda_j^a g \mathcal{G}^{ab} t_{kl}^b \frac{\delta}{\delta A_i^c} Z_{lj} \Big|_{A^c=0} = \delta \Lambda_j^a g \mathcal{G}^{ab} t_{kl}^b G_{ll'} \Gamma_{l'ij'} G_{j'j} \tag{45}$$

In this expression $G_{ll'}$ and $\Gamma_{l'ij'}$ represent the propagator and the three point vertex functions, respectively.

We represent $M_{k,i}$ diagrammatically in momentum space by the Feynman diagram shown in Figure 3.

FIG. 3



In this figure q stands for the external momentum flowing into and out of the diagram.

Before we can proceed further, we must consider first the diagonalization of the two point vertex. Let E_i^r and $\bar{E}_i^s$ be a complete set of orthonormal vectors which render diagonal the vertex Γ_{ij} :

$$\Gamma_{ij} = \sum_r E_i^r \, \Gamma_r \, E_j^r + \sum_s \bar{E}_i^s \, \bar{\Gamma}_s \, \bar{E}_j^s \tag{46}$$

Due to the orthonormality of these vectors, we have the eigenvalue equation (we now drop the summation convention on r):

$$\Gamma_{ij} \, E_j^r = \Gamma_r \, E_j^r \tag{47}$$

and a similar one for the vectors $\bar{E}_j^s$, with Γ_r replaced by $\bar{\Gamma}_s$

We now want to define physical states. We shall require that the vectors E_j^r , which correspond to these states, be eigenvectors of the eq. (47) with eigenvalue:

$$\Gamma_r = (k^2 + m_r^2) \, Z_r^{-1} \tag{48}$$

which exhibits a zero at the physical mass m_r^2.

Here Z_r is a finite wave functions renormalization constant determined by the requirement that the residues of the renormalized propagators at the physical poles be equal to 1.

In order that equation (47) be satisfied, the eigenvectors E_j^r must fulfil certain conditions. To obtain these, we apply on both sides of these equation the operator $\mathcal{D}_i^a$. Using the Lee identity (25) we find:

$$\Lambda_j^a E_j^r = -\Gamma_r \mathcal{G}^{ab} \Delta_j^b E_j^r \qquad (49)$$

We now remark that because of this relation the vanishing of $\Delta_j^b E_j^r$ implies the vanishing of $\Lambda_j^a E_j^r$, and vice-versa. In general, it is more natural to call physical particles those corresponding to the eigenvectors E_j^r satisfying [5]:

$$\Delta_j^b E_j^r = 0 \qquad (50a)$$

which implies:

$$\Lambda_j^a E_j^r = 0 \qquad (50b)$$

There is however one special case where (50b) arises more directly, which then implies the condition (50a). This case arises in the gauge $\Lambda_i^a A_i = 0$ which can be formally obtained in eq. (17) by writing $\Lambda_i \equiv \frac{n_i}{\sqrt{\beta}}$ and taking $\beta \to 0$. This interesting situation has been discussed in detail in reference (12).

As a side remark, we observe that for massless vector bosons satisfying: $\partial_j^a E_j^r = \Lambda_j^a E_j^r = 0$ with Λ space like, it is not possible to construct polarization vectors of finite norm when $\Lambda . p = 0$ [6]. However, since, as we shall show, the S-matrix elements are gauge independent, we can always avoid the singularity at $\Lambda . p = 0$ by an appropriate choice of the gauge parameter Λ.

Let us now consider equation (44) taken between physical states. Using (50b) we obtain:

$$E_i^r \, \delta\Gamma_{ij} \, E_j^r = 2 \, \Gamma_r \, E_i^r M_{i,j} E_j^r \qquad (51)$$

On the other hand we can also calculate $\delta \Pi_{ij}$ from eq. (46). In the modified gauge the appropriate eigenvectors will be:

$$\tilde{E}_i^r = E_i^r + \delta E_i^r \qquad (52a)$$

which are also orthonormal. Due to this fact we have:

$$E_i^r \delta E_i^r = 0 \qquad (52b)$$

With the help of (48), we then obtain:

$$E_i^r \delta \Pi_{ij} E_j^r = Z_r^{-1} (\delta m_r^2 - \delta Z_r \Pi_r) \qquad (53)$$

We now compare eq. (51) and (53). From equation (45) we see that $M_{i,j}$ does not have poles at the physical masses. Hence, on the mass shell, the left hand side of equation (53), as given by (51), vanishes. This implies that:

$$\delta m_r^2 = 0 \qquad (54)$$

i.e., that the masses of the physical particles are gauge independent. Furthermore, near the mass-shell, eq. (51) and (53) imply:

$$E_i M_{i,j} E_j^r = -\frac{1}{2} (Z^{-1} \delta Z)_r \qquad (55)$$

We will now proceed to study in more detail this equation since it yields important information concerning the wave function renormalization constant.

We will define the quantity $N^a_{\ell ij}$ as follows (see eq. 45):

$$M_{i,j} = \delta \Lambda^a_\ell N^a_{\ell ij} \qquad (56a)$$

where

$$N^a_{\ell ij} = g \mathcal{G}^{ab} t^b_{ik} \frac{\delta}{\delta A^c_j} Z_{k\ell} \Big|_{A^c=0} \qquad (56b)$$

We want to show that on mass shell, between physical states we have the relation:

$$E_i^r \Lambda_\ell^a N_{\ell ij}^a E_j^r = 0 \tag{57}$$

To this end, we make again use of the identity (22) written in terms of the generating functional of connected Green functions, i.e.:

$$\Lambda_\ell^a Z_{\ell k} = -i(\Lambda_j^a A_j^c) A_k^c - i\mathcal{G}^{ab}\{D_k^b + [D_e^b A_k^c + g t_{en}^b Z_{nk}]\Gamma_e\} \tag{58}$$

Using the antisymmetry of the matrices t^a we then obtain, at $A^c = 0$:

$$\Lambda_\ell^a N_{\ell ij}^a = i g^2 \mathcal{G}^{ab} t_{ik}^b G_{kn} t_{ne}^c \mathcal{G}^{ca} \Gamma_{ej} \tag{59}$$

which, in virtue of (47) vanishes on mass shell between physical states.

We are now in a position to investigate the contribution of gauge dependent terms with non-zero degree in Λ to the wave function renormalization constant. To isolate this contribution we consider a change of gauge which modifies only the modulus of Λ_i^a, but not its direction: $\Lambda_i^a \to (1+\varepsilon)\Lambda_i^a$ Clearly, under such a transformation a function of zero degree in Λ remain unchanged while one of degree different from zero will be modified. Using eq. (55), (56) and (57) we see that $(Z^{-1}\delta Z)_r$ vanishes in this case. It then follows that the wave function renormalization constant must be a function of zero degree in Λ.

Due to this circunstance we may now use power counting to investigate the degree of divergence of $(Z^{-1}\delta Z)_r$. From Figure 3, we see that this expression is at most logarithmically divergent, the divergent part being independent of the external momentum q. Remark that according to the previous discussion, between physical states on mass shell, the quantity $N^{a}_{\ell i j}$ is of degree minus one in Λ. Furthermore, due to the property (57), the divergent part of $N^{a}_{\ell i j}$ must in this case have the form (note that ℓ and therefore s must be Lorentz indices).

$$N^{a\,div}_{\ell i j} = \left(\delta_{\ell s} - \frac{\Lambda_\ell \Lambda_s}{\Lambda^2}\right) \bar{N}^{a}_{s i j} \tag{60}$$

Since $\bar{N}^{a}_{s i j}$ is of degree minus one in Λ and is independent of the momenta, it must be proporcional to $\frac{\Lambda_m}{\Lambda^2}$ where m is one of the indices s, i or j. When $m = s$, then clearly this implies the vanishing of $N^{a\,div}_{\ell i j}$. When m equals i or j, again $N^{a\,div}_{\ell i j}$ vanishes on mass shell between physical states, due to the condition (50b).

Consequently $(Z^{-1}\delta Z)^{div}_{r}$ vanishes, whence it follows, since Z_r is finite, that

$$(\delta Z)^{div}_{r} = 0 \tag{61}$$

This result has several interesting consequences. In particular, it implies that the divergent part of the residues of the propagators at the physical poles, which arises from the Feynman diagrams, must be a gauge independent quantity.

VI) S-MATRIX ELEMENTS

We will now discuss the S-matrix elements between physical states. We recal that, given the connected n-point Green functions $\hat{G}_{i_1 \ldots i_n}$, $(n \geq 3)$, the S-matrix elements for physical states are given by:

$$S^{\Lambda}_{r_1 \cdots r_n} = \prod_{k=1}^{n} Z_{r_k}^{-1/2} (k^2 + m_{r_k}^2) E_{i_k}^{r_k} G^{\Lambda}_{i_1 \cdots i_n} \Big|_{k^2 = -m_{r_k}^2} \qquad (62)$$

Similarly, using the fact that the physical masses are gauge independent (eq. 54) we obtain for the S-matrix elements in the modified gauge:

$$S^{\Lambda + \delta\Lambda}_{r_1 \cdots r_n} = \prod_{k=1}^{n} (Z_{r_k} + \delta Z_{r_k})^{-1/2} (k^2 + m_{r_k}^2)(E_{i_k}^{r_k} + \delta E_{i_k}^{r_k}) G^{\Lambda + \delta\Lambda}_{i_1 \cdots i_n} \Big|_{k^2 = -m_{r_k}^2} \qquad (63)$$

Keeping only infinitesimal terms, we obtain for $\delta S_{r_1 \quad r_n}$ the expression:

$$\delta S_{r_1 \cdots r_n} = \sum_s Z_{r_s}^{-1/2} (k^2 + m_{r_s}^2) [- \tfrac{1}{2} (Z^{-1} \delta Z)_{r_s} E_{i_s}^{r_s} + \delta E_{i_s}^{r_s}] \cdot$$

$$\cdot \prod_{k \neq s} Z_{r_k}^{-1/2} (k^2 + m_{r_k}^2) E_{i_k}^{r_k} G^{\Lambda}_{i_1 \cdots i_n} \Big|_{k^2 = -m_{r_k}^2} +$$

$$+ \prod_{k=1}^{n} Z_{r_k}^{-1/2} (k^2 + m_{r_k}^2) E_{i_k}^{r_k} \delta G_{i_1 \cdots i_n} \Big|_{k^2 = -m_{r_k}^2} \qquad (64)$$

In the Green function $G^{\Lambda}_{i_1 \quad i_n}$, which appears in the first term on the right hand side of eq. (64), let us single out the index i_s. We have:

$$G^{\Lambda}_{i_1 \cdots i_s \quad i_n} = G_{i_s j} \; G^{a}_{i_1 \cdots j \cdots i_n} \qquad (65)$$

where in the expression $G^{a}_{i_1 \cdots j \cdot i_n}$ the external line j is amputated. Now, since $G_{i_s j}$ is essentially the inverse of the two point vertex $\Gamma_{i_s j}$, we get, near the mass shell, using (46), the result:

$$G_{i_s j} \cong - E_{i_s}^{r_s} \; T_{r_s}^{-1} \; E_j^{r_s} \qquad (66)$$

We then see that, when this expression is multiplied by the factor $\delta E^{i_s}_{r_s}$, which appears in eq. (64) we get zero, in virtue of the relation (52b). Furthermore, using eq. (55)

together with (65) and (66), we can rewrite the first term on the right hand side of equation (64) so that $\delta S_{r_1 \dots r_n}$ becomes:

$$\delta S_{r_1 \dots r_n} = \prod_{k=1}^{n} Z_{r_k}^{-1/2} (k^2 + m_{r_k}^2)\, E_{i_k}^{r_k} \cdot \tag{67}$$

$$\cdot \Big[\sum_{P} M_{i_1, j_1}\, G_{j_1 i_2 \dots i_n} + \delta G_{i_1 \dots i_n} \Big]$$

We must now calculate $\delta G_{i_1 \dots i_n}$. We recall (see eq. 37 and 38) that

$$\delta Z = - \delta \Lambda_j^a \Lambda_k^a A_j^c A_k^c + M_k T_k \equiv - \delta \Lambda_j^a \Lambda_k^a A_j^c A_k^c + \Delta Z \tag{68}$$

The variations $\delta G_{i_1 \dots i_n}$ are obtained by taking functional derivatives of (68) with respect to J_i, at $J_i = 0$. Let us consider now the first term on the right hand side of eq. (68). We obtain:

$$\frac{\delta}{\delta J_{i_1}} \cdots \frac{\delta}{\delta J_{i_n}} \delta \Lambda_j^a \Lambda_k^a A_j^c A_k^c \Big|_{J=0} = \tag{69}$$

$$= \frac{1}{2} (\delta \Lambda_j^a \Lambda_k^a + \delta \Lambda_k^a \Lambda_j^a) \sum_{P} \sum_{s} G_{j i_1 \dots i_s} G_{k i_{s+1} \dots i_n}$$

The sum on s extends over all possible ways of dividing the n indices among the two Green functions and the sum on P extends over all non-trivial permutations of these indices. We now remark, using the identity (21) that the above expression contains only $(n-1)$ poles at the physical masses, when $1 < s < n-1$. Therefore, this contribution to eq. (67) taken on mass shell vanishes. When $s = 1$ or $s = n-1$ we see, using eq. (22) that although the above expression contains n poles at the physical masses, it is proportional to $\Delta^{a_k}_{i_k}$ for some k. Due to this fact, in virtue of (50a), this contribution also vanishes between physical states.

Consider now the second term in eq. (68). Performing the functional differentiations with respect to J_i, we find:

$$\Delta G_{i_1\ldots i_n} = -\sum_P \sum_{s=1}^{n-1} M_{i_1 j_1\ldots j_s} G_{j_1\ldots \tilde{i}_2}\ldots G_{j_s\ldots \tilde{i}_n} \tag{70}$$

The indices $\tilde{i}_2,\ldots \tilde{i}_n$ represent a given permutation of the set $i_2\ldots i_n$. The sum on s is taken over all possible ways of dividing these indices among the s Green functions. and the sum over p extends over all non-trivial permutations of the indices.

Equation (70) is represented graphically in Figure 4.

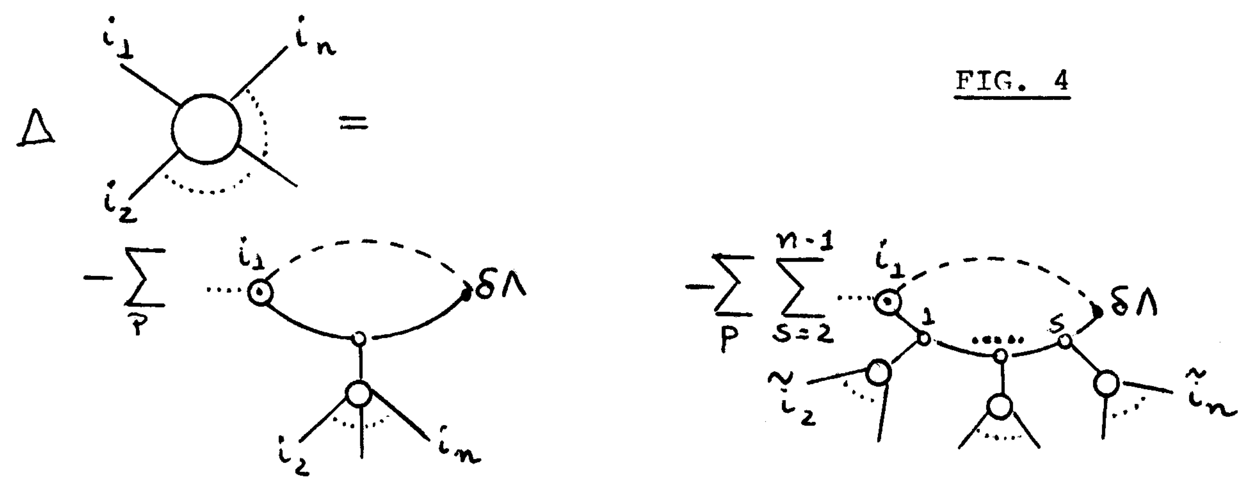


FIG. 4

In this figure, we have singled out the term corresponding to $s=1$, which contains a pole in the variable i_1 all other terms being free from physical poles in this variable. However, this term is precisely canceled by the first term in the brackets of eq. (67). Therefore:

$$\Delta S_{r_1\ldots r_n} = -\prod_{k=1}^{n} Z_{r_k}^{-1/2}(k^2+m_{r_k}^2)\, E^{r_k}_{i_k} \cdot \sum_P \sum_{s=2}^{n-1} M_{i_1 j_1\ldots j_s} G_{j_1\ldots \tilde{i}_2}\ldots G_{j_s\ldots \tilde{i}_n} \tag{71}$$

vanishes on mass-shell, whence it follows that the physical S matrix elements are gauge independent.

Given this fact, the unitarity of the physical S-matrix can be proved by showing that, in some suitable chosen gauge, the generating functional can be derived starting from a Hamiltonian formalism. This however has been done by S.Coleman[11] in the Arnowitt-Fickler[9] gauge, defined by $W_3^{\alpha}=0$.

We can therefore conclude that the S-matrix elements between physical states on mass shell are gauge independent and unitary.

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