Energetic Interpretation of the Solutions of the Einstein Field Equations for Spherically Symmetric Fluid Shells

Jorge L. deLyra*

Universidade de São Paulo
Instituto de Física
Rua do Matão, 1371,
05508-090 São Paulo, SP, Brazil

July 11, 2022

Abstract

We interpret the exact and quasi-exact solutions of General Relativity previously obtained for spherically symmetric shells of fluid matter, both liquid and gaseous, in terms of the energies involved. In order to do this we introduce certain integral expressions that are related to various parts of the energy. We then use these integral expressions in order to show that a certain parameter with dimensions of length, that in both cases was necessarily introduced into the solutions by the interface boundary conditions, is related to the binding energies of the gravitational systems.

In sequence, we use this representation of the gravitational binding energy in order to discuss the energetic stability of the new solutions found in the liquid case. We include in the stability discussion the well-known interior Schwarzschild solution for a liquid sphere, which can be obtained as a specific limit of the solutions that were previously obtained for the liquid shells. We show that this particular solution turns out to have zero binding energy and therefore to be a maximally unstable one, from the energetic point of view discussed here.

We also perform a numerical exploration of the energetic stability criterion of the liquid shell solutions, all of which have strictly positive binding energies, and show that indeed there is a particular subset of the solutions which are energetically stable. All these solutions have the form of shells with non-vanishing internal radii. This reduces the original three-parameter family of liquid shell solutions to a two-parameter family of energetically stable solutions.

The conclusion that inevitably follows from this exploration is that the requirement, which seems to be universally assumed in the literature, of the absence of any singularity whatsoever of any and all solutions at their centers, a requirement which is this case leads to the interior Schwarzschild solution, actually has the effect of selecting an energetically unstable solution, rather than the most stable one.

Keywords:
General Relativity, Einstein Equations, Energy Balance, Fluid Shells.

DOI: 10.5281/zenodo.6821946

*Email: delyra@lmail.if.usp.br ORCID: 0000-0002-8551-9711

1 Introduction

The issue of the energy in General Relativity is a difficult one, and its discussion even in specific examples quite often becomes involved and obscure. The difficulties start at the very foundations of the theory, with the impossibility of defining an energy-momentum tensor density for the gravitational field itself, a problem which apparently is related to the impossibility of localizing the energy of the gravitational field in the general case [1].

However, a recently discovered new class of static and time-independent exact and quasi-exact solutions [2,3] provides us with an opportunity to discuss the subject in a clear, precise and complete manner. It leads to a simple and clear characterization of all the energies involved in this class of solutions, as well as a characterization of the relations among them, which establishes an important connection with the fundamental concept of the conservation of energy. This is made possible by the fact that these solutions hold over the entire three-dimensional manifold, and not over just some part of it.

It is noteworthy that results similar to the ones we presented in [2] and [3] were also obtained for the case of neutron stars, with a Chandrasekhar-type equation of state [4], by Ni [5] and Neslušan [6]. Just as in [2,3], the analysis of that case also led to an inner vacuum region containing a singularity at the origin and a gravitational field which is repulsive with respect to that origin. This tends to indicate that these results are general at least to some considerable extent. It is to be expected that the ideas regarding the energy that we present here will be useful in that case as well.

This paper is organized as follows: in Section 2 we quickly review the new class of static and time-independent exact solutions for both gaseous and liquid shells, as well as the interior Schwarzschild solution, which can be obtained from the new liquid shell solutions in a certain limit; in Section 3 we establish certain general integral formulas for all the energies involved; in Section 4 we establish the general physical interpretation of the energies involved, including for both types of shell solutions, as well as for the interior Schwarzschild solution; in Section 5 we perform a small numerical exploration of the energetic stability of the liquid shell solutions, and in Section 6 we state our conclusions.

2 Review of the Shell Solutions

In two previous papers [2, 3] we established the solution of the Einstein field equations for the case of spherically symmetric shells of liquid and gaseous fluids located between radial positions r_1 and r_2 of the Schwarzschild system of coordinates. We will now quickly review these two solutions, emphasizing the similarities between them. In this work we will use the time-like signature $(+,-,-,-)$, following [1]. In terms of the coefficients of the metric, for an invariant interval given in terms of the Schwarzschild coordinates (t, r, θ, ϕ) by

$$ds^2 = e^{2\nu(r)}c^2dt^2 - e^{2\lambda(r)}dr^2 - r^2\left[d\theta^2 + \sin^2(\theta)d\phi^2\right], \tag{1}$$

where $\exp[\nu(r)]$ and $\exp[\lambda(r)]$ are two positive functions of only r, as was explained in [2] the Einstein field equations reduce to the set of three first-order differential equations

$$\left\{1 - 2\left[r\lambda'(r)\right]\right\} e^{-2\lambda(r)} = 1 - \kappa r^2\rho(r), \tag{2}$$

$$\left\{1 + 2\left[r\nu'(r)\right]\right\} e^{-2\lambda(r)} = 1 + \kappa r^2 P(r), \tag{3}$$

$$\left[\rho(r) + P(r)\right]\nu'(r) = -P'(r), \tag{4}$$

where $\rho(r)$ is the energy density of the matter, $P(r)$ is its isotropic pressure, the constant κ is given by $\kappa = 8\pi G/c^4$, G is the universal gravitational constant and c is the speed of light.

In these equations the primes indicate differentiation with respect to r. It is convenient for the analysis of the solutions to change variables in the field equations from the function $\lambda(r)$ to a function $\beta(r)$, which is defined to be such that

$$e^{2\lambda(r)} = \frac{r}{r - r_M \beta(r)}, \tag{5}$$

where $r_M = 2GM/c^2$ is the Schwarzschild radius associated to the total asymptotic gravitational mass M, which then implies that we have for the corresponding derivatives

$$2r\lambda'(r) = -r_M \frac{\beta(r) - r\beta'(r)}{r - r_M \beta(r)}. \tag{6}$$

Note that $\beta(r) = 0$ corresponds to $\lambda(r) = 0$ and therefore to $\exp[2\lambda(r)] = 1$ for the radial coefficient of the metric. In such cases the variations of the radial coordinate are equal to the variations of the corresponding proper radial lengths. Substituting the expressions in Equations (5) and (6) in the component field equation shown in Equation (2) a very simple relation giving the derivative of $\beta(r)$ in terms of $\rho(r)$ results,

$$\beta'(r) = \frac{\kappa r^2 \rho(r)}{r_M}. \tag{7}$$

Therefore, wherever $\rho(r) = 0$ we have that $\beta(r)$ is a constant. Since we must have that $\rho(r) \geq 0$, it therefore follows that $\beta(r)$ is a monotonically increasing function, which is a constant if and only if we are within a vacuum region. In addition to this, since according to the asymptotic boundary condition we must have that $\beta(\infty) = 1$, it also follows that $\beta(r)$ is limited from above by 1. Note that these facts are completely general for the spherically symmetric static case, irrespective of the type of fluid matter which is present within the matter region.

2.1 The Gaseous Shell Solutions

In a previous paper [3] we established the solution of the Einstein field equations for the case of a spherically symmetric shell of gaseous fluid located between the radial positions r_1 and r_2 of the Schwarzschild system of coordinates. These positions are *not* arbitrary, but rather are obtained as part of the solution of the problem. For this problem we assume the hypothesis that the gas satisfies the polytropic equation of state

$$P(r) = K \left[\rho(r)\right]^{1+1/n}, \tag{8}$$

where K is a positive constant and $n \geq 1$ is a real number that may be taken to be an integer or half-integer. For convenience we define the auxiliary quantity

$$F(r) = K \left[\rho(r)\right]^{1/n}, \tag{9}$$

in terms of which the equation of state becomes simply

$$P(r) = F(r)\rho(r). \tag{10}$$

As was shown in [3], given the field Equations (2) through (4) and the equation of state shown in Equation (8), the solution for $\lambda(r)$ is given by

$$\lambda(r) = \begin{cases} -\frac{1}{2}\ln\left(\frac{r+r_\mu}{r}\right) & \text{for} \quad 0 \leq r \leq r_1, \\ -\frac{1}{2}\ln\left[\frac{r-r_M\beta(r)}{r}\right] & \text{for} \quad r_1 \leq r \leq r_2, \\ -\frac{1}{2}\ln\left(\frac{r-r_M}{r}\right) & \text{for} \quad r_2 \leq r < \infty, \end{cases} \tag{11}$$

where once more $r_M = 2GM/c^2$, while for $\nu(r)$ we have

$$\nu(r) = \begin{cases} \frac{1}{2}\ln\left(\frac{1-r_M/r_2}{1+r_\mu/r_1}\right) + \frac{1}{2}\ln\left(\frac{r+r_\mu}{r}\right) & \text{for} \quad 0 \leq r \leq r_1, \\ \nu(r_2) - (n+1)\ln[1+F(r)] & \text{for} \quad r_1 \leq r \leq r_2, \\ \frac{1}{2}\ln\left(\frac{r-r_M}{r}\right) & \text{for} \quad r_2 \leq r < \infty. \end{cases} \tag{12}$$

The solution introduces into the system, through the interface boundary conditions, the new physical parameter r_μ with dimensions of length, which can be associated to a mass parameter μ in the same way that r_M is associated to M, namely by $r_\mu = 2G\mu/c^2$. As was also shown in [3], the determination of the function $\beta(r)$ in the matter region leads with no further difficulty to the determination of all the functions that describe both the matter and the geometry of the system, by means of the exact analytical relations

$$\rho(r) = \frac{r_M\beta'(r)}{\kappa r^2}, \tag{13}$$

$$P(r) = K\left[\frac{r_M\beta'(r)}{\kappa r^2}\right]^{1+1/n}, \tag{14}$$

$$F(r) = K\left[\frac{r_M\beta'(r)}{\kappa r^2}\right]^{1/n}, \tag{15}$$

$$\lambda(r) = \frac{1}{2}\ln\left[\frac{r}{r-r_M\beta(r)}\right], \tag{16}$$

$$\nu(r) = \nu(r_2) - (n+1)\ln[1+F(r)]. \tag{17}$$

The free parameters of the system are K, n and M, all of which describe the nature and state of the matter, and the value of $\beta'(r)$ at its point of maximum, which can also be seen to be related to the matter, since it determines the general scale of the matter energy density, as can be seen from Equation (7). Note that the radial positions r_1 and r_2 are not chosen by hand, and are defined as the positions where the energy matter density $\rho(r)$ becomes zero. As was shown in [3], the differential system has the property that once $\rho(r)$ hits the value zero during the integration, in either direction, it stays at zero from that point on, thus generating a vacuum region.

For all sets of parameters for which there is a solution of the differential problem the function $\beta'(r)$ has a single point of maximum within the matter region, which is the point where $\beta(r)$ has its single inflection point. As was also shown in [3], for all existing solutions it holds that $r_\mu > 0$. This strictly positive value of r_μ implies that the solution has a singularity at the origin. However, that singularity is not associated to an infinite concentration of matter, but rather, as explained in [2], to zero energy density at that point.

Both for the subsequent analysis and for the numerical approach, it is convenient to transform variables at this point, in order to write everything in terms of dimensionless variables and functions. In order to do this we must now introduce an arbitrary radial reference position $r_0 > 0$. For now the value of this parameter remains completely arbitrary, other than that it must be strictly positive, and has no particular physical meaning. It is only a mathematical device that allows us to define a dimensionless radial variable and a dimensionless parameter associated to the mass M by

$$\xi = \frac{r}{r_0}, \tag{18}$$

$$\xi_M = \frac{r_M}{r_0}, \tag{19}$$

as well as to define the dimensionless function of ξ, to assume the role of $\beta(r)$,

$$\gamma(\xi) = \xi_M \beta(r). \tag{20}$$

Note that the asymptotic condition that $\beta(r) \to 1$ for sufficiently large r translates here as the condition that $\gamma(\xi) \to \xi_M$ for sufficiently large ξ. Note also that, since $\beta(r)$ is a limited monotonic function, it follows from Equation (20) that similar conclusions can be drawn for $\gamma(\xi)$, which is therefore a monotonically increasing function which is limited from above, the upper limit in this case being the parameter ξ_M. We also have that $\gamma(\xi)$ is constant within vacuum regions. As was shown in [3] the function $\gamma(\xi)$, and thus the function $\beta(r)$ as well, is determined by the second-order ordinary differential equation

$$\pi'(\xi) = \pi(\xi) \left\{ \frac{2}{\xi} - \frac{n}{n+1} \frac{1}{\xi - \gamma(\xi)} \frac{1 + F(\xi,\pi)}{2F(\xi,\pi)} \left[\frac{\gamma(\xi)}{\xi} + F(\xi,\pi)\pi(\xi) \right] \right\}, \tag{21}$$

where $\pi(\xi) = \gamma'(\xi)$ is the derivative of $\gamma(\xi)$, the primes indicate derivatives with respect to ξ, and $F(\xi,\pi)$ is given by

$$F(\xi,\pi) = C \left[\frac{\pi(\xi)}{\xi^2} \right]^{1/n}, \tag{22}$$

where $C = K/\left(\kappa r_0^2\right)^{1/n}$ is a dimensionless constant. This differential system can be interpreted either as a second-order ordinary differential equation for $\gamma(\xi)$, or as one of a pair of first-order coupled ordinary differential equations determining $\gamma(\xi)$ and $\pi(\xi)$, the other equation being simply

$$\gamma'(\xi) = \pi(\xi). \tag{23}$$

This second interpretation is the one we adopted in [3]. This pair of first-order ordinary differential equations can be used for the numerical integration of this differential system, in order to obtain $\gamma(\xi)$ and $\beta(r)$, as we in fact did in that paper.

Note that in this formulation of the problem all dimensionfull physical quantities have vanished from view, and the problem is reduced to purely mathematical terms. All that is involved is a dimensionless function $\gamma(\xi)$ of a dimensionless variable ξ, its derivative $\pi(\xi)$, and two dimensionless positive real constants, n and C. A careful and complete study of this differential system, followed by a complete tabulation of all the properties of $\gamma(\xi)$, is all that stands between our current semi-analytical solution of the problem and what one could describe as a completely analytical solution of that problem. Some of the relevant properties of $\gamma(\xi)$ were in fact derived in [3].

2.2 The Liquid Shell Solutions

In another previous paper [2] we established the solution of the Einstein field equations for the case of a spherically symmetric shell of liquid fluid located between the arbitrarily chosen radial positions r_1 and r_2 of the Schwarzschild system of coordinates, with an energy density $\rho(r) = \rho_0$ which is constant with the radial coordinate r. In this case a completely analytical solution can be written down. This is a three-parameter family of solutions, which can be taken as any three of the four parameters r_1, r_2, M and ρ_0. The matter distribution is characterized by the radii r_1 and r_2, by its total asymptotic gravitational mass M, associated to the Schwarzschild radius r_M, and by a matter energy density ρ_0 which is constant with the radial Schwarzschild coordinate r within (r_1, r_2), and zero outside that interval.

It is interesting to note that this set of solutions can be interpreted as constituting the set of solutions of the case $n = 0$ of the polytropic problem discussed in the previous subsection, for in this case we have for the polytropic equation of state, written in an appropriate way,

$$\rho(r) = \overline{K}\,[P(r)]^{n/(n+1)}\,, \tag{24}$$

for some constant $\overline{K}$, which in the case $n = 0$ implies an energy density $\rho(r)$ which is constant with r. Given the field Equations (2) through (4), as presented in [2] the complete solution for $\lambda(r)$ is given by

$$\lambda(r) = \begin{cases} -\dfrac{1}{2}\ln\!\left(\dfrac{r + r_\mu}{r}\right) & \text{for} \quad 0 \leq r \leq r_1, \\[2ex] -\dfrac{1}{2}\ln\!\left[\dfrac{\kappa\rho_0\left(r_2^3 - r^3\right) + 3\,(r - r_M)}{3r}\right] & \text{for} \quad r_1 \leq r \leq r_2, \\[2ex] -\dfrac{1}{2}\ln\!\left(\dfrac{r - r_M}{r}\right) & \text{for} \quad r_2 \leq r < \infty, \end{cases} \tag{25}$$

where once again $r_M = 2GM/c^2$, while for $\nu(r)$ we have

$$\nu(r) = \begin{cases} \dfrac{1}{2}\ln\!\left(\dfrac{1 - r_M/r_2}{1 + r_\mu/r_1}\right) + \dfrac{1}{2}\ln\!\left(\dfrac{r + r_\mu}{r}\right) & \text{for} \quad 0 \leq r \leq r_1, \\[2ex] \dfrac{1}{2}\ln\!\left(\dfrac{r_2 - r_M}{r_2}\right) + \ln[z(r)] & \text{for} \quad r_1 \leq r \leq r_2, \\[2ex] \dfrac{1}{2}\ln\!\left(\dfrac{r - r_M}{r}\right) & \text{for} \quad r_2 \leq r < \infty, \end{cases} \tag{26}$$

in terms of a function $z(r)$ to be given shortly, and finally the pressure within the shell, that is, for $r_1 \leq r \leq r_2$, is given by

$$P(r) = \rho_0\,\frac{1 - z(r)}{z(r)}, \tag{27}$$

while it is zero outside the shell. Note that the value of $\lambda(r)$ within the matter region corresponds to the value of $\beta(r)$ given by

$$\beta(r) = 1 - \frac{\kappa\rho_0}{3r_M}\,\left(r_2^3 - r^3\right), \tag{28}$$

while in the outer vacuum region we have the constant value $\beta(r) = 1$, and in the inner vacuum region we have the constant value

$$\begin{aligned} \beta(r) &= 1 - \frac{\kappa\rho_0}{3r_M}\left(r_2^3 - r_1^3\right) \\ &= -\frac{r_\mu}{r_M}. \end{aligned} \tag{29}$$

This solution is valid under the condition that $r_2 > r_M$. Just as in the previous case, the solution introduces into the system, through the interface boundary conditions, the new physical parameter r_μ with dimensions of length, which once again can be associated to a mass parameter μ by $r_\mu = 2G\mu/c^2$. In all these expressions we have that r_μ is given in terms of the parameters characterizing the system by

$$r_\mu = \frac{\kappa\rho_0}{3}\left(r_2^3 - r_1^3\right) - r_M. \tag{30}$$

We also have that ρ_0 is determined algebraically in terms of r_1, r_2 and r_M as the solution of the transcendental algebraic equation

$$\begin{aligned} \sqrt{\frac{r_2}{3\left(r_2 - r_M\right)}} &= \sqrt{\frac{r_1}{\kappa\rho_0\left(r_2^3 - r_1^3\right) + 3\left(r_1 - r_M\right)}} + \\ &\quad + \frac{3}{2}\int_{r_1}^{r_2} dr\, \frac{\kappa\rho_0\, r^{5/2}}{\left[\kappa\rho_0\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)\right]^{3/2}}, \end{aligned} \tag{31}$$

and that the real function $z(r)$ is determined in terms of a non-trivial but straightforward elliptic real integral by the relation

$$\begin{aligned} z(r) &= \sqrt{\frac{\kappa\rho_0\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)}{r}} \times \\ &\quad \times \left\{\sqrt{\frac{r_2}{3\left(r_2 - r_M\right)}} + \frac{3}{2}\int_{r_2}^{r} ds\, \frac{\kappa\rho_0\, s^{5/2}}{\left[\kappa\rho_0\left(r_2^3 - s^3\right) + 3\left(s - r_M\right)\right]^{3/2}}\right\}. \end{aligned} \tag{32}$$

The relation shown in Equation (30) is a direct consequence of the field equations and of the interface boundary conditions associated with them. In [2] we proved that, so long as the pressure of the liquid is positive, we must have $r_\mu > 0$. In fact, the hypotheses of that proof can be weakened to require only that the pressure be strictly positive at a single point. Once again this strictly positive value of r_μ implies that the solution has a singularity at the origin. However, just as before that singularity is repulsive rather than attractive, and not associated to an infinite concentration of matter, but rather, as explained in [2], to zero energy density at that point.

2.3 The Interior Schwarzschild Solution

It is an interesting and somewhat remarkable fact that the well-known interior Schwarzschild solution [7,8] can be obtained from our solution for a liquid shell, even though the interior Schwarzschild solution has no singularity at the origin, while our solution always has such a singularity. Curiously enough, we must start by assuming that $r_\mu = 0$, even though we proved in [2] that one must have $r_\mu > 0$ in the shell solutions. The subtle point here is that the proof given in [2] relies on the existence of a shell with $r_1 > 0$, while in the case of the interior Schwarzschild solution we will have to use $r_1 = 0$, so that the shell becomes a filled sphere. If we start by first putting $r_\mu = 0$ and then make $r_1 = 0$ in Equation (30), we are led to the relation

$$\kappa\rho_0 = \frac{3r_M}{r_2^3}, \tag{33}$$

so that we may substitute $\kappa\rho_0$ in terms of r_M and the radius r_2 of the resulting sphere. Following the usual notation for the interior Schwarzschild solution, we now define a parameter R, with dimensions of length, such that $R^2 = r_2^3/r_M$, in terms of which we have

$$\kappa\rho_0 = \frac{3}{R^2}. \tag{34}$$

Note that the required condition that $r_2 > r_M$ is translated here as the condition that $R > r_2$. Making this substitution we have for $\lambda(r)$ inside the resulting sphere, directly from the line in Equation (25) for the matter region, in the case in which $r_\mu = 0$ and $r_1 = 0$,

$$\lambda(r) = -\frac{1}{2}\ln\left[1 - \left(\frac{r}{R}\right)^2\right], \tag{35}$$

for $r \leq r_2$, which implies that for the radial metric coefficient we have

$$\mathrm{e}^{-\lambda(r)} = \sqrt{1 - \left(\frac{r}{R}\right)^2}. \tag{36}$$

In order to obtain $\nu(r)$ inside the sphere we must first work out the function $z(r)$. Making the substitution of $\kappa\rho_0$ in terms of R in the result for $z(r)$ given in Equation (32) we get

$$z(r) = \sqrt{1 - \left(\frac{r}{R}\right)^2}\left[\sqrt{\frac{r_2}{r_2 - r_M}} + \frac{3}{2}\int_{r_2}^{r} ds\, \frac{s/R^2}{(1 - s^2/R^2)^{3/2}}\right]. \tag{37}$$

Is is now easy to see that in this case the remaining integral can be done, and we get

$$z(r) = \frac{3}{2} - \frac{1}{2}\sqrt{\frac{r_2}{r_2 - r_M}}\sqrt{1 - \left(\frac{r}{R}\right)^2}. \tag{38}$$

Using again the definition of R, which implies that we have $r_M/r_2 = (r_2/R)^2$, we may write this as

$$z(r) = \frac{3}{2} - \frac{1}{2}\sqrt{\frac{1 - (r/R)^2}{1 - (r_2/R)^2}}. \tag{39}$$

Note that we have $z(r_2) = 1$, which corresponds to $P(r_2) = 0$, so that the interface boundary conditions for $z(r)$ and $P(r)$ at r_2 are still satisfied. From this we may now obtain all the remaining results for the interior Schwarzschild solution. From the line in Equation (26) for the matter region, in the case in which $r_\mu = 0$ and $r_1 = 0$, we get for $\nu(r)$ in the interior of the sphere

$$\nu(r) = \frac{1}{2}\ln\left[1 - \left(\frac{r_2}{R}\right)^2\right] + \ln\left[\frac{3}{2} - \frac{1}{2}\sqrt{\frac{1 - (r/R)^2}{1 - (r_2/R)^2}}\right], \tag{40}$$

for $r \leq r_2$, which implies that for the temporal metric coefficient we have

$$\mathrm{e}^{\nu(r)} = \frac{3}{2}\sqrt{1 - \left(\frac{r_2}{R}\right)^2} - \frac{1}{2}\sqrt{1 - \left(\frac{r}{R}\right)^2}. \tag{41}$$

Finally, from Equation (27), in the case in which $r_\mu = 0$ and $r_1 = 0$, we get for the pressure $P(r)$ within the sphere

$$P(r) = \rho_0 \, \frac{\sqrt{1-(r/R)^2} - \sqrt{1-(r_2/R)^2}}{3\sqrt{1-(r_2/R)^2} - \sqrt{1-(r/R)^2}}. \tag{42}$$

These are indeed the correct results for the case of the interior Schwarzschild solution, including the well-known restriction that $r_2 > 9r_M/8$, which is required by the positivity of the left-hand side of Equation (41). Note that all the arguments of the logarithms and of the square roots are positive due to the conditions that $R > r_2 \geq r$. Note also that in the $r_1 \to 0$ limit the lines in Equations (25) and (26) for the case of the inner vacuum region become irrelevant, since this region reduces to a single point. On the other hand, the lines in Equations (25) and (26) for the case of the outer vacuum region do not change at all.

It is therefore apparent that the $r_1 \to 0$ limit of our liquid shell solutions does reproduce the interior Schwarzschild solution, so long as we adopt the value zero for r_μ. Our interpretation of these facts is that the $r_1 \to 0$ limit to the interior Schwarzschild solution is a *non-uniform* one, in which we have to leave out one point, the origin. In the $r_1 \to 0$ limit the singularity of the shell solutions becomes a strictly point-like one, and therefore a removable one, by a simple continuity criterion. This is certainly the case for the energy density $\rho(r) = \rho_0$, which in the limit is non-zero everywhere around the origin but at a single point, the origin itself. The same is true for the pressure $P(r)$, which in the limit is also non-zero around the origin but at the origin itself. Similar situations hold for $\lambda(r)$ and $\nu(r)$, as is not difficult to see numerically. It seems that all these functions converge in the $r_1 \to 0$ limit to functions with a point-like removable discontinuity at the origin.

We end this section by noting that a similar type of limit, in which we make $r_1 \to 0$ and $r_\mu \to 0$, can be applied to the solutions for gaseous shells [3] that we discussed before, and results in the filled-sphere solutions found by Tooper [9]. Therefore, all the Tooper solutions also correspond to the choice $r_\mu = 0$ which avoids the singularity at the origin, this being in fact the very criterion used by Tooper to define these solutions.

3 Integral Expressions for the Energies

It is possible to express the masses M and μ, which are associated to the parameters with dimensions of length $r_M = 2MG/c^2$ and $r_\mu = 2\mu G/c^2$ that appear in the exact solutions for gaseous and liquid shells described in Section 2, and hence to express the corresponding energies Mc^2 and μc^2, as integrals of the matter energy density $\rho(r)$ over coordinate volumes, in a way similar to what is usually done for M in the literature [4, 10], but leading to very different results in the case of the shell solutions. In order to do this in a simple and organized way, we must first change variables in the field equations from $\lambda(r)$ to $\beta(r)$, as was done at the first part of Section 2.

From the solutions it follows in either case, from Equations (11) and (25), that we have that $\beta(r) = 1 > 0$ in the outer vacuum region, and hence at r_2 in particular, and that we have that $\beta(r) = -r_\mu/r_M < 0$ in the inner vacuum region, and hence at r_1 in particular. Since $\beta(r)$ is a continuous function that goes from negative values at r_1 to positive values at r_2, it follows that there is a radial position r_z within the matter region where $\beta(r_z) = 0$, regardless of whether or not $\rho(r)$ is constant within the shell. At this particular radial position we also have that $\lambda(r_z) = 0$.

Let us now consider the integral of the matter energy density over a spherical coordinate volume within the matter region, where $\rho(r) \neq 0$, say from an arbitrary radial position r_a

such that $r_a \geq r_1$ to another radial position r_b such that $r_a < r_b \leq r_2$,

$$\int_{r_a}^{r_b} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, r^2 \sin(\theta)\rho(r) = 4\pi \int_{r_a}^{r_b} dr\, r^2 \rho(r), \tag{43}$$

where we integrated over the angles. Note that this is *not* an integral over the *proper* volume, but just an integral over the *coordinate* volume, since we are missing here the remaining factor $\exp[\lambda(r) + \nu(r)]$ of the Jacobian $\sqrt{-g}$. Since we have the three special radial positions r_1, r_z and r_2 where the values of $\beta(r)$ are known, let us consider now the integral of the energy density over the coordinate volume from r_z to r_2. Using Equation (7) we get

$$4\pi \int_{r_z}^{r_2} dr\, r^2 \rho(r) = 4\pi \frac{r_M}{\kappa} \int_{r_z}^{r_2} dr\, \beta'(r). \tag{44}$$

One can now see that the integral on the right-hand side is trivial, and since we have that $\beta(r_z) = 0$ and that $\beta(r_2) = 1$, we get

$$Mc^2 = 4\pi \int_{r_z}^{r_2} dr\, r^2 \rho(r), \tag{45}$$

where we have replaced κ and r_M by their values in terms of M, G and c. We have therefore an expression for the energy Mc^2 in terms of a coordinate volume integral of the energy density. Note however that the integral does not run over the whole matter region, since it starts at $r_z > r_1$ rather than at r_1. Therefore, if we consider the integral from r_1 to r_z, in a similar way we get

$$4\pi \int_{r_1}^{r_z} dr\, r^2 \rho(r) = 4\pi \frac{r_M}{\kappa} \int_{r_1}^{r_z} dr\, \beta'(r). \tag{46}$$

Once again one can see that the integral on the right-hand side is trivial, and since we have that $\beta(r_z) = 0$ and that $\beta(r_1) = -r_\mu/r_M$, we now get

$$\mu c^2 = 4\pi \int_{r_1}^{r_z} dr\, r^2 \rho(r), \tag{47}$$

where we have now replaced κ and r_μ by their values in terms of μ, G and c. We have therefore an expression for the energy μc^2 in terms of another coordinate volume integral of the energy density, this time over the remaining part of the matter region.

If we now consider the integral over the whole matter region, due to the additive property of the integrals over the union of disjoint domains, using Equations (45) and (47) we obtain the result that

$$4\pi \int_{r_1}^{r_2} dr\, r^2 \rho(r) = \mu c^2 + Mc^2. \tag{48}$$

This is a sum of energies, and is therefore also an energy, to which we will associate a mass parameter M_D, so that this energy is given by $M_D c^2$, and we have the relation

$$M_D c^2 = \mu c^2 + Mc^2, \tag{49}$$

where

$$M_D c^2 = 4\pi \int_{r_1}^{r_2} dr\, r^2 \rho(r). \tag{50}$$

We see therefore that the radial position r_z where $\beta(r_z) = 0$, and therefore $\lambda(r_z) = 0$, plays a particularly important role when it comes to the determination of the energies involved.

Note that this whole argument holds for any function $\rho(r)$ within the matter region, that is, for both the gaseous and liquid cases. For the specific case of liquid shells, with a constant $\rho(r) = \rho_0$, we find from Equation (28), which holds within the matter region, that in this case we have for the zero r_z of $\beta(r)$

$$r_z = \left(r_2^3 - \frac{3r_M}{\kappa\rho_0} \right)^{1/3}. \tag{51}$$

Note that, although all these integrals are written in terms of the energy density $\rho(r)$ of the matter, none of them represents just the energy of only the matter itself. In fact we must still interpret the meaning of each one of these expressions, which is what we will discuss in detail in the next section. However, one interpretation can be established at once. In either case $\rho(r)$ is the density of the total energy content of the matter, which includes the rest energy of the mass that constitutes the matter, a part which in all cases of interest comprises almost all the matter energy present, and some amount of thermal energy. In the case of gases we have within $\rho(r)$ a part which is the thermal energy associated to the thermal agitation of the gas particles. In the liquid case this extra amount of energy includes both some thermal energy and some relatively small amount of energy associated to the bindings between particles of fluid, which is associated to a certain amount of latent heat. We may therefore write that

$$\rho(r) = \rho_T(r) + \rho_U(r), \tag{52}$$

where $\rho_U(r)$ is the density of rest energy, and $\rho_T(r)$ is the remainder, which we will just call the density of thermal energy. In the language of Relativity, $\rho_T(r)$ is associated, essentially, to the small average increase in the masses of particles due to the average speed associated to the thermal agitation. We should keep in mind that in all cases of interest we always have that $\rho_T(r) \ll \rho_U(r)$. We therefore have for the expression for the mass parameter M_D in Equation (50)

$$\begin{aligned} M_D c^2 &= 4\pi \int_{r_1}^{r_2} dr\, r^2 \rho_T(r) + 4\pi \int_{r_1}^{r_2} dr\, r^2 \rho_U(r) \\ &= E_T + 4\pi \int_{r_1}^{r_2} dr\, r^2 \rho_U(r), \end{aligned} \tag{53}$$

where E_T is an energy associated to the thermal state of the matter, of the same order of magnitude of the total thermal energy, although not necessarily exactly equal to it. We will therefore associate a mass M_U to the integral of the density of rest energy, and write that

$$M_U c^2 = 4\pi \int_{r_1}^{r_2} dr\, r^2 \rho_U(r), \tag{54}$$

so that

$$M_D c^2 = E_T + M_U c^2, \tag{55}$$

implying that our identity in Equation (49) giving the relation among all the forms of energy becomes

$$E_T + M_U c^2 = \mu c^2 + M c^2, \tag{56}$$

where in all cases of interest we have that $E_T \ll M_U c^2$. We are now ready to work out the physical interpretation of all the terms in this relation among energies.

4 Physical Interpretation of the Energies

Of the four energies at play here, namely E_T, $M_U c^2$, μc^2 and $M c^2$, only the first and last ones have well-established meanings at this point. Since M is the asymptotic gravitational mass of the system, that is, the gravitational mass seen as the source of the gravitational field at large radial distances, the standard interpretation in General Relativity is that the energy Mc^2 is the total energy of this gravitational system, bound into the shell by the gravitational interactions, and which from now on we will simply call the *bound system*. It includes both the energy of the matter in the bound state and the (possibly negative) energy stored in the gravitational field itself, also in this bound state. The interpretation of the energy density $\rho(r)$ is that it is the amount of energy of the matter, per unit proper volume, as seen by a stationary local observer at the radial position r.

Our first task here is to establish the physical interpretation of the energy $M_U c^2$. In order to do this, the first thing to be done is to define an *unbound system* related to our bound system as defined above. This unbound system is what we get when we disperse all the infinitesimal matter elements of the shell to very large distances from each other, in order to eliminate all the gravitational interactions, thus producing a static set of particles at infinity. We will show here that the energy $M_U c^2$ is the total energy of this cold unbound system. We will do this by performing a mathematical transformation on the integral in Equation (54), which can be written as the following expression in terms of a volume integral,

$$M_U c^2 = \int_{r_1}^{r_2} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, r^2 \sin(\theta) \rho_U(r). \tag{57}$$

The transformation, applied to the right-hand side of this equation, will allow us to interpret the meaning of the left-hand side. This will be done in a general way, for any function $\rho_U(r)$ within the matter region. This transformation will consist in fact of the construction of a second integral, based on the concept of the Riemann sums of the volume integral shown in Equation (57).

Let us consider therefore an arbitrary Riemann partition of the integral in Equation (57), consisting of a large but finite number N of small cells δV_n with coordinate volume and linear coordinate dimensions below certain maximum values, where $n \in \{1, \ldots, N\}$. By definition of a Riemann partition the sum of all these volume elements is equal to the coordinate volume V of the shell,

$$V = \sum_{n=1}^{N} \delta V_n, \tag{58}$$

where we will assume that each volume element is at the spatial position $\vec{r}_n$ relative to the center of the shell, as illustrated in Figure 1. The energy $M_U c^2$ can therefore be written as the integration limit of the Riemann sum over this partition,

$$M_U c^2 = \lim_{N \to \infty} \sum_{n=1}^{N} \rho_U(r_n) \delta V_n, \tag{59}$$

where $r_n = |\vec{r}_n|$. We now consider the mathematical transformation in which we map each coordinate volume element δV_n at $\vec{r}_n$ onto an identical volume element $\delta V_n'$ at the coordinate

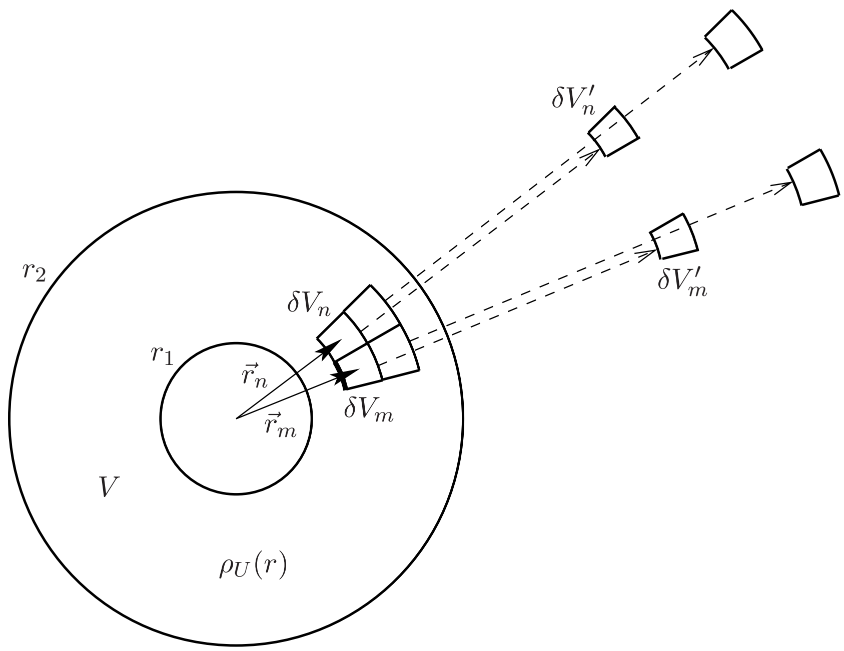

Figure 1: Illustration of the geometrical transformation of the integral over the shell.

position $\vec{r}\,'_n = \alpha\vec{r}_n$, for some large positive real number α, without changing the coordinate volume of the volume elements. The result is a new set of volume elements, all at large distances from each other, whose sum is still equal to the coordinate volume of the shell,

$$V = \sum_{n=1}^{N} \delta V'_n. \tag{60}$$

The geometrical transformation leading to the construction of the new integral is illustrated in Figure 1. Note that no actual physical transportation of the matter or of the rest energy within the volume elements δV_n of the shell is meant here, so that there are no actual physical transformations involved.

After defining the volume elements $\delta V'_n$ at large distances in this fashion, we now put within each one of these new volume elements exactly the same amount of mass and hence of rest energy that we have in the corresponding coordinate volume elements δV_n of the shell. This means putting into each volume element $\delta V'_n$ at infinity the same numbers of the same types of particles, as seen by a stationary local observer at the position $\vec{r}\,'_n$, that a stationary local observer at $\vec{r}_n$ sees within δV_n. However, unlike what happens within δV_n, the particles within $\delta V'_n$ are all at rest. In other words, we associate to each volume element at infinity the same value of the rest energy density $\rho_U(r'_n) = \rho_U(r_n)$ that we had for the corresponding volume element of the shell, where $r'_n = |\vec{r}\,'_n|$ and $r_n = |\vec{r}_n|$.

For large values of α these elements of mass and hence of rest energy within $\delta V'_n$ are all at large distances from each other, so as to render the gravitational interactions among them negligible. In the $\alpha \to \infty$ limit all the gravitational interactions among the volume elements $\delta V'_n$ go to zero. Besides, in the *integration* limit each element of mass and hence of rest energy so constructed tends to zero, so that the gravitational self-interactions within each volume element also go to zero. However, independently of either limit, by construction the total coordinate volume of the elements of volume at infinity remains equal to the coordinate volume of the shell. Therefore, by construction the corresponding sum of all the elements of rest energy at infinity is the same as the Riemann sum that appears in Equation (59),

$$\sum_{n=1}^{N} \rho_U(r'_n)\delta V'_n = \sum_{n=1}^{N} \rho_U(r_n)\delta V_n. \tag{61}$$

Now, at radial infinity spacetime is flat, so that the coordinate volume of each volume element $\delta V'_n$ coincides with its proper volume, and hence the energy element $\rho_U(r'_n)\delta V'_n$ is the total rest energy of that element of matter, so that the sum of all these rest energy elements is the total rest energy of the dispersed matter at infinity. In other words, once we take the integration limit the integral given in Equation (57) gives us the total rest energy of the system at infinity, which is free from all gravitational bindings, and which is also at zero temperature. Therefore we will name the quantity $M_U c^2$ the *total rest energy* of the *cold unbound system*. This is the total energy of the system when all gravitational interactions have been eliminated by increasing without limit the distances among its elements. This is in both analogy and contrast with the quantity Mc^2, which is the *total energy* of the *bound system*, after all its parts have been brought together to form the shell.

Note that this whole argument is general, in the sense that it is not limited to the case in which $\rho(r) = \rho_0$ is a constant. However, in the case of the liquid shells, since in that case $\rho(r) = \rho_0$ is indeed a constant, the total energy of the unbound system is just the product of ρ_0 by the coordinate volume V of the shell,

$$M_U c^2 = \rho_0 V. \tag{62}$$

Our next task here is to establish the physical interpretation of the energy μc^2. From Equation (56) we have that, up to the small thermal-related energy E_T, the energy parameter μc^2 is the difference between the total energy of the unbound system and the total energy of the bound system,

$$\mu c^2 - E_T = M_U c^2 - Mc^2, \tag{63}$$

and therefore we conclude that the difference on the left-hand side is equal to the gravitational *binding energy* of the system, which is given by the difference on the right-hand side. It is the amount of energy that must be given to the bound system in order to disperse its elements to infinity, thus eliminating all the gravitational bindings between those elements. It is also the amount of energy that must be dissipated by the unbound system during the process of its assembly into the bound system, when starting from the cold unbound system at infinity.

The theorem we proved in [2, 3], namely that we must have $r_\mu > 0$, which implies that $\mu c^2 > 0$, is actually necessary in order for the left-hand side of Equation (63) to be positive, as it must be if the right-hand side is to be finite, positive and non-zero, meaning that the bound system has a finite, positive and non-zero binding energy. This is, of course, closely related to the attractive nature of the gravitational interaction between particles. Here we see why we cannot completely eliminate E_T from the discussion, even if we have that $E_T \ll M_U c^2$. This is so because although both $M_U c^2$ and Mc^2 are typically much larger than the other terms in this equation, it is still possible that their difference is relatively small, and commensurate with the other terms.

We may therefore construct a direct interpretation of the energy μc^2 as a binding energy, by two different but related arguments. Considering first the process of dispersion of the bound system to infinity, we rewrite Equation (63) in the form

$$M_U c^2 = Mc^2 + \left(\mu c^2 - E_T\right), \tag{64}$$

which describes the energy of the cold unbound system at infinity as composed of the parts shown on the right-hand side. Therefore, μc^2 is the energy than must be given to the bound system of energy Mc^2 in order to disperse it to infinity, with the exception of the thermal-related energy E_T which is already present in the system, assuming here that this thermal-related energy can be used for this purpose without losses. Equivalently, if we think in terms of the assembly of the bound system, and rewrite Equation (63) in the form

$$Mc^2 = M_U c^2 - \left(\mu c^2 - E_T\right), \tag{65}$$

with gives the energy of the bound system as composed of the parts shown on the right-hand side. We may therefore conclude that μc^2 is the energy that the unbound system must dissipate, and therefore lose, during the assembly process, with the exception of the thermal-related energy E_T, which remains in the bound system after it is assembled. It is therefore fair to assert that μc^2 is essentially the binding energy of the system, if we include in the discussion the fact that the bound system must contain thermal energy if it is to be stable. In the all-important case of gaseous shells the presence of this thermal energy in the bound system is of course essential to maintain the pressure of the gas, which in turn is what keeps the gaseous shell stable against the attractive gravitational forces.

It is interesting to note that, although all these integrals are written in terms of the energy density $\rho(r)$ of the matter, the energy Mc^2 is *not* the energy $M_m c^2$ of just the matter within the bound system. That would be given by the integral with the full Jacobian factor $\sqrt{-g}$, where g is the determinant of $g_{\mu\nu}$, which in our case here results in

$$M_m c^2 = 4\pi \int_{r_1}^{r_2} dr\, r^2\, \mathrm{e}^{\lambda(r)+\nu(r)} \rho(r). \tag{66}$$

It is not difficult to verify that this energy is always smaller than $M_D c^2$, due to the fact that the exponent $\lambda(r)+\nu(r)$ is always negative within the matter region. In order to show this we just take the difference between the component field equations shown in Equations (3) and (2), thus obtaining

$$[\lambda(r) + \nu(r)]' = \frac{\kappa}{2}\, \mathrm{e}^{2\lambda(r)} r \left[\rho(r) + P(r)\right]. \tag{67}$$

Since all quantities appearing on the right-hand side are positive or zero, we may conclude that the derivative of the exponent is non-negative. However, we have that $\lambda(r_2)+\nu(r_2) = 0$, since this exponent is identically zero within the outer vacuum region, as can be seen from Equations (11) and (12), as well as from Equations (25) and (26). It follows that

$$\lambda(r) + \nu(r) < 0, \tag{68}$$

and therefore that

$$\mathrm{e}^{\lambda(r)+\nu(r)} < 1, \tag{69}$$

throughout the whole matter region, with the exception of the single radial position r_2 where the exponential is equal to one. Therefore, it follows for the two integrals that

$$4\pi \int_{r_1}^{r_2} dr\, r^2\, \mathrm{e}^{\lambda(r)+\nu(r)} \rho(r) < 4\pi \int_{r_1}^{r_2} dr\, r^2 \rho(r), \tag{70}$$

and therefore that $M_m c^2 < M_D c^2$. In general, in order to determine the difference between these two energies $M_m c^2$ has to be calculated numerically. Note that relations such as Equation (70) are also valid separately for $\rho_T(r)$ and for $\rho_U(r)$, and therefore for both the thermal-related energy and the rest energy separately. Here we get a glimpse of the reason why the thermal-related energy E_T is not exactly the thermal energy of the matter in the bound system, but in fact somewhat larger than it. Simply put, thermal energy also gravitates. If any amount of energy is to be delivered to the particles of the bound system in order to disperse that bound system to infinity, it must also do the work of dispersing itself to infinity as well, and hence gets there with its value somewhat decreased, since some more of it has to be surrendered to the gravitational field. This is fundamentally a consequence of the non-linearity of the theory.

4.1 Energetic Stability

This interpretation of the energy parameters involved leads right away to the idea that we may define a notion of *energetic stability* of the solutions obtained, in the general spirit of the principle of virtual work. Given certain constraints regarding some of the parameters of the solutions, we may obtain the parameter r_μ as a function of the remaining parameters of the system. Within this class of solutions, if there are two with different values of r_μ, which is monotonically increasing with the binding energy $\mu c^2 - E_T$, then in principle the constrained system will tend to go from the one with the smaller value of r_μ to the one with the larger value, given the existence of a permissible path between the two solutions. This type of analysis allows us to acquire some information about the dynamical behavior of the system, without having to find explicitly the corresponding time-dependent solutions.

Let us exemplify this with the liquid shell solutions, in a way that is physically illustrative. Since liquids can only exist at very low temperatures, in this case we may assume that $E_T \approx 0$, so that the temperature has no role to play, and thus we have that $M_D = M_U$ and that

$$\mu c^2 = M_U c^2 - M c^2. \tag{71}$$

In this case the system contains four parameters, namely r_1, r_2, r_M and ρ_0, of which only three are independent. As was explained in [2], these four parameters are related by the condition in Equation (31). Given any three of the parameters, that equation can be used to determine the fourth in terms of those three. Let us assume that we are given fixed values of both M and ρ_0, thus determining the local properties of the matter and the total amount of energy of the bound system. This is equivalent to fixing r_M and ρ_0, and therefore the result of solving Equation (31) is to establish r_1 as a function of r_2. We therefore are left with a collection of solutions parametrized by a single real parameter, the external radius r_2. We may then determine $r_\mu(r_2)$ and verify whether this function has a single local maximum at a certain value of r_2. This then identifies that particular solution which is stable, or that has the largest binding energy, among all others, given the constraints described.

Another approach, slightly more indirect, but perhaps ultimately simpler and more physically compelling, would be to keep constant the local parameter ρ_0 and the energy $M_U c^2$ of the unbound system. This fixes the local properties of the matter and the total energy of the unbound system that we start with, and we may then ask which is the solution that corresponds to the most tightly bound system that can be assembled from that unbound system. Since the energy of the unbound system is the product of ρ_0 by the coordinate volume V of the shell, as can be seen in Equation (62), keeping fixed both ρ_0

and M_U corresponds to keeping fixed at a value V_0 that coordinate volume, which is given by

$$V_0 = \frac{4\pi}{3}\left(r_2^3 - r_1^3\right). \tag{72}$$

This immediately determines r_2 as a simple function $r_2(r_1)$ of r_1. Then solving Equation (31) results in r_M being given as a function $r_M(r_1)$ of r_1 for the fixed value of ρ_0 and the fixed coordinate volume V_0. This corresponds to the energy of the bound system with internal radius r_1, for the given fixed values of ρ_0 and V_0. The minimum of the function $r_M(r_1)$ gives us the value of r_1 that corresponds to the most tightly bound system that can be assembled from a given unbound system. Other solutions in the same family, with other values of r_1, will tend to decay into this one, given a permissible decay path between the two solutions involved. We will execute this program numerically in Section 5.

We saw that in the case of the interior Schwarzschild solution we have the value zero for r_μ. This implies that the resulting solution, in the low-temperature case, has zero gravitational binding energy, and that its energy is the same as the energy of the corresponding cold unbound system, which is a very strange and even bizarre situation indeed. This means that the resulting solution is not only energetically unstable, but that it is in fact *maximally* energetically unstable, since the bound system cannot possibly have more energy than the unbound system. Given a permissible path, in principle one would be able to disperse the matter distribution of the interior Schwarzschild solution, taking every element of matter do infinity, without giving any energy at all to the system. This makes this particular solution quite unrealistic, and may be one reason why it has never proved to be a very useful one.

In the context of the second numerical approach described above, in which ρ_0 and M_U are kept constant, and limiting ourselves to only the spherically symmetric dynamics, the instability of the interior Schwarzschild solution means that, if some small perturbation created even an infinitesimal vacuum bubble at the origin, then this bubble would spontaneously grow until its radius r_1 assumed the value that minimizes the function $r_M(r_1)$. This is the physical meaning of the fact that the singularity at the origin is a repulsive one.

5 Numerical Exploration of the Binding Energy

Here we will explore numerically the issues of the binding energy and of the energetic stability of the low-temperature liquid shell solutions. In this exploration we will keep fixed the local energy density parameter ρ_0, as well as the total energy $M_U c^2$ of the unbound system. Our objective will be then to determine the existence and the parameters of the maximally bound liquid shell solution. We will do this by calculating the energy Mc^2 of the bound system and showing that it has a point of minimum as a function of r_1. Since we keep fixed the parameter ρ_0, and since the energy of the unbound system is given by $M_U c^2 = \rho_0 V_0$, this implies that we also keep fixed the coordinate volume V_0 of the shell, given in Equation (72), which immediately establishes r_2 as a given function of r_1,

$$r_2(r_1) = \left(r_1^3 + \frac{3V_0}{4\pi}\right)^{1/3}. \tag{73}$$

Therefore, of the three free parameters of our solutions, which can be taken to be r_1, r_2 and ρ_0, one is being kept fixed and another is a given function, so that we are left with only one free parameters, which we will take to be r_1. Under these circumstances we have that r_M, and therefore both the mass M and the energy Mc^2 of the bound system, are functions of r_1, with values that are left to be determined numerically.

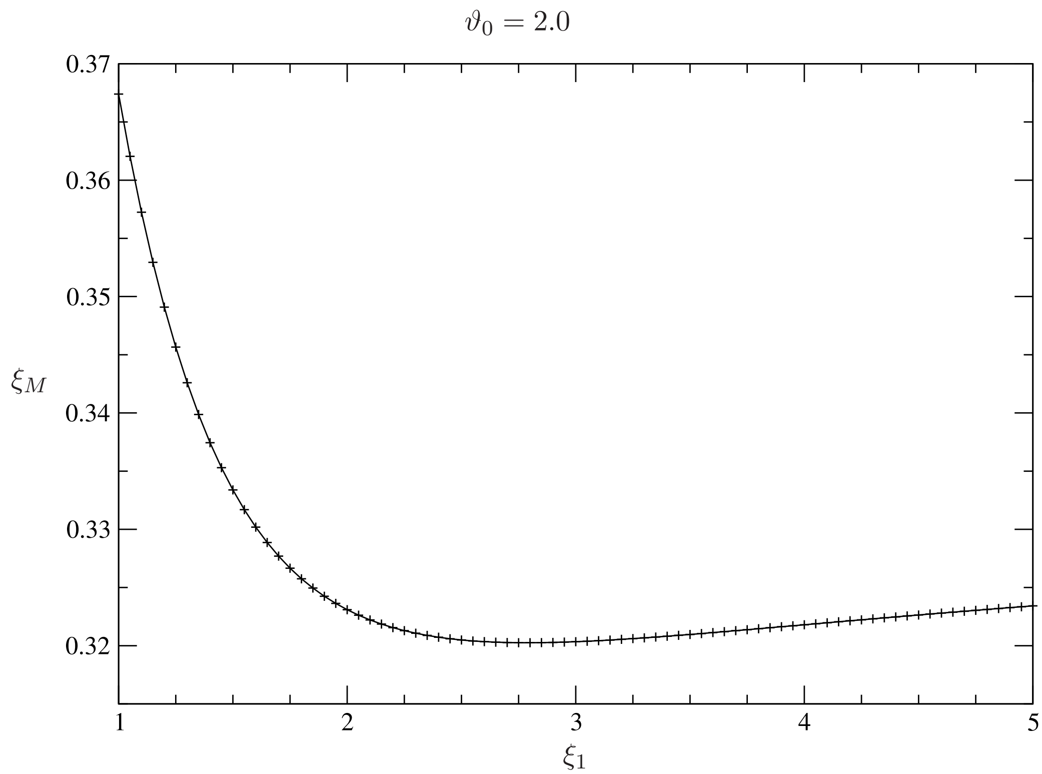

Figure 2: Graph of the energy of the bound system as a function of ξ_1, for a fixed energy of the unbound system, given by $\vartheta_0 = 2$, and with ξ_1 in $[1, 5]$.

In order to perform the numerical work it is convenient to first rescale the variables, creating a set of equivalent dimensionless variables, as was already mentioned in Section 2. Since under these conditions $\kappa\rho_0$ is a constant which has dimensions of inverse square length, we will define a constant r_0 with dimensions of length by

$$r_0 = \frac{1}{\sqrt{\kappa\rho_0}}. \tag{74}$$

Having now the known constant r_0, we use it in order to define the set of dimensionless parameters given by

$$\begin{aligned} \xi_1 &= \frac{r_1}{r_0}, \\ \xi_2 &= \frac{r_2}{r_0}, \\ \xi_M &= \frac{r_M}{r_0}, \\ \vartheta_0 &= \frac{3V_0}{4\pi r_0^3}, \end{aligned} \tag{75}$$

where ϑ_0 is the ratio between the coordinate volume V_0 of the shell and the volume of an Euclidean sphere of radius r_0. The expression in Equation (73) giving r_2 as a function of r_1 is now translated as

$$\xi_2(\xi_1) = \left(\vartheta_0 + \xi_1^3\right)^{1/3}. \tag{76}$$

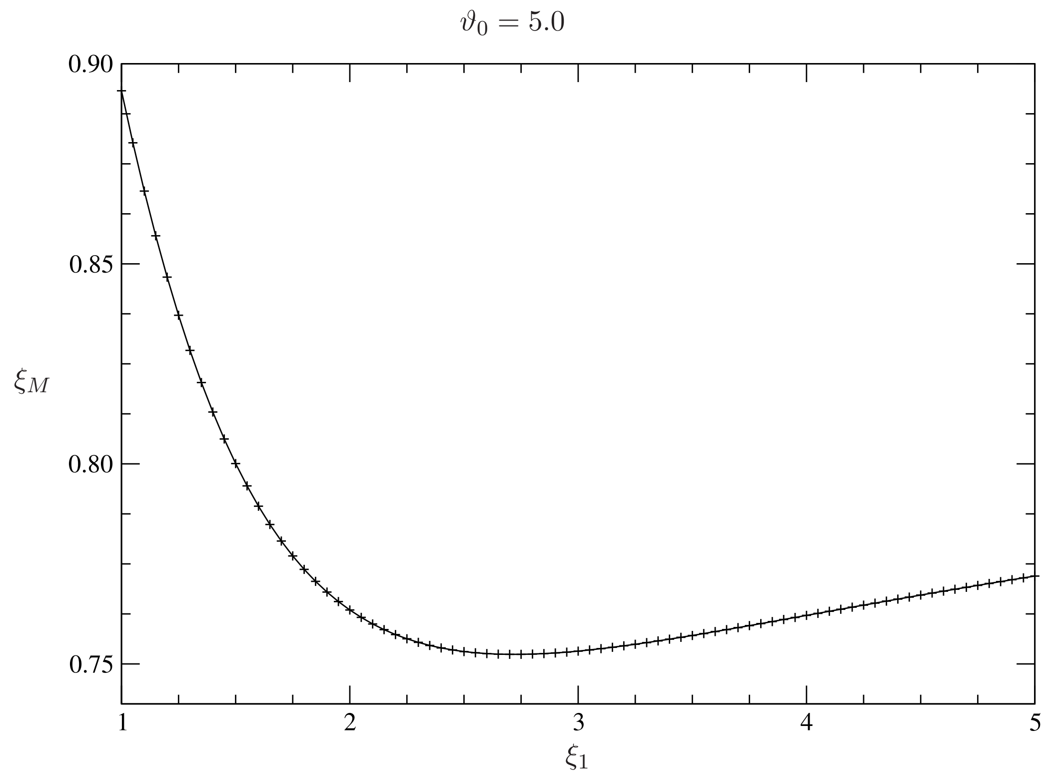

Figure 3: Graph of the energy of the bound system as a function of ξ_1, for a fixed energy of the unbound system, given by $\vartheta_0 = 5$, and with ξ_1 in $[1, 5]$.

Note, for subsequent use, that this can also be written as $\xi_2^3 - \xi_1^3 = \vartheta_0$. The relation which we must now use in order to determine ξ_M is that given in Equation (31), which upon rescalings by r_0 can be written as

$$\sqrt{\frac{\xi_2}{3\,(\xi_2 - \xi_M)}} = \sqrt{\frac{\xi_1}{\xi_2^3 - \xi_1^3 + 3\,(\xi_1 - \xi_M)}} + \frac{3}{2}\int_{\xi_1}^{\xi_2} d\xi\, \frac{\xi^{5/2}}{\left[\xi_2^3 - \xi^3 + 3\,(\xi - \xi_M)\right]^{3/2}}, \qquad (77)$$

where we changed variables in the integral from r to $\xi = r/r_0$. Substituting for ϑ_0 where possible we have the following non-trivial algebraic equation that determines ξ_M, and therefore r_M, in terms of ξ_1,

$$\sqrt{\frac{\xi_1}{\vartheta_0 + 3\,(\xi_1 - \xi_M)}} - \sqrt{\frac{\xi_2}{3\,(\xi_2 - \xi_M)}} + \frac{3}{2}\int_{\xi_1}^{\xi_2} d\xi\, \frac{\xi^{5/2}}{\left[\xi_2^3 - \xi^3 + 3\,(\xi - \xi_M)\right]^{3/2}} = 0. \qquad (78)$$

Our objective here is to solve this equation in order to get $\xi_M(\xi_1)$, given a fixed value of ϑ_0 and with ξ_2 given by Equation (76). Note that, due to the homogeneous scalings leading from the dimensionfull quantities to the dimensionless ones, shown in Equation (75), each solution of this equation is valid for any strictly positive value of ρ_0, which no longer appears explicitly. The same is true of the graphs to be generated using this equation. Given a value of ϑ_0, the corresponding graph represents the results for all the possible strictly positive values of the energy density ρ_0.

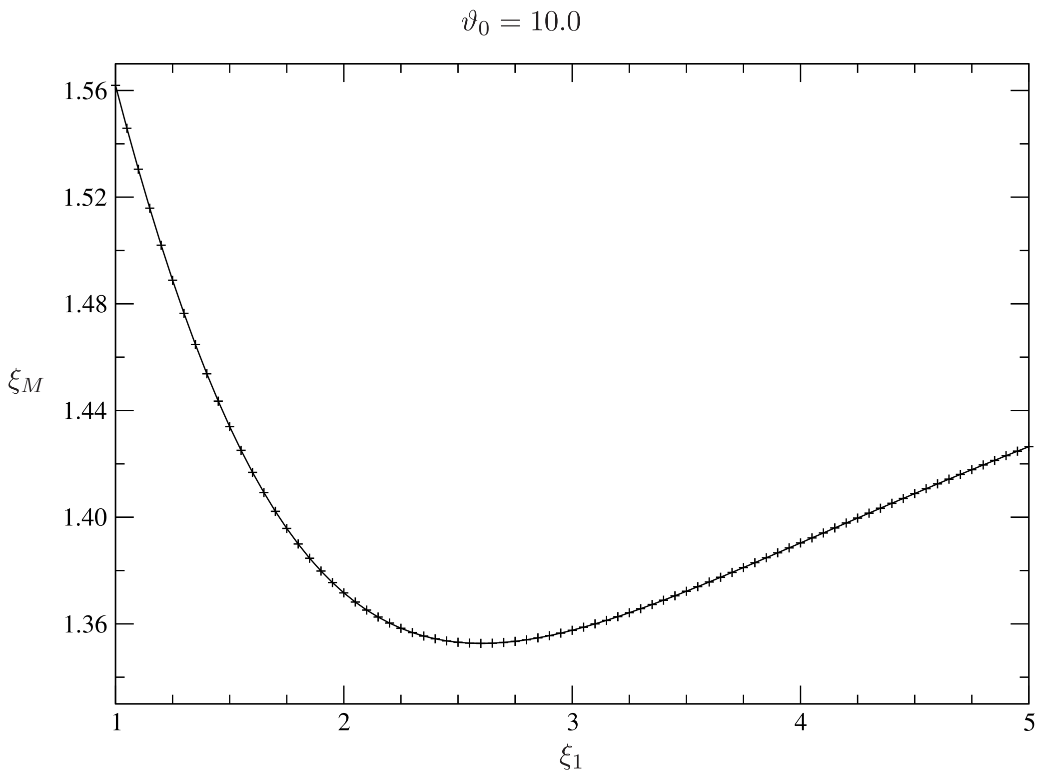

Figure 4: Graph of the energy of the bound system as a function of ξ_1, for a fixed energy of the unbound system, given by $\vartheta_0 = 10$, and with ξ_1 in $[1, 5]$.

There are two main numerical tasks here, the calculation of the integral and the resolution of this algebraic equation for ξ_M. The integral can be readily and efficiently calculated by a cubic interpolation method, using the values of the integrand and of its derivative at the two ends of each integration interval. So long as we can return the value of the integral without too much trouble, Equation (78) can be readily and efficiently solved by an exponential sandwich (or bisection) method [11]. There are two readily available and robust initial upper and lower bounds for the value of ξ_M, the minimum possible lower bound being zero, and the maximum possible upper bound being the energy of the unbound system, since we must have that $Mc^2 < M_U c^2$, which in terms of the dimensionless parameters translates as $\xi_M < \vartheta_0/3$. We may therefore start the process with a lower bound $\xi_{M\ominus} = 0$ and an upper bound $\xi_{M\oplus} = \vartheta_0/3$ for ξ_M. In practice, the efficiency of this algorithm may be highly dependent on the use of a tighter pair of initial bounds.

A few examples of the functions obtained in this way can be seen in Figures 2 through 5, which show ξ_M as a function of ξ_1, for fixed values of the energy of the unbound system, that is, for various fixed values of ϑ_0. Each graph consists of 81 data points. In order to ensure good numerical precision we used 10^6 integration intervals in the domain $[\xi_1, \xi_2]$. The exponential sandwich was iterated until a relative precision of the order of 10^{-12} was reached. The four graphs shown were generated on a high-end PC in approximately 25 hours, 15 hours, 62 hours and 154 hours, respectively, without too much preoccupation with efficiency. As one can see, the graphs clearly display minima of ξ_M, which are located at certain values of ξ_1. At these minima the pairs of values (ξ_1, ξ_2) are given approximately, in each case, by $(2.79, 2.87)$, $(2.72, 2.93)$, $(2.60, 3.02)$ and $(2.35, 3.21)$, respectively.

The minima of these functions give us the value of ξ_1 that corresponds to the most

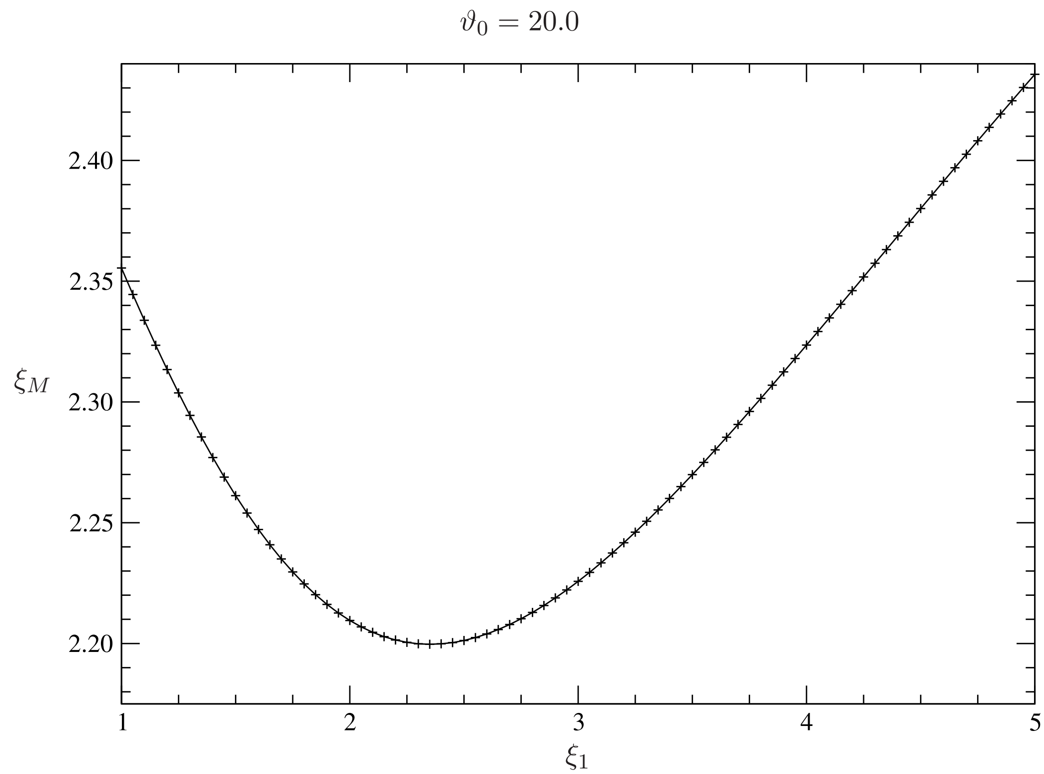

Figure 5: Graph of the energy of the bound system as a function of ξ_1, for a fixed energy of the unbound system, given by $\vartheta_0 = 20$, and with ξ_1 in $[1, 5]$.

tightly bound system that can be assembled from the given unbound system in each case. With the given values of ρ_0 and $M_U c^2$, in each case this establishes the value of r_1 for the most tightly bound and therefore energetically stable solution, and hence determines the values of r_2, r_M and of all the functions describing both the spacetime geometry and the state of the matter for that stable solution. The limiting value of ξ_M when $\xi_1 \to 0$, not shown in these graphs, corresponds to the interior Schwarzschild solution and thus to the energy of the unbound system in each case, which in terms of the variables shown in the graphs is given by $\vartheta_0/3$. The $\xi_1 \to \infty$ limit to the other side rises fairly slowly and does not seem to approach this same value asymptotically, a situation that is probably due to the fact that an infinitesimally thin shell at infinity still has some binding energy, as compared to the corresponding set of isolated infinitesimal masses of point particles.

6 Conclusions

In this paper we have established the energetic interpretation of the exact and quasi-exact solutions obtained in previous papers for spherically symmetric shells of liquid and gaseous fluids [2, 3]. All the energies involved were precisely characterized, including the total energies of the unbound systems, the total energies of the bound systems, the gravitational binding energies, and the thermal and rest energies associated to the matter. This led to a characterization of the stability of the bound systems in terms of their binding energies. In the case of liquid fluids we have identified a two-parameter family of energetically stable solutions, within the original three-parameter family of solutions. In a few cases these stable solutions were identified numerically. It is to be expected that the interpretations of the energies that were introduced here will be useful in other cases, such as those involving polytropes, white dwarfs and neutron stars.

In order to accomplish this analysis, integral expressions for all the energies involved were presented, as integrals of the matter energy density over various coordinate volumes. All these expressions hold in general, for both liquid and gaseous fluids. A particular radial position r_z within the matter region, at which we have $\lambda(r_z) = 0$ and therefore $\exp[\lambda(r_z)] = 1$ for the radial coefficient of the metric, was identified as playing a special role in relation to the integral expressions for the various energies. This is the single finite radial position where the three-dimensional space is neither stretched nor contracted, as compared to the behavior of the radial coordinate r. The existence of an inner region where the three-dimensional space is contracted rather than stretched is a new feature, characteristic of the shell solutions, and absent from the other known solutions.

The energetic interpretation was extended to the case of the two-parameter family of interior Schwarzschild solutions for filled spheres [7,8], which can be obtained as a particular limit of the liquid shell solutions, and which turn out to be maximally unstable ones. This means that there is a strong tendency of the solution for a filled liquid sphere to spontaneously generate an internal vacuum region and thus become a liquid shell solution. This is clearly connected to the repulsive character of the gravitational field around the origin, in the case of the shell solutions, pushing matter and energy away from that origin, as was discussed and characterized in the previous paper [2]. Any small perturbation of the interior Schwarzschild solution will trigger this mechanism into action, thus leading to an energetic decay from that filled sphere solution to a stable shell solution.

The crucial development leading to all this was the introduction of the parameter r_μ in the previous papers, which was shown there to be necessarily strictly positive in all cases, for the correct resolution of the differential equations and the satisfaction of the corresponding interface boundary conditions, as implied by the Einstein field equations. The apparently traditional routine of choosing $r_\mu = 0$ in order to eliminate the singularity at the origin not only is often incompatible with the correct resolution of the differential system but, when it is not thus incompatible, it is tantamount to selecting a solution which has no binding energy at all and is therefore maximally unstable from the energetic point of view. Both from the purely mathematical point of view and from the physical point of view, this is more often than not the incorrect choice, which we are simply not at liberty to make in the general case.

Acknowledgments

The author would like to thank his friends Prof. C. E. I. Carneiro and Mr. Rodrigo de A. Orselli for their helpful criticism and careful reading of the manuscript.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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