# Accelerated Coordinate Systems. Fictitious Forces. Foucault Pendulum.

C.H.Furukawa (furukawa@if.usp.br), F.D.Saad (fuad@if.usp.br) and M.Cattani (mcattani@if.usp.br)
Institute of Physics of the University of São Paulo (IFUSP)

**Abstract.** Particles motion is briefly analyzed in inertial and non inertial coordinate systems. Special attention was dedicated to rotating coordinate systems to describe the famous Foucault pendulum. Accelerated rectilinear motion was also briefly described.
***Key words***: *inertial and non inertial coordinate systems; Foucault pendulum.*

## (I)Introduction.

Our analysis is performed in the Galilean limit, that is, when particles velocities are much smaller than the light velocity. A reference system of coordinates is called **inertial frame of reference** when it is not undergoing acceleration.[1] In this system, a physical object with zero net force acting on it moves with a constant velocity or, in other words, it is a frame in which Newton's first law of motion holds. All inertial frames are in a state of constant, rectilinear motion with respect to one another; in other words, an accelerometer moving with any of them would detect zero acceleration. Measurements in one inertial frame can be converted to measurements in another by a simple transformation. In Classical Physics in a **non-inertial reference frame** the physics vary depending on the acceleration of that frame with respect to an inertial frame, and the usual physical forces must be supplemented by *fictitious forces*.[1] For example, a ball dropped towards the ground does not go exactly straight down because the Earth is rotating, which means the frame of reference of an observer on Earth is not inertial. As will be seen in what follows, the physics must account for the *Coriolis effect* (in this case thought as a force) to predict the horizontal motion. Another example of such a fictitious force associated with rotating reference frames is the centrifugal effect, or centrifugal force. In **Section 1** is considered the general case of relative accelerated frames. In **Section 2** are analyzed rotating coordinate systems showing *inertial forces*, that is, *centrifugal* and *Coriolis forces.* In **Section 3** is seen the Harmonic Oscillator motion in inertial and non-inertial frames. In **Section 4** is analyzed the motion of the famous Foucault Pendulum. In **Appendix** was studied the harmonic motion in an accelerated referential system.

## (1)Accelerated Coordinate Systems.

Let us consider two coordinate systems S and S* with origins O and O*, respectively. The system with origin O is taken as *fixed in space* .[1] The distance of the origin O* from O is given by **h**. The relation between the *rectangular* coordinates **r** and **r*** is given by

$$\mathbf{r} = \mathbf{r}^* + \mathbf{h} \qquad \text{and} \qquad \mathbf{r}^* = \mathbf{r} - \mathbf{h} \qquad \textbf{(1.1).}$$

Taking the origin O as fixed and O* moving with respect to O we get, differentiating **r** ,given by **Eq.(1,1)**

$$\mathbf{v} = d\mathbf{r}/dt = d\mathbf{r}^*/dt + d\mathbf{h}/dt = \mathbf{v}^* + \mathbf{v}_h \qquad \textbf{(1.2),}$$

where **v** and **v*** are the velocities of the moving point relative to O and O*, respectively, and $\mathbf{v}_h$ the velocity of O* relative to O. Noting that the coordinate system S* is obtained by a *translation* of the S. The relation between relative accelerations is

$$\mathbf{a} = d^2\mathbf{r}/dt^2 = d^2\mathbf{r}^*/dt^2 + d^2\mathbf{h}/dt^2 = \mathbf{a}^* + \mathbf{a}_h \qquad \textbf{(1.3).}$$

Newton´s equations of motion, in the fixed coordinate system S, for a particle with mass m subjected to a force **F** is given by

$$m\,\mathbf{a} = m\, d^2\mathbf{r}/dt^2 = \mathbf{F} \qquad \textbf{(1.4).}$$

Using Eqs.(1.3) and (1.4) we can obtain

$$m\, d^2\mathbf{r}^*/dt^2 + m\,\mathbf{a}_h = \mathbf{F} \qquad \textbf{(1.5).}$$

If O* is moving with a constant velocity relative to O, then $\mathbf{a}_h = 0$, and we have,

$$m\, d^2\mathbf{r}^*/dt^2 = \mathbf{F} \qquad \textbf{(1.6).}$$

Thus, Newton´s equations of motion, if they hold in any coordinate system, hold also in any other coordinate system moving with constant velocity relative to the first. This is the **Newtonian Principle of Relative.[1]**

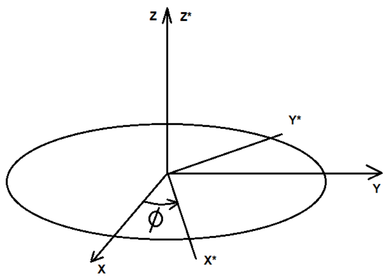
For any motion of O* we can write **Eq.(1.5)** in the form

$$m\, d^2\mathbf{r}^*/dt^2 = \mathbf{F} - m\,\mathbf{a}_h \qquad \textbf{(1.7),}$$

where the term $- m\,\mathbf{a}_h$ can be interpreted (*called*) as a **fictitious force.** We can treat the motion of a mass m relative to a moving coordinate system using Newton´s equations of motion if we add this fictitious force to the actual force which acts. From the classical mechanics point of view, it is not a force at all which is defined by **F**.

## (2)Rotating Coordinate Systems.

Let us consider two coordinate systems, S(x,y,z)(fixed ) and S*(x*,y*,z*), with coincident origins O and O*, whose axes are rotated relative to one another. Let us now suppose that S* is rotating around the z-axis through the origin O, with angular velocity ω (see **Figure(2.1)**).




**Figure(2.1).** S*system rotating around the z-axis of the inertial system S. The coordinates (x,y) and (x*,y*) axes are co-planar.

We define the *vector angular velocity* **ω** as a vector of magnitude ω along the z-axis in the direction of advance of a right-hand screw rotating with S* system.[1]

So, when a vector **B** is at rest in S*, its starred derivative $d\mathbf{B}^*/dt = 0$ and it can be shown that its unstarred derivative is given by

$$d\mathbf{B}/dt = \boldsymbol{\omega} \times \mathbf{B} \qquad \textbf{(2.2).}$$

Taking **ω = constant** it can be also shown[1] that for any vector **A** and **A*** we have**,**

$$d\mathbf{A}/dt = d\mathbf{A}^*/dt + \boldsymbol{\omega} \times \mathbf{B} \qquad \textbf{(2.3),}$$

which is the fundamental relationship between time derivatives for rotating systems.[1]

For velocities $\mathbf{v} = \mathbf{dr}/dt$, $\mathbf{v^*} = \mathbf{dr^*}/dt$ and accelerations $\mathbf{a} = \mathbf{dv}/dt$ and $\mathbf{a^*} = \mathbf{dv^*}/dt$ are satisfied the following equations,[1] remembering that $\mathbf{a}_h = 0$ since O ≡ O*,

$$\mathbf{dr}/dt = \mathbf{dr^*}/dt + \boldsymbol{\omega} \times \mathbf{r} \qquad \textbf{(2.4),}$$

$$\mathbf{d^2r}/dt^2 = \mathbf{d^2r^*}/dt^2 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (\mathbf{dr^*}/dt) \qquad \textbf{(2.5).}$$

**Eq. (2.5)** is called **Coriolis´Theorem**.

If we assume that Newton´s law of motion $\mathbf{F} = m\mathbf{a} = m\,\mathbf{d^2r}/dt^2$ holds in S, we have in S*[1]

$$m\,\mathbf{d^2r^*}/dt^2 + m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times (\mathbf{dr^*}/dt) = \mathbf{F} \qquad \textbf{(2.6).}$$

Transposing the second and third terms to the right side, we obtain an equation of motion similar in form to Newton´s equation of motion:

$$m\,\mathbf{d^2r^*}/dt^2 = \mathbf{F} - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times (\mathbf{dr^*}/dt) \qquad \textbf{(2.7).}$$

The second term on the right is called ***centrifugal force*** (that is, "away from the center") and the third term is called ***Coriolis force.*** Note that **F** is a **real force** and another ones are **"ficticious"** forces.

## (3)Harmonic Oscillator in Inertial and Non-Inertial Frames.

Let us consider now a 3-dim inertial Cartesian system S ("fixed in space") and a non-inertial Cartesian system S*(accelerated relatively to S). They have the same origins O and O* and the same z an z* axis (see **Fig.1**). S* is rotating around the vertical z-axis with constant angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{k}$.

Let us assume that in S ("fixed") we have an harmonic oscillator, with mass m and frequency $\psi$, vibrating along the x-axis, obeying the equations

$$\mathbf{x(t)} = A\cos(\psi t)\,\mathbf{i} \quad \text{and} \quad \mathbf{v(t)} = -A\psi\sin(\psi t)\,\mathbf{i} \qquad \mathbf{(3.1),}$$

where $\psi = (k/m)^{1/2}$, k is the spring elastic constant and the period $T = 2\pi/\psi$.

Note that the spring moves along the x-axis in the fixed (x,y) plane and that coordinates (x,y) and (x*,y*) axes are co-planar.

As S* is rotating around the vertical axis with angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{k}$, we have $d\mathbf{r^*}/dt = d\mathbf{r}/dt + \boldsymbol{\omega} \times \mathbf{r}$ (**Eq.(2.4)**). When $\boldsymbol{\omega} \times \mathbf{r}$ is very small we have $d\mathbf{r^*}/dt \approx d\mathbf{r}/dt$ and

$$\mathbf{v^*}(t) = v(t)\cos(\omega t)\,\mathbf{i^*} - v(t)\sin(\omega t)\,\mathbf{j^*} = A\psi\sin(\psi t)[\cos(\omega t)\,\mathbf{i^*} - \sin(\omega t)\,\mathbf{j^*}] \qquad (3.2),$$

taking into account that at for t = 0 we have v(0) = 0.

According **Eqs.(3.1),** in S the particle moves harmonically along the x-axis in the plane (x,z). On the other hand, in S*, from Eq.(3.2), we verify that its motion is along the axis x* and y* which are performing rotations around the z-axis.

## (4)The Foucault Pendulum.

A very interesting application of the theory of rotating coordinate systems is the famous problem of the **Foucault Pendulum**. In **Figure (4.1)** is seen the inertial coordinate system S(x,y,z) with the Earth center at the origin O. The Earth is rotating with angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$ around the north-south **z-axis**

The pendulum is in the system S*(x*,y*,z*) (inside a "box")fixed in an horizontal plane of the Earth (see **Fig.(4.1)**).The pendulum with length $\ell$, fixed at the point P along the **z*-axis,** has a ball with mass m. This mass is hanging from a string arranged to swing freely in the vertical plane inside the "box" frame fixed in the Earth northern hemisphere. The **z*-axis** forms an angle $\theta$ ("colatitude") with **z-axis** of S.

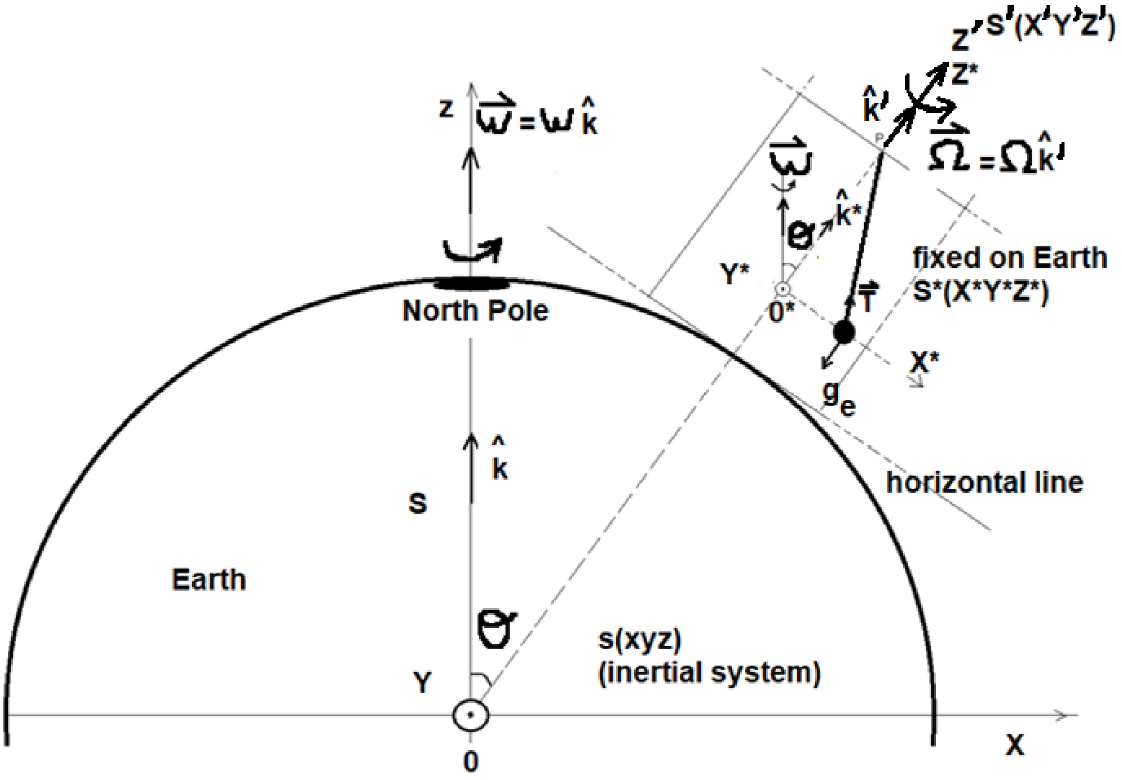



**Figure (4.1).** Rotating Earth with angular velocity **ω** around the z-axis. The pendulum is shown inside the non-inertial system S* fixed in the Earth. Due to the Coriolis force it will rotate around the k′- axis of S′, as will be shown below.

Indicating by **T** the tension in the string we can verify, following **Eq.(2.7)**,that the equation of motion of the ball in the S* system becomes given by,[1]

$$m d^2\mathbf{r}^*/dt^2 = \mathbf{T} + m\,\mathbf{g}_e - 2m\,\boldsymbol{\omega} \times (d\mathbf{r}^*/dt) \qquad \mathbf{(4.1)},$$

putting $\mathbf{g}_e = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \approx \mathbf{g}$.[1] If the Coriolis force were not present, **Eq.(4.1)** would be the equation for a simple pendulum on a non rotating Earth.

According to **Eq.(4.1),** The horizontal component of this force is perpendicular to the velocity $d\mathbf{r}^*/dt$. Thus, it will be impossible to the pendulum to continue to swing in the fixed vertical plane $(x^*,z^*)$. In order to solve this problem including the Coriolis term, is used the experimental result as a clue.[1] That is, is introduced a new coordinate system $S'(x',y',z')$. The z′-axis would be coincident with the z*-axis and the S′ would have a rotation around this axis described by a constant angular velocity $\boldsymbol{\Omega} = \Omega\,\mathbf{k}'$.

In these conditions we have

$$m\, d^2\mathbf{r}^*/dt^2 = \mathbf{T} + m\mathbf{g} - 2m\boldsymbol{\omega} \times (d\mathbf{r}^*/dt). \qquad \mathbf{(4.2).}$$

As $\omega \approx 7.5\ 10^{-5}$ rad/s, for velocities $dr^*/dt < 8$ Km/h the Coriolis force is very small, less than 0.1% of the gravitational force. Since the Coriolis force $2m\boldsymbol{\omega} \times (d\mathbf{r}^*/dt)$ is perpendicular to the oscillation "box plane" $(x^*,z^*)$, will be impossible for the pendulum to swing in this fixed vertical plane.[1] This force would be responsible by a ball rotation with angular velocity around the **z*-axis.** As can be seen from Figure **(4.1)** the angular velocity Ω would be given by [1]

$$\boldsymbol{\Omega} = \Omega\, \mathbf{k}^* = -\,\omega \cos\theta\, \mathbf{k}^* \qquad \textbf{(4.3),}$$

where ω is the angular velocity rotation of the Earth and θ the colatitude angle.

So, Ω is the angular velocity of the **precessing** S´ **system (**"box frame" **)** relative to the earth and θ is the angle between **k*** and **k´** axis, shown in **Figure(4.1).** The ball rotates, describing a circle in the (x*,y*) plane, around the z*-axis with the tangential velocity **v*= ω x r*.**[2]

At the **equator**, that is, when $\theta = \pi/2 \rightarrow \cos\theta = 0$, we have $\Omega = 0$. That is, there is no pendulum precession: the swinging "frame" S* remains fixed.

At the **north or south pole** when $\theta = \pm\,\pi$ we have $\Omega = \pm\,\omega$. That is, the pendulum swings in a vertical plane fixed in space while the earth turns beneath it. In our laboratory the case when the pendulum is hanged at the **north pole** of motion can be exemplified taking into account the apparatus seen in **Figs.(4.2 and 4.3)**. When the pendulum is initially put to oscillate along the support plane and the platform (which is attached to the circular support) is put to rotate we verify that the pendulum continue to swing fixed in space along the initial position of the platform plane.

**Figure (4.2).** Pendulum hanged at the north pole of the circular support oscillating in its plane which is rotating around the z*-axis.

In **Figure (4.3)** are shown the pendulum and circular support positions after 180° angular rotation of the platform.

**Figure (4.3).** Pendulum and circular support positions after 180º angular rotation of the platform.

If φ is the Earth **latitude angle** we have Ω = 360º sinφ/day [1], where latitudes north and south of the equator are defined as positive and negative, respectively. A "pendulum day" is the time needed for the plane of a freely suspended Foucault pendulum to complete an apparent rotation about the local vertical. For example, a Foucault pendulum at 30° south latitude rotates counterclockwise 360° in two days.

Using enough wire length, the described circle can be wide enough that the tangential displacement along the measuring circle of between two oscillations can be visible by eye, rendering the Foucault pendulum a spectacular experiment: for example, the original Foucault pendulum in Paris Observatory (France) moves circularly, with a heavy ball with m = 28 Kg, ℓ = 67 m and 6 m pendulum amplitude, by about 5mm each period.[1] In reference [1] is also shown the video of the Foucault pendulum in motion at **COSI** (**Center of Science and Industry**)which is a science museum and research center in Columbus (Ohio, University).

## Appendix. Harmonic Oscillation in an Accelerated System.

Let us consider an harmonic oscillator, with mass m and elastic constant k, inside a wagon train that moves with velocity much smaller than the light velocity, but with acceleration $a_h$ relative to a station**.** The station is fixed in an inertial system S and the oscillator is fixed in an accelerated non-inertial system S*. Thus, according to Eq.(1.7), in S* the motion described by

$$m\, d^2\ell^*/dt^2 = F - m\, a_h \qquad \text{(A.1)},$$

where $\ell^*$ is the spring compression in S* and $F = -k\ell + ma$ is the force observed in S. Since $\ell = \ell^*$ and $a = a_h$ we verify that

$$m\, d^2\ell^*/dt^2 = -k\, \ell^* \qquad \text{(A.2)}.$$

This implies that, as the frequency $\omega = (k/m)^{1/2}$ it is not modified by the acceleration, time intervals measured in inertial and non-inertial systems are equal. Remember that, according to the Theory of Relativity, time intervals in inertial and non-inertial systems can be different for velocities close to the light velocity.[3,4]

## REFERENCES

[1]Keith R.Symon."Mechanics". Addison-Wesley(1953).
[2]https://en.wikipedia.org/wiki/Foucault_pendulum
[3]R.M.Eisberg."Fundamentals of Modern Physics" John Wiley & Sons(1961).
[4]M.Cattani. "Comments On Topics of General Relativity and Special Relativity". http://publica-sbi.if.usp.br/PDFs/pd1711 (07/**2018**)

**Acknowledgements.** *The authors thank the librarian Virginia de Paiva for the invaluable assistance for publication of this paper.*