Non-Abelian gauge theories with composite fields in the background field method

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Abstract

Non-Abelian gauge theories with composite fields are examined in the background field method. Generating functionals of Green's functions for a Yang–Mills theory with composite and background fields are introduced, including the generating functional of vertex Green's functions (effective action). The corresponding Ward identities are obtained, and the issue of gauge dependence is investigated. A gauge variation of the effective action is found in terms of a nilpotent operator depending on the composite and background fields. On-shell independence from the choice of gauge fixing for the effective action is established. In the study of the Ward identities and gauge dependence, finite field-dependent BRST transformations with a background field are introduced and employed on a systematic basis. On the one hand, this involves the consideration of (modified) Ward identities with a field-dependent anticommuting parameter, also depending on a non-trivial background. On the other hand, the issue of gauge dependence is studied with reference to a finite variation of the gauge Fermion. The concept of a joint introduction of composite and background fields to non-Abelian gauge theories is exemplified by the Gribov–Zwanziger theory, including the case of a local BRST-invariant horizon, and also by the Volovich–Katanaev model of two-dimensional gravity with dynamical torsion.

1 Introduction

The use of background [1, 2, 3] and composite [5, 4] fields has gained considerable attention in quantum field theory. The background field method [1, 2, 3] reformulates the quantization of Yang–Mills theories under a background gauge condition [3, 6, 7], in a manner which provides an effective action invariant under the gauge transformations of background fields and reproduces physical results with considerable simplifications in calculating the Feynman diagrams, which allows one to study a wide range of quantum properties in gauge theories [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]; see also [18, 19, 20, 21, 22] for recent developments. The interest in composite fields is due to the fact that the effective action for composite fields (see [4] for a review) introduced in [5] has found diverse applications to quantum

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field models such as [23, 24], including the early Universe, the inflationary Universe, the Standard Model and SUSY theories [25, 26, 27, 28]. It seems to be of particular interest in this regard to study the problem of BRST-invariant renormalizability in Yang–Mills theories, which includes the $N=1$ SUSY formulations [29, 30, 31], the functional renormalization group [32, 33, 34, 35, 36], and the Gribov horizon [37, 38], by using the concept of soft BRST symmetry [39, 40, 41, 50, 52] and the local composite operator technique [57, 58] in arbitrary backgrounds [59].

In this paper, we address the issue of quantum non-Abelian gauge theories including both composite and background fields, with a systematic treatment based on Yang–Mills theories quantized using the Faddeev–Popov method [60] in the presence of both composite and background fields. A combined treatment of Yang–Mills fields A_μ with composite and background ones calls for a joint introduction of these ingredients on a systematic basis. We suggest to use the symmetry principle as such a systematic guideline. Thus, suppose that a generating functional $Z(J,L)$ of Green's functions with composite fields is given, depending on sources J_A for the usual fields ϕ^A, and also on sources L_m for the composite fields $\sigma^m(\phi)$. One may ask how some background fields B_μ can be introduced in such a way as to produce an extended functional $Z(B,J,L)$ which reflects the symmetries inherent in $Z(J,L)$. Suppose, on the other hand, that a generating functional $Z(B,J)$ of Green's functions in the background field method is given, also featuring some symmetries, and then one may ask how composite fields $\sigma^m(\phi,B)$ with sources L_m can be introduced for the resulting $Z(B,J,L)$ to inherit the original symmetries. These two approaches prove to be equivalent in the sense outlined in the following preliminary exposition.

In the first approach, one is given a generating functional $Z(J,L)$,

$$Z(J,L)=\int d\phi\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}(\phi)+J_A\phi^A+L_m\sigma^m(\phi)\right]\right\}, \tag{1.1}$$

corresponding to the Faddeev–Popov action $S_{\mathrm{FP}}(\phi)$ of a Yang–Mills theory with composite fields $\sigma^m(\phi)$. Then a background field B_μ can be introduced by *localizing* the inherent global symmetry of $Z(J,L)$ under $SU(N)$ transformations ("rotations" for J_A and tensor transformations for L_m) in such a way that $Z(B,J,L)$ defined as[1]

$$Z(B,J,L)=\left.Z(J,L)\right|_{\partial_\mu\to D_\mu(B)} \tag{1.2}$$

turns out to be invariant under local $SU(N)$ transformations of the sources J_A, L_m, accompanied by gauge transformations of the field B_μ with an associated covariant derivative $D_\mu(B)$. Besides, the original $S_{\mathrm{FP}}(\phi)$ becomes modified to the Faddeev–Popov action $S_{\mathrm{FP}}(\phi,B)$ of the background field method and related to $S_{\mathrm{FP}}(\phi)$ by so-called *background* and *quantum* transformations of this method (see Section 2).

In the second approach, one is given a generating functional $Z(B,J)$ constructed using the background field method for Yang–Mills theories,

$$Z(B,J)=\int d\phi\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}(\phi,B)+J_A\phi^A\right]\right\}, \tag{1.3}$$

which implies that $S_{\mathrm{FP}}(\phi,B)$ is given by

$$S_{\mathrm{FP}}(\phi,B)=\left.S_{\mathrm{FP}}(\phi)\right|_{\partial_\mu\to D_\mu(B)}\ .$$

[1] More specifically, the replacement $\partial_\mu\to D_\mu(B)$ is described by the relations (2.16), (2.17) of Section 2. This rule is unambiguous for local fields σ^m without higher derivatives, $\sigma^m=\sigma^m(\phi,\partial\phi)$. For more details, see Section 2.

One can then introduce some composite fields $\sigma^m(\phi, B)$ on condition that the resulting generating functional

$$Z(B, J, L) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\rm FP}(\phi, B) + J_A\phi^A + L_m\sigma^m(\phi, B)\right]\right\} \tag{1.4}$$

inherit the symmetry of $Z(B, J)$ under local $SU(N)$ rotations of the sources J_A accompanied by gauge transformations of the background field B_μ with the covariant derivative $D_\mu(B)$. This symmetry condition for $Z(B, J, L)$ is met by a local $SU(N)$ tensor transformation law imposed on $\sigma^m(\phi, B)$, which is provided by B_μ entering into the composite fields $\sigma^m(\phi, B)$ via the covariant derivatives $D_\mu(B)$ and implies

$$\sigma^m(\phi, B) = \sigma^m(\phi)\big|_{\partial_\mu \to D_\mu(B)} \tag{1.5}$$

for certain $\sigma^m(\phi)$, which brings us back to the first approach.

In the main part of the present work, we choose the first approach as a starting point of our systematic treatment assuming the composite fields to be *local*, and then, in the remaining part, we show how the first and second approaches can be extended beyond the given assumptions by considering some examples. The principal research issues to be addressed are as follows:

1. Introduction of generating functionals of Green's functions with composite and background fields in Yang–Mills theories; investigation of the related symmetry properties;

2. Extension of finite field-dependent BRST (FD BRST) transformations [42, 40] to the case of background field dependence;

3. Investigation of the Ward identities and gauge dependence for the above generating functionals on a basis of finite FD BRST transformations;

4. Introduction of background and composite fields field into the Gribov–Zwanziger theory [38];

5. Introduction of composite fields into a quantized Volovich–Katanaev model [61].

The paper is organized as follows. In Section 2, we introduce a generating functional of Green's functions with composite and background fields in Yang–Mills theories. Section 3 is devoted to the corresponding Ward identities and the properties of the generating functional of vertex Green's functions (effective action). Thus, the effective action $\Gamma_{\rm eff}(B, \Sigma)$, depending on the background field B_μ and a set of tensor auxiliary fields Σ^m associated with σ^m, is found to exhibit a gauge symmetry under the gauge transformations of B_μ along with the local $SU(N)$ transformations of Σ^m. The study of the Ward identities systematically utilizes the concept of finite FD BRST transformations [42, 40], first suggested[2] in [43, 44] and now depending also on the background field B_μ. In Section 4, we use the finite FD BRST transformations and the related (modified) Ward identities to analyze the dependence of the generating functionals of Green's functions upon a choice of gauge-fixing. In doing so, we evaluate a finite gauge variation of the effective action in terms of a nilpotent operator depending on the composite and background fields, and also determine the conditions of on-shell gauge-independence. Loop expansion properties and a one-loop effective action with composite and background fields are examined in Section 5. In Section 6, we consider an example of the Gribov–Zwanziger theory [38], which is a quantum Yang–Mills theory including the Gribov horizon [37], using a local and a non-local BRST-invariant representations, in terms of composite fields. The quantum theory [38] is then extended by introducing a background field, which modifies our first approach

[2]See [51] for a field-antifield BV formalism and [45, 46, 47, 48, 49] for extended $N \geq 2$ BRST symmetries.

(1.1), (1.2) beyond the case of local composite fields, along with a study of the gauge-independence problem. In Section 7, we consider an example of the two-dimensional gravity with dynamical torsion by Volovich and Katanaev [62], quantized according to the background field method in [61] and featuring a gauge-invariant background effective action. As a modification of our second approach (1.3), (1.4), (1.5), when extended beyond the Yang–Mills case, the quantized two-dimensional gravity [61] is generalized to the presence of composite fields, and the corresponding effective action with composite and background fields is found to be gauge-invariant, in a way similar to the Yang–Mills case. Section 8 presents a summary of our results. Appendices A, B, C support the consideration of the respective Yang–Mills, Gribov–Zwanziger and Volovich–Katanaev models.

We use DeWitt's condensed notation [63]. The Grassmann parity and ghost number of a quantity F are denoted by $\epsilon(F)$, $\mathrm{gh}(F)$, respectively, and $[F, G\}$ stands for the supercommutator of any quantities F, G with a definite Grassmann parity, $[F, G\} = FG - (-1)^{\epsilon(F)\epsilon(G)} GF$. Unless specifically indicated by an arrow, derivatives with respect to fields and sources are understood as left-hand ones.

2 Generating Functional of Green's Functions

Consider a generating functional $Z(J, L)$ corresponding to the Faddeev–Popov action $S_{\mathrm{FP}}(\phi)$ of a Yang–Mills theory with local composite fields,

$$Z(J, L) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}(\phi) + J_A\phi^A + L_m\sigma^m(\phi)\right]\right\}, \tag{2.1}$$

where L_m are sources to the composite fields $\sigma^m(\phi)$,

$$\sigma^m(\phi) = \sum_{n=2} \frac{1}{n!}\Lambda^m_{A_1\ldots A_n}\phi^{A_n}\ldots\phi^{A_1}\ , \tag{2.2}$$

and J_A are sources to the fields $\phi^A = (A^i, b^\alpha, \bar{c}^\alpha, c^\alpha)$ composed by gauge fields A^i, (anti)ghost fields $\bar{c}^\alpha$, c^α, and Nakanishi–Lautrup fields b^α, with the following distribution of Grassmann parity and ghost number:

	ϕ^A	J_A	σ^m	L_m
ϵ	$(0,0,1,1)$	$\epsilon\left(\phi^A\right)$	$\epsilon(\sigma^m)$	$\epsilon(\sigma^m)$
gh	$(0,0,-1,1)$	$-\mathrm{gh}\left(\phi^A\right)$	$\mathrm{gh}(\sigma^m)$	$-\mathrm{gh}(\sigma^m)$

The Faddeev–Popov action $S_{\mathrm{FP}}(\phi)$,

$$S_{\mathrm{FP}}(\phi) = S_0(A) + \Psi(\phi)\overleftarrow{s},$$

is given in terms of a gauge-invariant classical action $S_0(A)$, invariant, $\delta S_0(A) = 0$, under infinitesimal gauge transformations $\delta A^i = R^i_\alpha(A)\xi^\alpha$ with a closed algebra of gauge generators $R^i_\alpha(A)$,

$$R^i_{\alpha\, j}(A)\, R^j_\beta(A) - R^i_{\beta\, j}(A)\, R^j_\alpha(A) = F^\gamma_{\alpha\beta}R^i_\gamma(A)\,, \quad F^\gamma_{\alpha\beta} = \mathrm{const}, \quad R^i_{\alpha,\, j} \equiv R^i_\alpha\frac{\overleftarrow{\delta}}{\delta A^j}\ ,$$

and a nilpotent Slavnov variation $\overleftarrow{s}$ applied to a gauge Fermion $\Psi(\phi)$, $\epsilon(\Psi) = -\mathrm{gh}(\Psi) = 1$,

$$S_{\mathrm{FP}}(\phi) = S_0(A) + \Psi(\phi)\overleftarrow{s}, \quad \Psi(\phi) = \bar{c}^\alpha\chi_\alpha(\phi)\,, \ \overleftarrow{s}^{\,2} = 0, \tag{2.3}$$

where

$$\phi^A\overleftarrow{s} = \left(R^i_\alpha(A)\, c^\alpha, 0, b^\alpha, 1/2F^\alpha_{\beta\gamma}c^\gamma c^\beta\right).$$

For the explicit field content

$$
\begin{aligned}
i &= (x,p,\mu), \quad \alpha = (x,p)\,, \quad \mu = 0,\dots,D-1, \quad p = 1,\dots,N^2-1, \\
\phi^A &= \left(A^{p|\mu}, b^p, \bar{c}^p, c^p\right), \quad \left(A^\mu, b, \bar{c}, c\right) \equiv T^p \left(A^{p|\mu}, b^p, \bar{c}^p, c^p\right), \quad [T^p, T^q] = f^{pqr} T^r,
\end{aligned}
$$

the field variations $\phi^A \overleftarrow{s}$ have the form

$$
\begin{aligned}
\left(A_\mu, b, \bar{c}, c\right) \overleftarrow{s} &= \left(\left[D_\mu(A), c\right], 0, b, g/2\,[c,c]_+\right), \\
\left(A^p_\mu, b^p, \bar{c}^p, c^p\right) \overleftarrow{s} &= \left(D^{pq}_\mu(A)\, c^q, 0, b^p, g/2 f^{pqr} c^q c^r\right),
\end{aligned} \tag{2.4}
$$

where

$$
D_\mu(A) \equiv \partial_\mu + g A_\mu\,, \quad D^{pq}_\mu(A) = \delta^{pq}\partial_\mu + g f^{prq} A^r_\mu\,,
$$

The classical action $S_0(A)$ has the form (in the adjoint representation with Hermitian T^p),

$$
\begin{aligned}
S_0(A) &= \frac{1}{2g^2}\int d^D x\ \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = \frac{1}{4}\int d^D x\ F^p_{\mu\nu}F^{p|\mu\nu}, \quad \mathrm{Tr}\left(T^p T^q\right) = \frac{1}{2}\delta^{pq}, \\
F_{\mu\nu} &\equiv \left[D_\mu(A), D_\nu(A)\right], \quad F^p_{\mu\nu} = \partial_\mu A^p_\nu - \partial_\nu A^p_\mu + g f^{prs} A^r_\mu A^s_\nu\,,
\end{aligned} \tag{2.5}
$$

and the gauge Fermion $\Psi(\phi) = \bar{c}^\alpha \chi_\alpha(\phi)$ with gauge-fixing functions $\chi_\alpha(\phi) = \chi^p(\phi(x))$ reads

$$
\Psi(\phi) = \int d^D x\ \bar{c}^p \chi^p(\phi) = 2\int d^D x\ \mathrm{Tr}\left[\bar{c}\chi(\phi)\right], \quad \chi(\phi) = T^p\chi^p(\phi)\,. \tag{2.6}
$$

The Faddeev–Popov action $S_{\mathrm{FP}}(\phi)$ is invariant under two kinds of global transformations: BRST transformations [64, 65], $\delta_\lambda \phi^A = \phi^A \overleftarrow{s}\lambda$, with an anticommuting parameter λ, $\epsilon(\lambda) = \mathrm{gh}(\lambda) = 1$, and $SU(N)$ rotations (finite $\phi^A \xrightarrow{U} \phi'^A$ and infinitesimal $\delta_\varsigma \phi^A$) with even parameters ς^p,

$$
\begin{aligned}
\left(A_\mu, b, \bar{c}, c\right) \xrightarrow{U} \left(A_\mu, b, \bar{c}, c\right)' &= U\left(A_\mu, b, \bar{c}, c\right)U^{-1}, \quad U = \exp\left(-gT^p\varsigma^p\right), \quad \varsigma^p = \mathrm{const}, \\
\delta_\varsigma\left(A^p_\mu, b^p, \bar{c}^p, c^p\right) &= g f^{prq}\left(A^r_\mu, b^r, \bar{c}^r, c^r\right)\varsigma^q,
\end{aligned} \tag{2.7}
$$

or, in a tensor form, via the adjoint representation with a matrix $M^{pq}(\varsigma)$,

$$
\left(A^p_\mu, b^p, \bar{c}^p, c^p\right)' = M^{pq}(\varsigma)\left(A^q_\mu, b^q, \bar{c}^q, c^q\right), \quad M^{pq}(\varsigma) = \delta^{pq} + g f^{pqr}\varsigma^r + O\left(\varsigma^2\right).
$$

The classical action $S_0(A)$ in $S_{\mathrm{FP}}(\phi) = S_0(A) + \Psi(\phi)\overleftarrow{s}$ is invariant under $A_\mu \xrightarrow{U} A'_\mu$ as a particular case ($\xi^p(x) = \mathrm{const}$) of invariance under the finite form $A_\mu \xrightarrow{V} A'_\mu$ of gauge transformations

$$
A'_\mu = V A_\mu V^{-1} + g^{-1} V\left(\partial_\mu V^{-1}\right), \quad D_\mu(A') = V D_\mu(A) V^{-1}, \quad V = \exp\left(-gT^p\xi^p\right)\,, \quad \xi^p = \xi^p(x)\,, \tag{2.8}
$$

whereas the invariance of $\Psi(\phi)\overleftarrow{s}$ under $\phi^A \xrightarrow{U} \phi'^A$ is implied by the explicit form of $\overleftarrow{s}$ and the fact that the gauge functions $\chi^p(\phi)$ are local and constructed[3] from the fields ϕ^A, structure constants f^{pqr} and derivatives ∂_μ, for instance, in Landau and Feynman gauges,

$$
\chi^p_{\mathrm{L}}(\phi) = \partial^\mu A^p_\mu\,, \quad \chi^p_{\mathrm{F}}(\phi) = b^p + \partial^\mu A^p_\mu\,. \tag{2.9}
$$

[3]Note that $F^{pq} = f^{prq}F^r$, $f^{pqr}F^qG^r = [F,G\}^p$, $\partial_\mu F^p = [\partial_\mu, F]^p$, $F^pG^p \sim \mathrm{Tr}(FG)$, $F_1^{pp_1}F_2^{p_1p_2}\cdots F_n^{p_{n-2}p} = \mathrm{Tr}(F_1\cdots F_n)$ for any quantities F^p and G^p carrying the index p, with $F' = UFU^{-1}$, $G' = UGU^{-1}$. Given this, the explicit form (2.4) of the field variations $\phi^A\overleftarrow{s}$ implies the property $(\chi\overleftarrow{s})' = U(\chi\overleftarrow{s})U^{-1}$.

so that, in particular, $\chi^p(\phi)$ transform as $SU(N)$ vectors, with $\Psi(\phi)$ being invariant under $\phi^A \stackrel{U}{\to} \phi'^A$,

$$\chi(\phi') = U\chi(\phi)U^{-1}, \quad \delta\chi^p(\phi) = gf^{prq}\chi^r(\phi)\varsigma^q.$$

By similar reasonings, one can see that local composite fields $\sigma^m(\phi)$, also constructed from the fields ϕ^A, structure constants f^{pqr} and derivatives ∂_μ,

$$\sigma^m(\phi) = \sigma^{p_1\cdots p_k|\mu_1\cdots\mu_l}(\phi(x)), \quad m = (x, p_1\cdots p_k, \mu_1\cdots\mu_l),$$

in the path integral (2.1) for $Z(J,L)$ transform under $\phi^A \stackrel{U}{\to} \phi'^A$ as tensors with respect to the indices $p_1,\ldots,p_k$, namely,

$$\begin{aligned} \sigma'^{\,p_1\cdots p_k|\mu_1\cdots\mu_l} &= M^{p_1q_1}\cdots M^{p_kq_k}\sigma^{q_1\cdots q_k|\mu_1\cdots\mu_l}, \\ \delta_\varsigma\sigma^{p_1\cdots p_k|\mu_1\cdots\mu_l} &= g\sum_{r_s\in\{r_1,\ldots,r_k\}} f^{p_sr_sq}\sigma^{p_1\cdots r_s\cdots p_k|\mu_1\cdots\mu_l}\varsigma^q \equiv gf^{\{p\}\hat{r}q}\sigma^{p_1\cdots\hat{r}\cdots p_k|\mu_1\cdots\mu_l}\varsigma^q, \end{aligned} \tag{2.10}$$

which generalizes the transformation of a vector in the index p. As a consequence, the exponential in the path integral (2.1) for $Z(J,L)$ is invariant under $\phi^A \stackrel{U}{\to} \phi'^A$, accompanied by global transformations of the sources J_A, L_m,

$$(J_A, L_m) \stackrel{U}{\to} (J_A, L_m)', \quad J_A = (J^p_{(A)\mu}, J^p_{(b)}, J^p_{(\bar{c})}, J^p_{(c)}), \quad L_m = L^{p_1\cdots p_i}_{\mu_1\cdots\mu_j}, \tag{2.11}$$

in a tensor and infinitesimal form:

$$\begin{aligned} &(J^{p|\mu}_{(A)}, J^p_{(b)}, J^p_{(\bar{c})}, J^p_{(c)})' = M^{pq}(J^{q|\mu}_{(A)}, J^q_{(b)}, J^q_{(\bar{c})}, J^q_{(c)}), \qquad L'^{\,p_1\cdots p_i}_{\mu_1\cdots\mu_j} = M^{p_1q_1}\cdots M^{p_iq_i}L^{q_1\cdots q_i}_{\mu_1\cdots\mu_j}, \\ &\delta_\varsigma(J^{p|\mu}_{(A)}, J^p_{(b)}, J^p_{(\bar{c})}, J^p_{(c)}) = gf^{prq}(J^{r|\mu}_{(A)}, J^r_{(b)}, J^r_{(\bar{c})}, J^r_{(c)})\varsigma^q, \quad \delta_\varsigma L^{p_1\cdots p_i}_{\mu_1\cdots\mu_j} = gf^{\{p\}\hat{r}q}L^{p_1\cdots\hat{r}\cdots p_i}_{\mu_1\cdots\mu_j}\varsigma^q. \end{aligned} \tag{2.12}$$

which provides the invariance of the source term $J_A\phi^A + L_m\sigma^m(\phi)$.

Let us introduce an additional gauge field $B_\mu = B^p_\mu T^p$ which transforms as in (2.8),

$$B_\mu \stackrel{V}{\to} B'_\mu = VB_\mu V^{-1} + g^{-1}V\left(\partial_\mu V^{-1}\right), \tag{2.13}$$

with the inherent property

$$D_\mu(B') = VD_\mu(B)V^{-1}, \quad D_\mu(B) \equiv \partial_\mu + gB_\mu, \tag{2.14}$$

and subject the exponential in the path integral (2.1) to the following modification:

$$\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}(\phi) + J\phi + L\sigma(\phi)\right]\right\}\Big|_{\partial_\mu\to D_\mu(B)} \equiv \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}(\phi,B) + J\phi + L\sigma(\phi,B)\right]\right\}, \tag{2.15}$$

where the replacement $\partial_\mu \to D_\mu(B)$ is understood as

$$\partial_\mu \to D_\mu(B): \quad [\partial_\mu, \bullet] \to [D_\mu(B), \bullet] \Rightarrow [D_\mu(A), \bullet] \to [D_\mu(A+B), \bullet], \tag{2.16}$$

in particular,

$$F_{\mu\nu}(A) \to [D_\mu(A+B), D_\nu(A+B)] = F_{\mu\nu}(A+B). \tag{2.17}$$

The quantum action $S_{\mathrm{FP}}(\phi)$ in (2.15) is then replaced by the background action $S_{\mathrm{FP}}(\phi,B)$ of the form (2.22), (2.23), while as regards the replacement $\sigma^m(\phi) \to \sigma^m(\phi,B)$ one should notice the following.

Namely, in the case of local composite fields without higher derivatives, $\sigma^m = \sigma^m(\phi, \partial\phi)$, where ∂_μ enter only via the structures $[\partial_\mu, \phi]$ and $[D_\mu(A), \phi]$ in a matrix form, the introduction of B_μ according to (2.16) is unambiguous. In the case of higher derivatives, $\sigma^m = \sigma^m(\phi, \partial\phi, \ldots, \partial\cdots\partial\phi)$, the introduction of B_μ is not unique, since, prior to including the background field to $\partial_{\mu_1}\cdots\partial_{\mu_n}\phi$, these structures can be modified by adding terms with a difference of cross derivatives. Such extra terms are zero in the absence of a background; however, they are non-vanishing (and arbitrary) in the presence of a background:

$$[\partial_\mu, \partial_\nu] = 0, \quad [D_\mu(B), D_\nu(B)] \neq 0.$$

At the same time, the general considerations below remain valid irrespective of a particular representation of $\sigma^m(\phi, B)$ according to (2.16).

Due to the transformation property (2.14) of the derivative $D_\mu(B)$, the generating functional $Z(B, J, L)$ modified by the field B_μ according to (2.15), (2.16),

$$Z(B, J, L) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}(\phi, B) + J_A\phi^A + L_m\sigma^m(\phi, B)\right]\right\}, \tag{2.18}$$

is invariant under a set of local transformations,

$$\begin{aligned} \delta_\xi B^p_\mu &= D^{pq}_\mu(B)\,\xi^q, \\ \delta_\xi(J^{p|\mu}_{(A)}, J^p_{(b)}, J^p_{(\bar{c})}, J^p_{(c)}) &= gf^{prq}(J^{r|\mu}_{(A)}, J^r_{(b)}, J^r_{(\bar{c})}, J^r_{(c)})\xi^q, \\ \delta_\xi L^{p_1\cdots p_k}_{\mu_1\cdots\mu_l} &= gf^{\{p\}\hat{r}q}L^{p_1\cdots\hat{r}\cdots p_k}_{\mu_1\cdots\mu_l}\xi^q, \end{aligned} \tag{2.19}$$

given by the gauge transformations (2.13) of the field B_μ combined with a localized form $U(\varsigma) \to V(\xi)$ of the transformations (2.11), (2.12) for the sources J_A, L_m with infinitesimal parameters ξ^p,

$$(B_\mu, J_A, L_m) \xrightarrow{V} (B_\mu, J_A, L_m)'.$$

The invariance property $Z(B', J', L') = Z(B, J, L)$ can be established by applying to the transformed path integral $Z(B', J', L')$ a compensating change of the integration variables:

$$\delta_\xi\left(A^p_\mu, b^p, \bar{c}^p, c^p\right) = gf^{prq}\left(A^r_\mu, b^r, \bar{c}^r, c^r\right)\xi^q, \quad (A_\mu, b, \bar{c}, c) \xrightarrow{V} (A_\mu, b, \bar{c}, c)',$$

whose Jacobian equals to unity in view of the antisymmetry of the structure constants. The invariance of $Z(B, J, L)$ can be recast in the form

$$\begin{aligned} \int d^Dx\ \Bigg\{&\left[D^{pq}_\mu(B)\,\xi^q\right]\frac{\overrightarrow{\delta}}{\delta B^p_\mu} + g\xi^q f^{\{p\}\hat{r}q}L^{p_1\cdots\hat{r}\cdots p_k}_{\mu_1\cdots\mu_l}\frac{\overrightarrow{\delta}}{\delta L^{p_1\cdots p_k}_{\mu_1\cdots\mu_l}} \\ &+ g\xi^q f^{prq}\left(J^{r|\mu}_{(A)}\frac{\overrightarrow{\delta}}{\delta J^{p|\mu}_{(A)}} + J^r_{(b)}\frac{\overrightarrow{\delta}}{\delta J^p_{(b)}} + J^r_{(\bar{c})}\frac{\overrightarrow{\delta}}{\delta J^p_{(\bar{c})}} + J^r_{(c)}\frac{\overrightarrow{\delta}}{\delta J^p_{(c)}}\right)\Bigg\}\, Z(B, J, L) = 0. \end{aligned} \tag{2.20}$$

To interpret the generating functional $Z(B, J, L)$ in (2.18), notice that the modified Faddeev–Popov action $S_{\mathrm{FP}}(\phi, B)$ constructed by the rule (2.15), (2.16), (2.17) is invariant under the finite local transformations

$$(A_\mu, b, \bar{c}, c) \xrightarrow{V} V(A_\mu, b, \bar{c}, c)\,V^{-1}, \quad B_\mu \xrightarrow{V} VB_\mu V^{-1} + g^{-1}V\partial_\mu V^{-1} \tag{2.21}$$

and takes the form

$$S_{\mathrm{FP}}(\phi, B) = S_0(A + B) + \Psi(\phi, B)\overleftarrow{s}_{\mathrm{q}}, \tag{2.22}$$

where

$$
\begin{aligned}
S_0(A+B) &= S_0(A)\big|_{[\partial_\mu,\bullet]\to[D_\mu(B),\bullet]} = S_0(A)\big|_{D_\mu(A)\to D_\mu(A+B)}\ , \\
\Psi(\phi,B) &= \Psi(\phi)\big|_{[\partial_\mu,\bullet]\to[D_\mu(B),\bullet]}\ , \quad \overleftarrow{s}_{\mathrm{q}} = \overleftarrow{s}\big|_{D_\mu(A)\to D_\mu(A+B)}\ ,
\end{aligned}
\tag{2.23}
$$

namely,

$$
\begin{aligned}
(A_\mu, b, \bar{c}, c)\overleftarrow{s}_{\mathrm{q}} &= \left([D_\mu(A+B), c], 0, b, g/2\,[c,c]_+\right)\ , \\
\left(A^p_\mu, b^p, \bar{c}^p, c^p\right)\overleftarrow{s}_{\mathrm{q}} &= \left(D^{pq}_\mu(A+B)\,c^q, 0, b^p, g/2 f^{pqr}c^q c^r\right)\ .
\end{aligned}
\tag{2.24}
$$

For instance, the Landau and Feynman gauges (2.9) are modified to the background gauges

$$
\chi^p_{\mathrm{L}}(\phi,B) = D^{pq}_\mu(B)\,A^{q|\mu}\ , \quad \chi^p_{\mathrm{F}}(\phi,B) = b^p + D^{pq}_\mu(B)\,A^{q|\mu}. \tag{2.25}
$$

In the background field method, a quantum action $S_{\mathrm{FP}}(\phi,B)$ constructed according to (2.22), (2.24) is known as the Faddeev–Popov action with a background field B_μ. The quantum action $S_{\mathrm{FP}}(\phi,B)$ is invariant under global transformations of ϕ^A, with a nilpotent generator $\overleftarrow{s}_{\mathrm{q}}$ and an anticommuting parameter λ:

$$
\delta S_{\mathrm{FP}}(\phi,B) = 0, \quad \delta\phi^A = \phi^A\overleftarrow{s}_{\mathrm{q}}\lambda, \quad \overleftarrow{s}_{\mathrm{q}}\overleftarrow{s}_{\mathrm{q}} = 0, \quad \epsilon(\lambda) = \mathrm{gh}(\lambda) = 1. \tag{2.26}
$$

At the infinitesimal level, the local transformations (2.21) for the fields A_μ, B_μ are known as *background transformations*, and the transformations of A_μ, B_μ corresponding to the modified Slavnov variation $\overleftarrow{s}_{\mathrm{q}}$ in (2.22), (2.24), (2.26) are known as *quantum transformations*,

$$
\begin{aligned}
\text{background}:&\quad \delta_{\mathrm{b}}A_\mu = g\,[A_\mu, T^p\xi^p]\,, &&\delta_{\mathrm{b}}B_\mu = [D_\mu(B), T^p\xi^p]\,, \\
\text{quantum}:&\quad \delta_{\mathrm{q}}A_\mu = [D_\mu(A+B), T^p\xi^p]\,, &&\delta_{\mathrm{q}}B_\mu = 0,
\end{aligned}
$$

whereas the classical action $S_0(A+B)$ is left invariant by both of these types of transformations. In this connection, the family of background gauges $\chi^p(\phi,B) = \tilde{\chi}^p(A,B) + (\alpha/2)\,b^p$, parameterized by $\alpha \neq 0$ and defined according to (2.23),

$$
\chi^p(A,B) = \chi^p(A)\big|_{[\partial_\mu,\bullet]\to[D_\mu(B),\bullet]}\ ,
$$

with the Nakanishi–Lautrup fields b^p integrated out of (2.18) by the shift $b^p \to b^p + \alpha^{-1}\tilde{\chi}$ at the vanishing sources, $J = L = 0$, reduces the vacuum functional $Z(B)$ to the form (for future convenience, we denote $A \equiv Q$),

$$
\begin{aligned}
Z(B) &= \int dQ\, d\bar{c}\, dc\, \exp\left\{\frac{i}{\hbar}\left[S_0(Q+B) + S_{\mathrm{gf}}(Q,B) + S_{\mathrm{gh}}(Q,B;\bar{c},c)\right]\right\}, \\
S_{\mathrm{gf}}(Q,B) &= -\frac{1}{2\alpha}\int d^Dx\ \tilde{\chi}^p\tilde{\chi}^p, \quad S_{\mathrm{gh}}(Q,B) = \int d^Dx\ \bar{c}^p\ \delta_{\mathrm{q}}\tilde{\chi}^p\big|_{\xi\to c}\ ,
\end{aligned}
\tag{2.27}
$$

where the gauge-fixing term $S_{\mathrm{gf}} = S_{\mathrm{gf}}(Q,B)$ is invariant under the background transformations, $\delta_{\mathrm{b}}S_{\mathrm{gf}} = 0$, due to $\delta_{\mathrm{b}}\tilde{\chi}^p = gf^{prq}\tilde{\chi}^r\xi^q$, which may be employed to define the quantum action in background gauges $\tilde{\chi}^p(Q,B)$ depending on the quantum Q and background B fields with the associated background and quantum transformations (see also [3]),

$$
\begin{aligned}
&\delta_{\mathrm{b}}B^p_\mu = D^{pq}_\mu(B)\,\xi^q, &&\delta_{\mathrm{b}}Q^p_\mu = gf^{prq}Q^r_\mu\xi^q, \\
&\delta_{\mathrm{q}}B^p_\mu = 0, &&\delta_{\mathrm{q}}Q^p_\mu = D^{pq}_\mu(Q+B)\,\xi^q.
\end{aligned}
\tag{2.28}
$$

By construction, the quantum action and the integrand of $Z(B)$ in (2.27) are invariant under the residual local transformations (2.21), namely,

$$(Q_\mu,\bar{c},c)\overset{V}{\to}V\,(Q_\mu,\bar{c},c)\,V^{-1}\;,\quad B_\mu\overset{V}{\to}VB_\mu V^{-1}+g^{-1}V\partial_\mu V^{-1}, \tag{2.29}$$

which translates infinitesimally, $\delta\,(B_\mu,Q_\mu,\bar{c},c)$, to the background transformations δ_{b} accompanied by some compensating transformations of the ghost fields:

$$\delta\left(B^p_\mu,\;Q^p_\mu,\;\bar{c}^p,\;c^p\right)=\left(\delta_{\mathrm{b}}B^p_\mu,\;\delta_{\mathrm{b}}Q^p_\mu,\;gf^{prq}\bar{c}^r\xi^q,\;gf^{prq}c^r\xi^q\right).$$

Given this, we interpret $Z\,(B,J,L)$ defined according to (2.1), (2.15), (2.16), (2.18) as a generating functional of Green's functions for Yang–Mills theories with composite fields in the background field method, or as a generating functional of Green's functions with composite and background fields for such theories. As we shall see below, this interpretation provides the existence of a corresponding gauge-invariant effective action with composite and background fields.

3 Ward Identities, Effective Action

Let us now present the corresponding generating functionals of connected and vertex Green's functions with composite and background fields and examine their properties. To this end, we first introduce an extended generating functional $Z\,(B,J,L,\phi^*)$,

$$\begin{aligned} Z\,(B,J,L,\phi^*) &= \int d\phi\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{ext}}\,(\phi,\phi^*,B)+J_A\phi^A+L_m\sigma^m\,(\phi,B)\right]\right\}\\ &\equiv \int \mathcal{I}^{\Psi}_{\phi,\phi^*,B}\exp\left\{\frac{i}{\hbar}\left[J_A\phi^A+L_m\sigma^m\,(\phi,B)\right]\right\}, \end{aligned} \tag{3.1}$$

with an extended quantum action $S_{\mathrm{ext}}\,(\phi,\phi^*,B)$ given by

$$\begin{aligned} S_{\mathrm{ext}}\,(\phi,\phi^*,B) &= S_{\mathrm{FP}}\,(\phi,B)+\phi^*_A(\phi^A\overleftarrow{s}_{\mathrm{q}}),\\ \phi^*_A(\phi^A\overleftarrow{s}_{\mathrm{q}}) &= \int d^Dx\left(A^{*p}_\mu D^{pq|\mu}\,(A+B)\,c^q+\bar{c}^{*p}b^p+(g/2)\,f^{pqr}c^{*p}c^qc^r\right),\\ &= 2\int d^Dx\;\mathrm{Tr}\left(A^*_\mu\left[D^\mu\,(A+B)\,,c\right]+\bar{c}^*b+(g/2)\,c^*\left[c,c\right]_+\right), \end{aligned}$$

where ϕ^*_A, $\epsilon\,(\phi^*_A)=\epsilon\left(\phi^A\right)+1$, $\mathrm{gh}\,(\phi^*_A)=-\mathrm{gh}\left(\phi^A\right)$, is a set of antifields introduced as sources to the variations $\phi^A\overleftarrow{s}_{\mathrm{q}}$,

$$\phi^*_A=\left(A^{*p}_\mu,b^{*p},\bar{c}^{*p},c^{*p}\right),\quad\left(A^*_\mu,b^*,\bar{c}^*,c^*\right)\equiv T^p\left(A^{*p}_\mu,b^{*p},\bar{c}^{*p},c^{*p}\right).$$

Due to the invariance property (2.26) of $S_{\mathrm{FP}}\,(\phi,B)$, the extended quantum action $S_{\mathrm{ext}}\,(\phi,\phi^*,B)$ satisfies the identity

$$S_{\mathrm{ext}}\,(\phi,\phi^*,B)\,\overleftarrow{s}_{\mathrm{q}}=0,$$

which can be recast in the form of a master equation,

$$(S_{\mathrm{ext}},S_{\mathrm{ext}})=0,\quad(F,G)\equiv F\frac{\overleftarrow{\delta}}{\delta\phi^A}\frac{\overrightarrow{\delta}}{\delta\phi^*_A}G-(-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}\,G\frac{\overleftarrow{\delta}}{\delta\phi^A}\frac{\overrightarrow{\delta}}{\delta\phi^*_A}F,$$

or, equivalently,

$$(-1)^{\epsilon(\phi^A)}\,\frac{\overrightarrow{\delta}}{\delta\phi^A}\frac{\overrightarrow{\delta}}{\delta\phi^*_A}\exp\left[(i/\hbar)\,S_{\mathrm{ext}}\right]=0,$$

which holds due to the complete antisymmetry of the structure constants f^{pqr}.

Let us make in the integrand (3.1) a finite FD BRST transformation (see Appendix A.1 for details) with a generator $\overleftarrow{s}_{\mathrm{q}}$ given by (2.24) and a Grassmann-odd functional parameter $\lambda(\phi,B)$,

$$\phi^A\to\phi'^A=\phi^A+\phi^A\overleftarrow{s}_{\mathrm{q}}\lambda(\phi,B), \tag{3.2}$$

where $\lambda(\phi,B)$ is related to a finite change [40, 42] of the gauge fermion $\Delta\Psi\,(\phi,B)$ depending also on the background field. For a finite constant λ, the following invariance property holds true:

$$\mathcal{I}^{\Psi}_{\phi+\phi\overleftarrow{s}_{\mathrm{q}}\lambda,\phi^*,B}=\mathcal{I}^{\Psi}_{\phi,\phi^*,B}\ ,\quad \lambda\frac{\overleftarrow{\delta}}{\delta\phi}=\lambda\frac{\overleftarrow{\delta}}{\delta B}=0, \tag{3.3}$$

whereas the FD parameter $\lambda(\phi,B)$ in the Jacobian $\mathrm{Sdet}\,\|\delta\phi'/\delta\phi\|=[1+\lambda(\phi,B)\overleftarrow{s}_q]^{-1}$ of (A.1) for the change of variables (3.2), given the choice of $\lambda(\phi,B)=\lambda(\phi,B|\Delta\Psi)$ in the form

$$\begin{aligned}\lambda(\phi,B\,|\Delta\Psi)\ &=\ \Delta\Psi(\phi,B)\left\{[\Delta\Psi(\phi,B)]\,\overleftarrow{s}_{\mathrm{q}}\right\}^{-1}\left(\exp\left\{-\frac{i}{\hbar}\,[\Delta\Psi(\phi,B)]\,\overleftarrow{s}_{\mathrm{q}}\right\}-1\right)\\ &=\ -\frac{i}{\hbar}\Delta\Psi(\phi,B)+o(\Delta\Psi),\end{aligned} \tag{3.4}$$

implies the independence of the extended vacuum functional, $Z_{\Psi}\,(B,\phi^*)=Z\,(B,0,0,\phi^*)$, from a finite variation of an admissible gauge condition, $\Psi(\phi,B)\to\Psi(\phi,B)+\Delta\Psi(\phi,B)$, namely,

$$Z_{\Psi}\,(B,\phi^*)=Z_{\Psi+\Delta\Psi}\,(B,\phi^*)\Longleftrightarrow\mathcal{I}^{\Psi}_{\phi+\phi\overleftarrow{s}_{\mathrm{q}}\lambda,\phi^*,B}=\mathcal{I}^{\Psi+\Delta\Psi}_{\phi,\phi^*,B}\ . \tag{3.5}$$

The latter property, once finite FD BRST transformations are applied to the integrand of the generating functional (3.1), leads to a *modified Ward identity*, suggested for the first time within the BV formalism in [51], now with respect to a functional $Z\,(B,J,L,\phi^*)$ extended by antifields and a background field:

$$\left\langle\left[1+\frac{i}{\hbar}J_A\left(\phi^A\overleftarrow{s}_{\mathrm{q}}\right)\lambda(\phi,B)\right]\left[1+\lambda(\phi,B)\overleftarrow{s}_{\mathrm{q}}\right]^{-1}\right\rangle_{\Psi,B,J,L,\phi^*}=1\ . \tag{3.6}$$

Here, the notation $\langle\mathcal{D}\rangle_{\Psi,B,J,L,\phi^*}$, with a certain functional $\mathcal{D}\,(\phi,\phi^*,B)$, implies a source-dependent expectation value corresponding to a gauge-fixing functional $\Psi(\phi,B)$,

$$\langle\mathcal{D}\rangle_{\Psi,B,J,L,\phi^*}=Z^{-1}(B,J,L,\phi^*)\int d\phi\ \mathcal{D}\,(\phi,\phi^*,B)\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{ext}}\,(\phi,\phi^*,B)+J_A\phi^A+L_m\sigma^m\,(\phi,B)\right]\right\}, \tag{3.7}$$

with the normalization $\langle 1\rangle_{\dots}=1$, where the dots stand for Ψ,B,J,L,ϕ^* as in (3.6). Using the familiar rules $\langle\phi^A\rangle_{\dots}=Z^{-1}\frac{\hbar}{i}\overrightarrow{\delta}/\delta J_A Z$ and $\langle\phi^A\overleftarrow{s}_{\mathrm{q}}\rangle_{\dots}=Z^{-1}\frac{\hbar}{i}\delta/\delta\phi^*_A Z$, one presents the modified identity (3.6) in the form

$$\left\{\hat{\omega}\lambda\left(\frac{\hbar\,\overrightarrow{\delta}}{i\delta J},B\right)+\left[\sum_{n=1}(-1)^n\left(\lambda\overleftarrow{s}_{\mathrm{q}}\right)^n\left(\frac{\hbar\,\overrightarrow{\delta}}{i\delta J},B\right)\right]\left[1+\hat{\omega}\lambda\left(\frac{\hbar\,\overrightarrow{\delta}}{i\delta J},B\right)\right]\right\}Z=0, \tag{3.8}$$

with a nilpotent Grassmann-odd operator $\hat{\omega}$,

$$\hat{\omega}=\left[J_A+L_m\sigma^m_{,A}\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J},B\right)\right]\frac{\overrightarrow{\delta}}{\delta\phi^*_A},\quad \hat{\omega}^2=0. \tag{3.9}$$

In deriving the λ-dependent identity (3.8), we have used the expansion $(1+x)^{-1}=1+\sum_{n>0}(-1)^n x^n$ with $x=\lambda\overleftarrow{s}_{\mathrm{q}}$. Notice that, instead of the monomial $(\lambda\overleftarrow{s}_{\mathrm{q}})^n$, we can equivalently apply $[(\hbar/i)\lambda_{,A}\overrightarrow{\delta}/\delta\phi^*_A]^n$ with $\lambda_{,A}=\lambda\overleftarrow{\delta}/\delta\phi^A$ under the sign of functional integral, which leads to another representation of the identity (3.8),

$$\left\{\hat{\omega}\lambda\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J},B\right)+\left(\sum_{n=1}(-1)^n\left[\lambda_{,A}\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J},B\right)\frac{\hbar\overrightarrow{\delta}}{i\delta\phi^*_A}\right]^n\right)\left[1+\hat{\omega}\lambda\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J},B\right)\right]\right\}Z=0. \tag{3.10}$$

For an infinitesimal FD parameter λ, the identity (3.10) acquires the form

$$\left[\hat{\omega}\lambda\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J},B\right)-\frac{\hbar}{i}\lambda_{,A}\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J},B\right)\frac{\overrightarrow{\delta}}{\delta\phi^*_A}\right]Z=0. \tag{3.11}$$

For a constant λ, namely, $\lambda\overleftarrow{s}_{\mathrm{q}}=0$, the relation (3.8) contains the usual Ward identity [66], depending parametrically on a background field:

$$\hat{\omega}Z\left(B,J,L,\phi^*\right)=0. \tag{3.12}$$

The generating functional $W\left(B,J,L,\phi^*\right)$ of connected Green's functions, $W=(\hbar/i)\ln Z$, satisfies a related modified Ward identity,

$$\hat{\Omega}\langle\lambda(B)\rangle W+\left[\sum_{n=1}(-1)^n\langle(\lambda(B)\overleftarrow{s}_{\mathrm{q}}\rangle)^n\right]\left[1+\hat{\Omega}\langle\lambda\left(B\right)\rangle\right]W=0, \tag{3.13}$$

$$\hat{\Omega}=\left[J_A+L_m\sigma^m_{,A}\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J}+\frac{\overrightarrow{\delta}W}{\delta J},B\right)\right]\frac{\overrightarrow{\delta}}{\delta\phi^*_A},\quad \langle\lambda(B)\rangle\equiv\lambda\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J}+\frac{\overrightarrow{\delta}W}{\delta J},B\right), \tag{3.14}$$

deduced by a unitary transformation of the operator $\hat{\omega}$ in (3.9),

$$\hat{\Omega}=\hat{U}^{-1}\hat{\omega}\hat{U},\quad \hat{U}=\exp\left(i/\hbar W\right). \tag{3.15}$$

Once again, an infinitesimal FD λ reduces the identity (3.13), or, equivalently, (3.10), to the form

$$\left[\hat{\Omega}\langle\lambda(B)\rangle-\frac{\hbar}{i}\langle\lambda_{,A}(B)\rangle\frac{\overrightarrow{\delta}}{\delta\phi^*_A}\right]W=0. \tag{3.16}$$

In turn, for a constant λ the relation (3.13) contains the usual Ward identity [66]

$$\hat{\Omega}W=\left[J_A+L_m\sigma^m_{,A}\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J}+\frac{\overrightarrow{\delta}W}{\delta J},B\right)\right]\frac{\overrightarrow{\delta}}{\delta\phi^*_A}W=0. \tag{3.17}$$

As we introduce a generating functional $\Gamma\left(B,\phi,\Sigma,\phi^*\right)$ of vertex Green's functions with composite fields (on a background) by using a double Legendre transformation [66],

$$\Gamma\left(B,\phi,\Sigma,\phi^*\right)=W\left(B,J,L,\phi^*\right)-J_A\phi^A-L_m\left[\sigma^m\left(\phi,B\right)+\Sigma^m\right]\ , \tag{3.18}$$

where

$$\phi^A = \frac{\overrightarrow{\delta} W}{\delta J_A},\ \Sigma^m = \frac{\overrightarrow{\delta} W}{\delta L_m} - \sigma^m \left(\frac{\overrightarrow{\delta} W}{\delta J}, B \right),\ -J_A = \Gamma \frac{\overleftarrow{\delta}}{\delta \phi^A} + L_m \sigma^m_{,A}(\phi, B),\ -L_m = \Gamma \frac{\overleftarrow{\delta}}{\delta \Sigma^m}\ ,$$

the modified Ward identity (3.13) acquires the form (see Appendix A.2 for details)

$$\hat{\omega}_\Gamma \langle\langle \lambda(B) \rangle\rangle \Gamma + \left\{ \sum_{n=1} (-1)^n \left[\langle\langle \lambda(B) \overleftarrow{s}_{\mathrm{q}} \rangle\rangle \right]^n \right\} \left[1 + \hat{\omega}_\Gamma \langle\langle \lambda\left(B\right) \rangle\rangle \right] \Gamma = 0, \tag{3.19}$$

$$\begin{aligned}
\hat{\omega}_\Gamma = {} & (\Gamma, \bullet) + \left[\sigma^m_{,A}(\hat{\phi}, B) - \sigma^m_{,A}(\phi, B) \right] \frac{\overrightarrow{\delta}\Gamma}{\delta \phi^*_A} \frac{\overrightarrow{\delta}}{\delta \Sigma^m} \\
& - \frac{i}{\hbar} \left(\left[\Gamma \frac{\overleftarrow{\delta}}{\delta \Sigma^m} \left(\sigma^m_{,C}(\hat{\phi}, B) \frac{\overrightarrow{\delta}\Gamma}{\delta \phi^*_C} \right), \Phi^{\mathsf{a}} \right\} \frac{\overrightarrow{\delta}}{\delta \Phi^{\mathsf{a}}} + \left[\Gamma \frac{\overleftarrow{\delta}}{\delta \Sigma^m} \left(\sigma^m_{,C}(\hat{\phi}, B) \frac{\overrightarrow{\delta}\Gamma}{\delta \phi^*_C} \right),\ \sigma^n(\phi, B) \right\} \frac{\overrightarrow{\delta}}{\delta \Sigma^n} \right) \\
& + \frac{i}{\hbar} (-1)^{\epsilon(\sigma^n) + \epsilon(\phi^D)} \sigma^n_{,D}(\phi, B) \left[\Gamma \frac{\overleftarrow{\delta}}{\delta \Sigma^m} \left(\sigma^m_{,C}(\hat{\phi}, B) \frac{\overrightarrow{\delta}\Gamma}{\delta \phi^*_C} \right), \phi^D \right\} \frac{\overrightarrow{\delta}}{\delta \Sigma^n} \\
& + (-1)^{\epsilon(\sigma^m) + \epsilon(\phi^D)\epsilon(\phi^A)} \left[\sigma^m_{,D}(\phi, B), \Gamma \frac{\overleftarrow{\delta}}{\delta \Sigma^n} \sigma^n_{,A}(\hat{\phi}, B) \right\} \left(G''^{-1} \right)^{A\mathsf{a}} \left(\frac{\overrightarrow{\delta}}{\delta \Phi^{\mathsf{a}}} \frac{\overrightarrow{\delta}\Gamma}{\delta \phi^*_D} \right) \frac{\overrightarrow{\delta}}{\delta \Sigma^m},
\end{aligned} \tag{3.20}$$

where $\langle\langle \lambda(B) \rangle\rangle \equiv \lambda(\hat{\phi}, B)$, and the following notation is used:

$$\begin{aligned}
\hat{\phi}^A &\equiv \phi^A + i\hbar \left(G''^{-1} \right)^{A\mathsf{a}} \overrightarrow{\delta} / \delta \Phi^{\mathsf{a}}\ , \quad \left(G'' \right)_{\mathsf{ab}} \equiv \overrightarrow{\delta} F_{\mathsf{b}} / \delta \Phi^{\mathsf{a}}\ , \\
\Phi^{\mathsf{a}} &= \left(\phi^A, \Sigma^m \right), \quad F_{\mathsf{a}} = \left(\Gamma_{,A} - \Gamma_{,n} \sigma^n_{,A}\left(\phi, B\right), \Gamma_{,m} \right), \quad \Gamma_{,\mathsf{a}} \equiv \Gamma \overleftarrow{\delta} / \delta \Phi^{\mathsf{a}}\ .
\end{aligned} \tag{3.21}$$

For a constant infinitesimal parameter λ, the modified Ward identity (3.19) is reduced to the usual one [66], however, now with a background included,

$$\frac{1}{2} \left(\Gamma, \Gamma \right) = -\Gamma \frac{\overleftarrow{\delta}}{\delta \Sigma^m} \left[\sigma^m_{,A}(\hat{\phi}, B) - \sigma^m_{,A}(\phi, B) \right] \frac{\overrightarrow{\delta}}{\delta \phi^*_A} \Gamma. \tag{3.22}$$

The extended generating functional $Z\left(B, J, L, \phi^*\right)$ of Green's functions (3.1) and the related functional $W\left(B, J, L, \phi^*\right)$ exhibit an invariance under the local transformations (2.19) accompanied by the following transformations of the antifields:

$$\delta_\xi \left(A^{*p}_\mu, b^{*p}, \bar{c}^{*p}, c^{*p} \right) = g f^{prq} \left(A^{*r}_\mu, b^{*r}, \bar{c}^{*r}, c^{*r} \right) \xi^q. \tag{3.23}$$

Then the effective action $\Gamma_{\text{eff}}\left(B, \Sigma\right)$ with composite and background fields defined as

$$\Gamma_{\text{eff}}\left(B, \Sigma\right) = \Gamma\left(B, \phi, \Sigma, \phi^*\right)\big|_{\phi = \phi^* = 0} \tag{3.24}$$

satisfies an identity (see Appendix A.3) related to (2.20),

$$\int d^D x\ \left\{ \left[D^{pq}_\mu\left(B\right) \xi^q \right] \frac{\overrightarrow{\delta}}{\delta B^p_\mu} + g f^{\{p\}\hat{r}q} \Sigma^{p_1 \cdots \hat{r} \cdots p_k}_{\mu_1 \cdots \mu_l} \xi^q \frac{\overrightarrow{\delta}}{\delta \Sigma^{p_1 \cdots p_k}_{\mu_1 \cdots \mu_l}} \right\} \Gamma_{\text{eff}}\left(B, \Sigma\right) = 0, \tag{3.25}$$

and is thereby invariant under the local transformations

$$\delta_\xi B^p_\mu = D^{pq}_\mu\left(B\right) \xi^q, \quad \delta_\xi \Sigma^{p_1 \cdots p_k}_{\mu_1 \cdots \mu_l} = g f^{\{p\}\hat{r}q} \Sigma^{p_1 \cdots \hat{r} \cdots p_k}_{\mu_1 \cdots \mu_l} \xi^q, \tag{3.26}$$

which consist of the initial gauge transformations for the background field B^p_μ and of the local $SU(N)$ transformations for the fields $\Sigma^{p_1\cdots p_k}_{\mu_1\cdots\mu_l}$.

Returning once again to the modified Ward identities, we point out that, once the composite fields $\sigma^m(\phi,B)$ are absent, the formulas (3.8), (3.13), (3.19) are reduced to those involving the respective functionals $Z|_{L=0}$, $W|_{L=0}$, $\Gamma|_{\Sigma=\sigma=0}$, which presents a new form of λ-dependent Ward identities, additional to the usual Ward identities ($\lambda = \text{const}$) for these functionals. The deduction of the modified Ward identities (3.8), (3.13), (3.19) for the generating functionals Z, W, Γ of Green's functions with composite and background fields, implying the respective usual Ward identities (3.12), (3.17), (3.22) by means of finite FD BRST transformations, comprises the results of this section that have a generic character. Finally, it should be noted that we have assumed the existence of a "deep" gauge-invariant regularization preserving the Ward identities (see, e.g., [29]), as we expect the corresponding renormalized generating functionals to obey the same properties as the unrenormalized ones.

4 Gauge Dependence Problem

Let us study the gauge dependence of the generating functionals $Z\left(B,J,L,\phi^*\right)$, $W\left(B,J,L,\phi^*\right)$, $\Gamma\left(B,\phi,\Sigma,\phi^*\right)$ of Green's functions with composite and background fields. In this regard, the representations (3.4) and (3.8) also provide a relation which describes the gauge dependence of $Z(B,J,L,\phi^*) = Z_\Psi$ for a finite change $\Psi \to \Psi + \Delta\Psi$:

$$\Delta Z_\Psi = Z_{\Psi+\Delta\Psi} - Z_\Psi = \hat{\omega}\lambda\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J}, B\,|{-\Delta\Psi}\right) Z_\Psi = \frac{i}{\hbar}\hat{\omega}\Delta\Psi\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J}, B\right) Z_\Psi + o(\Delta\Psi). \tag{4.1}$$

The corresponding finite change $\Delta W_\Psi = \Delta W\left(B,J,L,\phi^*\right)$ can be presented as follows,[4] with account taken of (3.15) and of the usual Ward identity (3.17) for W_Ψ, namely,

$$\Delta W_\Psi = \frac{\hbar}{i}\hat{\Omega}\lambda\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J} + \frac{\overrightarrow{\delta}W}{\delta J}, B\,|{-\Delta\Psi}\right) = \hat{\Omega}\Delta\Psi\left(\frac{\hbar\overrightarrow{\delta}}{i\delta J} + \frac{\overrightarrow{\delta}W}{\delta J}, B\right) + o(\Delta\Psi), \tag{4.2}$$

where $\hat{\Omega}$, given by (3.14), is nilpotent, $\hat{\Omega}^2 = 0$, as a consequence of $\hat{\omega}^2 = 0$.

To obtain a finite change $\Delta\Gamma\left(B,\phi,\Sigma,\phi^*\right)$, we note that

$$\Delta\Gamma\left(B,\phi,\Sigma,\phi^*\right) = \Delta W\left(B,J,L,\phi^*\right),$$

as a general property of the Legendre transformation in the case of its dependence on an external parameter η, namely, $\Delta\Gamma\left(\eta\right) = \Delta W\left(\eta\right)$. Then $\Delta\Gamma\left(B,\phi,\Sigma,\phi^*\right)$ admits the representation

$$\begin{aligned} \Delta\Gamma &= \frac{\hbar}{i}\hat{\omega}_\Gamma\langle\langle\lambda\left(B| - \Delta\Psi\right)\rangle\rangle = \delta\Gamma + o\left(\langle\langle\Delta\Psi\rangle\rangle\right), \\ \langle\langle\lambda\left(B| - \Delta\Psi\right)\rangle\rangle &\equiv \lambda(\hat{\phi}, B| - \Delta\Psi), \quad \delta\Gamma \equiv \hat{\omega}_\Gamma\langle\langle\Delta\Psi\rangle\rangle, \end{aligned} \tag{4.3}$$

where $\hat{\phi}^A$ is given by (3.21), while the operator $\hat{\omega}_\Gamma$ is a Legendre transform of $\hat{\Omega}$ in (3.14), and thereby inherits the property of nilpotency: $\hat{\omega}^2_\Gamma = 0$.

[4] From $\Delta Z = \hat{A}Z$, with a certain operator $\hat{A}$, it follows that $\Delta W = \frac{\hbar}{i}\left\langle\hat{A}_W\right\rangle = \frac{\hbar}{i}\hat{A}_W 1$, with $\hat{A}_W$ given by $\hat{A}_W = e^{-i/\hbar W}\hat{A}e^{i/\hbar W}$.

From (4.3), it follows, according to [66, 67], that the generating functional $\Gamma(B, \phi, \Sigma, \phi^*)$ of vertex Green's functions is gauge-independent, $\delta\Gamma = 0$, on the extremals

$$\frac{\delta\Gamma}{\delta\phi^A} = \frac{\delta\Gamma}{\delta\Sigma^m} = 0, \tag{4.4}$$

so that the effective action $\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(B, \Sigma)$ with composite and background fields (3.24) is gauge-independent, $\delta\Gamma_{\text{eff}} = 0$, on the extremals (4.4), which is the principal result of this section.

5 Loop Expansion

Now we examine the procedure of a loop expansion for the effective action (EA) with composite and background fields. The initial relation

$$\begin{aligned} \exp\left\{\frac{i}{\hbar}\Gamma(B, \Sigma, \phi, \phi^*)\right\} &= \exp\left\{\frac{i}{\hbar}\Gamma_{,m}\Sigma^m\right\}\int d\widetilde{\phi}\exp\left\{\frac{i}{\hbar}\left[S_{\text{ext}}\left(\widetilde{\phi}, \phi^*, B\right) - F_A(B, \Sigma, \phi, \phi^*)\right.\right.\\ &\quad \left.\left.\times\left(\widetilde{\phi}^A - \phi^A\right) - \Gamma_{,m}\left(\sigma^m\left(\widetilde{\phi}, B\right) - \sigma^m(\phi, B)\right)\right]\right\}, \end{aligned} \tag{5.1}$$

according to (3.1), (3.18), with allowance for the notation (3.21) and a shift of variables[5] involving $\widetilde{\phi}^A$,

$$\widetilde{\phi}^A \to \widetilde{\phi}^A + \phi^A, \tag{5.2}$$

acquires the form (see (3.21) for the notation F_A)

$$\begin{aligned} \exp\left\{\frac{i}{\hbar}\Gamma(B, \Sigma, \phi, \phi^*)\right\} &= \exp\left\{\frac{i}{\hbar}\Gamma_{,m}\Sigma^m\right\}\int d\widetilde{\phi}\exp\left\{\frac{i}{\hbar}\left[S_{\text{ext}}\left(\widetilde{\phi} + \phi, \phi^*, B\right)\right.\right.\\ &\quad \left.\left.-F_A\widetilde{\phi}^A - \Gamma_{,m}\left(\sigma^m\left(\widetilde{\phi} + \phi^A, B\right) - \sigma^m(\phi, B)\right)\right]\right\}. \end{aligned} \tag{5.3}$$

The representation (5.3) examined at a vanishing background B^μ reduces to the EA with composite fields [68], albeit in the case of arbitrary (not limited to scalars) fields ϕ^A, with a dependence of the composite fields σ^m on ϕ^A being generally more then quadratic.

We further assume the representation

$$\Gamma(B, \Sigma, \phi, \phi^*) = S_{\text{ext}}(\phi, \phi^*, B) + \hbar\Gamma^{(1)}(B, \Sigma, \phi, \phi^*) + \Gamma_2(B, \Sigma, \phi, \phi^*), \tag{5.4}$$

with a one-loop effective action $\Gamma^{(1)}$ and a functional Γ_2 of order $\hbar^2$, which includes all the two-particle-irreducible vacuum graphs depending on the antifields ϕ^*, with the vertices determined by a functional $S_{\text{int}}(\widetilde{\phi}, \phi, \ldots) = S_{\text{int}}(\widetilde{\phi}, \phi, \phi^*, B)$ given by the interaction part of the quantum action and the non-quadratic part due to the composite fields:

$$\begin{aligned} S_{\text{int}}(\widetilde{\phi}, \phi, \ldots) &\equiv S_{\text{ext}}\left(\widetilde{\phi} + \phi, \ldots\right) - S_{\text{ext}}(\phi, \ldots) - S_{\text{ext}}(\phi, \ldots)\left(\frac{\overleftarrow{\delta}}{\delta\phi^A} + \frac{1}{2}\frac{\overleftarrow{\delta}}{\delta\phi^A}\frac{\overleftarrow{\delta}}{\delta\phi^B}\widetilde{\phi}^B\right)\widetilde{\phi}^A\\ &\quad -F_A\widetilde{\phi}^A - \Gamma_{,m}\left[\sigma^m\left(\widetilde{\phi} + \phi, B\right) - \sigma^m(\phi, B)\right]. \end{aligned} \tag{5.5}$$

[5] The change of variables (5.2) is identified with the background-quantum splitting used in [68], where the background component ϕ^A is not to be confused with B^μ.

From the representation (2.2), (3.1), we obtain the relation

$$\sum_{n\geq 2}\frac{1}{n!}\Lambda^{m}_{A_n\ldots A_1}\left(\frac{\hbar}{i}\right)^{n-1}\prod_{k=1}^{n}\frac{\overrightarrow{\delta}}{\delta J_{A_k}}Z\left(B,J,L,\phi^*\right)=\frac{\overrightarrow{\delta}}{\delta L_m}Z\left(B,J,L,\phi^*\right), \tag{5.6}$$

which can be recast[6] in terms of the generating functional $W\left(B,J,L,\phi^*\right)$, $W=(\hbar/i)\ln Z$,

$$\sum_{n\geq 2}\frac{1}{n!}\Lambda^{m}_{A_n\ldots A_1}\left(\frac{\hbar}{i}\right)^{n-1}\left[\prod_{k=1}^{n}\frac{\overrightarrow{\delta}}{\delta J_{A_k}}W+\theta_{n,2}n\frac{i}{\hbar}\left(\prod_{k=1}^{n-1}\frac{\overrightarrow{\delta}}{\delta J_{A_k}}\right)W\frac{\overrightarrow{\delta}}{\delta J_{A_n}}W\right.$$
$$\left.+\ldots+\left(\frac{i}{\hbar}\right)^{n-1}\prod_{k=1}^{n}\left(\frac{\overrightarrow{\delta}}{\delta J_{A_k}}W\right)\right]=\frac{\overrightarrow{\delta}}{\delta L_m}W, \tag{5.7}$$

with the Heaviside symbol $\theta_{n,k}=\{0\ (n\leq k), 1\ (n>k)\}$, as well as in terms of the EA:

$$\sum_{n\geq 2}\frac{1}{n!}\Lambda^{m}_{A_n\ldots A_1}\left(\frac{\hbar}{i}\right)^{n-1}\left[\prod_{k=1}^{n-2}\left(G''^{-1}\right)^{A_k\mathsf{a}_k}\frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}_k}}\left(G''^{-1}\right)^{A_{n-1}A_n}\right.$$
$$+\theta_{n,2}n\frac{i}{\hbar}\phi^{A_1}\left(\prod_{k=2}^{n-2}\left(G''^{-1}\right)^{A_k\mathsf{a}_k}\frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}_k}}\left(G''^{-1}\right)^{A_{n-1}A_n}\right)$$
$$\left.+\ldots+n\left(G''^{-1}\right)^{A_1A_2}\left(\frac{i}{\hbar}\right)^{n-2}\left(\prod_{k=3}^{n-3}\left(G''^{-1}\right)^{A_k\mathsf{a}_k}\frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}_k}}\right)\left(G''^{-1}\right)^{A_{n-1}A_n}\right]=\Sigma^m. \tag{5.8}$$

From (5.3), we then find a representation for the one-loop approximation,

$$\Gamma^{(1)}\left(B,\Sigma,\phi,\phi^*\right)-\Gamma_{,m}\Sigma^m=\frac{i}{2}\mathrm{sTr}\ \ln\left[S''_{\mathrm{ext}}\left(\phi,\phi^*,B\right)-\Gamma_{,m}(\sigma^m)''\left(\phi^A,B\right)\right]|_{\widetilde{\phi}=0}\ , \tag{5.9}$$

being a functional Clairaut type equation (see [69] for details), with the variables ϕ, ϕ^*, B treated as parameters. The equation (5.9) can be solved as follows:

$$2\Gamma^{(1)}\left(B,\Sigma,\phi,\phi^*\right)=\mathrm{sTr}\ \ln\left[(\Sigma^m\Upsilon_m)^{ab}\bar{S}''_{\mathrm{ext}|bc}\right]-i\mathrm{sTr}\ \ln\left[(\Sigma^m\Upsilon_m)^{ab}\right]+i\mathrm{sTr}\ \ln\left(\bar{S}''_{\mathrm{ext}|\beta\gamma}\right)-i\delta(0)(n_+-n_-), \tag{5.10}$$

using a division of the discrete part of indices $(A;B)=(x,a,\alpha;y,b,\beta)$, $m=(x,\tilde{m})$, in the form $a,b=1,\ldots,n=(n_+,n_-)$ and $\alpha,\beta=n+1,\ldots,N=(N_+,N_-)$, with the property $\tilde{m}=1,\ldots,\frac{1}{2}n(n+1)\leq\frac{1}{2}N(N+1)$ implied by the supermatrices $(\sigma^m)''{}_{cd}$, as we impose on the supermatrices $(\Upsilon_m)^{ab}$ in (5.10) the following condition:

$$(\Upsilon_m)^{ab}(\sigma^m)''{}_{cd}\left(\phi,B\right)|_{\widetilde{\phi}=0}=\frac{1}{2}\left(\delta^a_c\delta^b_d+\delta^b_c\delta^a_d\right). \tag{5.11}$$

The supermatrices $\bar{S}''_{\mathrm{ext}|bc}$ and $\bar{S}''_{\mathrm{ext}|\beta\gamma}$ are given by

$$\bar{S}''_{\mathrm{ext}|bc}(x,y)=S''_{\mathrm{ext}|bc}(x,y)-\int dz\ dz'\ S''_{\mathrm{ext}|b\gamma}(x,z)\left(S''^{-1}_{\mathrm{ext}}\right)^{\gamma\delta}(z,z')S''_{\mathrm{ext}|\delta c}(z',y),$$
$$S''_{\mathrm{ext}|BC}=\frac{\overrightarrow{\delta}}{\delta\phi^C}S_{\mathrm{ext}}\frac{\overleftarrow{\delta}}{\delta\phi^B}=\begin{pmatrix}S''_{\mathrm{ext}|bc} & S''_{\mathrm{ext}|b\gamma}\\ S''_{\mathrm{ext}|\beta c} & S''_{\mathrm{ext}|\beta\gamma}\end{pmatrix}(x,y),\quad (B;C)=(b,\beta,x;c,\gamma,y)\,. \tag{5.12}$$

For $A=(x,a)$, the third term in (5.10) is vanishing, $n=N$, with $S''_{\mathrm{ext}|BC}\equiv S''_{\mathrm{ext}|bc}=\bar{S}''_{\mathrm{ext}|bc}$, as in [69], albeit for a model featuring gauge invariance.

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[6]The dots "..." in (5.6) stand for a number of terms containing more than two derivatives $\frac{\overrightarrow{\delta}}{\delta J_{A_k}}$ entering as multipliers. These contributions are related to the terms in (5.7), (5.8) which are also indicated by dots.

6 Gribov–Zwanziger Theory

Let us extend the case of background fields to the concept of Gribov horizon [37], implemented in the Gribov–Zwanziger model [38] by using a composite field. We propose three descriptions for the Gribov horizon introduction. To do so, we consider a Euclidean form[7] of the Faddeev–Popov action $S_{\mathrm{FP}}(\phi)$ for a Yang–Mills theory (2.3), (2.5), (2.6) in Landau gauge $\chi^p_{\mathrm{L}}(\phi) = \partial^\mu A^p_\mu$ and examine a non-local horizon functional $H(A)$,

$$H(A) = \gamma^2 \int d^D x \left[\int d^D y \; f^{prt} g A^r_\mu(x) \left(K^{-1}\right)^{pq}(x;y) \, f^{qst} g A^{s|\mu}(y) + D\left(N^2-1\right) \right], \tag{6.1}$$

where K^{-1} is the inverse,

$$\int d^D z \left(K^{-1}\right)^{pr}(x;z)\,(K)^{rq}(z;y) = \int d^D z \,(K)^{pr}(x;z)\left(K^{-1}\right)^{rq}(z;y) = \delta^{pq}\delta(x-y)\,, \tag{6.2}$$

of the Faddeev–Popov operator K in terms of the gauge condition $\partial^\mu A^p_\mu = 0$,

$$K^{pq}(x;y) = \left(\delta^{pq}\partial^2 + g f^{prq} A^r_\mu \partial^\mu\right)\delta(x-y)\,, \quad K^{pq}(x;y) = K^{qp}(y;x)\,, \tag{6.3}$$

and γ is a Gribov thermodynamic parameter [38]. The latter is introduced in a self-consistent way by solving a gap equation (horizon condition) for a Gribov–Zwanziger action $S_{\mathrm{GZ}} = S_{\mathrm{GZ}}(\phi)$,

$$\frac{\partial E_{\mathrm{vac}}}{\partial\gamma} = 0, \quad \exp\left(-\hbar^{-1}E_{\mathrm{vac}}\right) \equiv \int d\phi \; \exp\left(-\hbar^{-1}S_{\mathrm{GZ}}\right),$$

where E_{vac} is the vacuum energy, and the action S_{GZ} is given by

$$S_{\mathrm{GZ}}(\phi) = S_{\mathrm{FP}}(\phi) - H(A)\,. \tag{6.4}$$

A generating functional of Green's functions $Z_H(J,L)$ with composite fields for the quantum theory in question can be presented in terms of a Faddeev–Popov action shifted by a constant value, $S_{\mathrm{FP}}(\phi) - H(0)$,

$$\begin{aligned} Z_H(J,L) &= \left. Z_H(J,\mathcal{L})\right|_{L_0=1}, \quad \mathcal{L}_{\mathsf{M}} = (L_0, L_m)\,, \\ Z_H(J,\mathcal{L}) &= \int d\phi \; \exp\left\{-\hbar^{-1}\left[S_{\mathrm{FP}}(\phi) - H(0) + J_A\phi^A + \mathcal{L}_{\mathsf{M}}\sigma^{\mathsf{M}}(A)\right]\right\}, \end{aligned} \tag{6.5}$$

where $\mathcal{L}_{\mathsf{M}} = (L_0, L_m)(x)$, $\epsilon(L_0) = \mathrm{gh}(L_0) = 0$, are sources to composite fields $\sigma^{\mathsf{M}}(A) = (\sigma^0, \sigma^m)(A)$, and $\sigma^0(A) \equiv \sigma(A)$ is a non-local field,

$$\sigma(A)(x) = \gamma^2 \int d^D y \; f^{trp} g A^r_\mu(x)\,(\tilde{K}^{-1})^{pq}(x;y)\, f^{qst} g A^{s|\mu}(y)\,, \tag{6.6}$$

with $\tilde{K}^{-1}$ being the inverse,

$$\int d^D z \; (\tilde{K}^{-1})^{pr}(x;z)\,(\tilde{K})^{rq}(z;y) = \int d^D z \; (\tilde{K})^{pr}(x;z)\,(\tilde{K}^{-1})^{rq}(z;y) = \delta^{pq}\delta(x-y)\,, \tag{6.7}$$

[7] We use the metric signature $\eta_{\mu\nu} = (-,+,\ldots,+)$ and carry out a Wick rotation, $x^0 \to ix^0$, $A^{p|0} \to iA^{p|0}$, $S_{\mathrm{FP}} \to iS_{\mathrm{FP}}$. In Euclidean metric, $A_\mu = A^\mu$, we maintain the summation convention $A_\mu B_\mu = A_\mu B^\mu$.

of an operator $\tilde{K}$ defined for a quantity $F^p = F^p(x)$,

$$\int d^D y \; (\tilde{K})^{pq}(x;y)\, F^q(y) \equiv [\partial_\mu, [D^\mu(A), F]]^p(x)\,, \quad A_\mu(x) = T^p A^p_\mu(x)\,, \quad F(x) = T^p F^p(x)\,, \tag{6.8}$$

which results in

$$\tilde{K}^{pq}(x;y) = \partial^\mu D^{pq}_\mu(A)\,\delta(x-y)\,, \tag{6.9}$$

and therefore reduces to the operator K of (6.3) as one takes into account the Landau gauge condition, due to $S_{\mathrm{FP}}(\phi)$, in the path integral (6.5). Note in conclusion that one cannot absorb the constant term $H(0)$ into $\sigma(A)(x)$ while preserving the basic definition (2.2) for composite fields. Besides, it should be noted that the horizon and therefore also the field $\sigma(A)(x)$ in (6.6) are not BRST-invariant.

6.1 Background Horizon Term

Let us now extend the generating functional $Z_H(J,\mathcal{L})$ with a non-local composite field (6.5), (6.6), (6.7), (6.8) to the case of a background field B_μ equipped with a covariant derivative $D_\mu(B)$ having the gauge properties (2.13), (2.14), by using the approach (1.2), (2.16) as adapted to Euclidean QFT, which implies a modification of derivatives $\partial_\mu \to D_\mu(B)$ in (6.5), according to

$$Z_H(B,J,\mathcal{L}) = \left. Z_H(J,\mathcal{L})\right|_{\partial_\mu \to D_\mu(B)} = \int d\phi \exp\left\{-\hbar^{-1}\left[S_{\mathrm{FP}}(\phi,B) - H(0) + J_A\phi^A + \mathcal{L}_{\mathsf{M}}\sigma^{\mathsf{M}}(A,B)\right]\right\}, \tag{6.10}$$

where $S_{\mathrm{FP}}(\phi,B)$ is the Faddeev–Popov action in the Landau background gauge $\chi^p_{\mathrm{L}}(\phi,B) = 0$, see (2.25), and $\sigma^0(A,B) \equiv \sigma(A,B)$ is a non-local composite field on a background:

$$\sigma(A,B)(x) = \gamma^2 \int d^D y \; f^{trp} g A^r_\mu(x)\, (\tilde{K}^{-1}_B)^{pq}(x;y)\, f^{qst} g A^{s|\mu}(y)\,. \tag{6.11}$$

Here, $\tilde{K}^{-1}_B$ is a modified operator $\tilde{K}^{-1}$ as in (6.2), with the corresponding inverse $\tilde{K}_B$ determined by the replacement $\tilde{K} \to \tilde{K}_B$,

$$\int d^D y \; (\tilde{K})^{pq}_B(x;y)\, F^q(y) \equiv [D_\mu(B), [D^\mu(A+B), F]]^p(x)\,, \quad F(x) = F^p(x) T^p, \tag{6.12}$$

and having the manifest form

$$(\tilde{K})^{pq}_B(x;y) = D^{pr}_\mu(B)\, D^{rq|\mu}(A+B)\,\delta(x-y)\,. \tag{6.13}$$

In the particular case, cf. (6.5),

$$Z_H(B,J,L) = \left. Z_H(B,J,\mathcal{L})\right|_{L_0=1}\,,$$

we arrive at the generating functional

$$Z_H(B,J,L) = \int d\phi \exp\left\{-\hbar^{-1}\left[S_{\mathrm{GZ}}(\phi,B) + J_A\phi^A + L_m\sigma^m(A,B)\right]\right\}, \tag{6.14}$$
$$S_{\mathrm{GZ}}(\phi,B) \equiv S_{\mathrm{FP}}(\phi,B) - H(A,B)\,,$$

with a non-local functional $H(A,B)$ given by

$$H(A,B) = \gamma^2 \int d^D x \left[\int d^D y \; f^{prt} g A^r_\mu(x) \left(K^{-1}_B\right)^{pq}(x;y)\, f^{qst} g A^{s|\mu}(y) + D\left(N^2-1\right)\right], \tag{6.15}$$

where K_B^{-1} is the inverse of an operator K_B as in (6.7) which is identical with the operator $\tilde{K}_B$ in (6.12) being expressed, due to $S_{\mathrm{FP}}(\phi, B)$ in (6.14), by using the background gauge condition $D_\mu^{pq}(B)\, A^{q|\mu} = 0$ and the properties of f^{pqr}, including the Jacobi identity,

$$K_B(x;y) = \left[\partial^2 + g(\partial_\mu B^\mu) + g(A_\mu + 2B_\mu)\partial^\mu + g^2(A_\mu + B_\mu) B^\mu\right]\delta(x-y)\,, \tag{6.16}$$

where A_μ, B_μ are matrices with the elements $\left(A_\mu^{pq}, B_\mu^{pq}\right) = f^{prq}\left(A_\mu^r, B_\mu^r\right)$, and $K_B(x;y)$ is related to $\tilde{K}_B(x;y)$ in (6.13) by the equality (see Appendix B.1)

$$K_B(x;y) = \tilde{K}_B(x;y) - g\left[D_\mu(B), A^\mu\right]\delta(x-y) = D^\mu(A+B)\, D_\mu(B)\,\delta(x-y)\,. \tag{6.17}$$

The operator K_B is an extension of the original operator K in (6.3) and exhibits the properties

$$K_B|_{B=0} = K, \quad (K_B)^{pq}(x;y) = (K_B)^{qp}(y;x)\,, \tag{6.18}$$

where the latter can be verified by a straightforward calculation:

$$\int d^Dy\ \left[(K_B)^{pq}(x;y) - (K_B)^{qp}(y;x)\right] F^q(y) = g f^{prq}\left[D_\mu^{rs}(B)\, A^{s|\mu}\right] F^q(x) = 0.$$

In view of (6.18), we interpret $S_{\mathrm{GZ}}(\phi, B)$ as a Gribov–Zwanziger action on a background B_μ, with a non-local background horizon term $H(A, B)$ given by (6.15), (6.17).

Since the consideration involves the action functionals $S_{\mathrm{FP}}(\phi)$ and $S_{\mathrm{FP}}(\phi, B)$, which are invariant under respective global $SU(N)$ transformations and localized $SU(N)$ transformations combined with gauge transformations for B_μ^p, it is natural to analyze the behavior of the generating functionals $Z_H(J)$ and $Z_H(B, J)$ with respect to these transformations. For such a purpose, it is convenient to recast $Z_H(B, J)$ in a local form by extending the configuration space along the lines of [70]. Namely, we introduce a set of commuting $(\bar{\varphi}_\mu^{pq}, \varphi_\mu^{pq})$ and anticommuting $(\bar{\omega}_\mu^{pq}, \omega_\mu^{pq})$ auxiliary fields, where $\bar{\varphi}_\mu^{pq}$ and φ_μ^{pq} are mutually complex-conjugate,

	$\bar{\varphi}_\mu^{pq}$	φ_μ^{pq}	$\bar{\omega}_\mu^{pq}$	ω_μ^{pq}
ϵ	0	0	1	1
gh	0	0	-1	1

This allows one to construct the parameterization

$$\exp\left\{\hbar^{-1}\left[H(A,B) - H(0,B)\right]\right\} = \int d\bar{\varphi}\ d\varphi\ d\bar{\omega}\ d\omega \exp\left[-\hbar^{-1} S_\gamma(A, B; \bar{\varphi}\ \varphi, \bar{\omega}, \omega)\right], \tag{6.19}$$

where

$$S_\gamma = \int d^Dx\ \left[-\bar{\varphi}_\mu^{rp} K_B^{pq} \varphi^{rq|\mu} + \bar{\omega}_\mu^{rp} K_B^{pq} \omega^{rq|\mu} + i\gamma g\ f^{prq} A^{r|\mu}\left(\bar{\varphi}_\mu^{pq} + \varphi_\mu^{pq}\right)\right], \tag{6.20}$$

as we imply

$$K_B^{pq}\varphi_\mu^{rq}(x) = \int d^Dy\ K_B^{pq}(x,y)\,\varphi_\mu^{rq}(y)\,, \quad K_B^{pq}\omega_\mu^{rq}(x) = \int d^Dy\ K_B^{pq}(x,y)\,\omega_\mu^{rq}(y)\,.$$

The auxiliary fields are regarded as BRST doublets [70], so that the Slavnov variation, being in our case the operator $\overleftarrow{s}_{\mathrm{q}}$ of (2.24), can be extended as follows:

$$\left(\bar{\varphi}_\mu^{pq}, \varphi_\mu^{pq}, \bar{\omega}_\mu^{pq}, \omega_\mu^{pq}\right)\overleftarrow{s}_{\mathrm{q}} = \left(0, \omega_\mu^{pq}, -\bar{\varphi}_\mu^{pq}, 0\right).$$

These transformations, however, do not provide invariance for the functional S_γ:

$$S_\gamma \overleftarrow{s}_{\mathrm{q}} \neq 0.$$

In the extended configuration space $\Phi = (\phi, \bar{\varphi},\ \varphi, \bar{\omega}, \omega)$, the generating functional $Z_H(B,J)$ given by the restriction $L_m = 0$ in (6.14) acquires the form

$$Z_H(B,J) = \int d\Phi \exp\left\{-\hbar^{-1}\left[S_{\mathrm{GZ}}(\Phi,B) + J_A\phi^A\right]\right\}, \tag{6.21}$$

where the local action $S_{\mathrm{GZ}}(\Phi,B)$ of the Gribov–Zwanziger theory on a background reads (notice the antisymmetry of f^{pqr})

$$\begin{aligned} S_{\mathrm{GZ}}(\Phi,B) &= S_{\mathrm{FP}}(\phi,B) - H(0,B) - i\gamma g \int d^Dx \ \mathrm{Tr}\ A^\mu \left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right) \\ &\quad + \int d^Dx\ d^Dy\ \mathrm{Tr}\left[-\bar{\varphi}^\mu(x) K_B(x,y) \varphi^{\mathrm{T}}_\mu(y) + \bar{\omega}^\mu(x) K_B(x,y)\omega^{\mathrm{T}}_\mu(y)\right]. \end{aligned} \tag{6.22}$$

The action $S_{\mathrm{GZ}}(\Phi,B)$ is invariant under the global $SU(N)$ transformations

$$\left(A_\mu, B_\mu, b, \bar{c}, c, \bar{\varphi}_\mu, \bar{\omega}_\mu, \omega^{\mathrm{T}}_\mu\right) \stackrel{U}{\to} U\left(A_\mu, B_\mu, b, \bar{c}, c, \bar{\varphi}_\mu, \bar{\omega}_\mu, \omega^{\mathrm{T}}_\mu\right) U^{-1}. \tag{6.23}$$

Indeed, due to the unitarity of U, we also have

$$\bar{\varphi}_\mu \stackrel{U}{\to} U\bar{\varphi}_\mu U^{-1} \Longrightarrow \varphi^{\mathrm{T}}_\mu \stackrel{U}{\to} U\varphi^{\mathrm{T}}_\mu U^{-1},$$

and the manifest expression (6.16) for $K_B(x,y)$ consistently implies

$$K_B(x,y) \stackrel{U}{\to} UK_B(x;y)U^{-1} = (K')_B(x;y), \quad (K')_B(x;y) = (K')^{\mathrm{T}}_B(y;x). \tag{6.24}$$

The infinitesimal form of field transformations (6.23), given by

$$\delta_\varsigma F^{pq} = gf^{prs}F^{rq}\varsigma^s + gf^{qrs}F^{pr}\varsigma^s, \quad F^{pq} = \left(A_\mu, B_\mu, b, \bar{c}, c, \bar{\varphi}_\mu, \varphi^{\mathrm{T}}_\mu, \bar{\omega}_\mu, \omega^{\mathrm{T}}_\mu\right)^{pq}, \tag{6.25}$$

produces a unit Jacobian (notice the antisymmetry of f^{pqr}) in the integration measure of (6.21) and leaves invariant the functional $Z_H(B,J)$ under infinitesimal global $SU(N)$ transformations of the background field B_μ and the sources J_A, having the adjoint representation form

$$\delta_\varsigma G^{pq} = gf^{prs}G^{rq}\varsigma^s + gf^{qrs}G^{pr}\varsigma^s, \quad G^{pq} = \left(B_\mu, J_{\mu(A)}, J_{(b)}, J_{(\bar{c})}, J_{(c)}\right)^{pq}. \tag{6.26}$$

This behavior of $Z_H(B,J)$ includes the invariance of the restricted generating functional $Z_H(J)$ under the global $SU(N)$ transformations of the sources and can also be established directly in the non-local form by using the properties of $(K^{-1}_B)(x;y)$,

$$(K^{-1}_B)(x,y) \stackrel{U}{\to} U(K^{-1}_B)(x;y)U^{-1}, \quad \left(K^{-1}\right)_B(x;y) = (K^{-1})^{\mathrm{T}}_B(y;x),$$

implied by (6.2), (6.24) and providing a global $SU(N)$ invariance of the original $H(A) = H(A,0)$ and the background-modified $H(A,B)$ horizon functionals in (6.1), (6.15),

$$(A_\mu, B_\mu) \stackrel{U}{\to} (A_\mu, B_\mu)' = U(A_\mu, B_\mu)U^{-1}, \quad H(A', B') = H(A,B).$$

Notably, it turns out that the global $SU(N)$ invariance of the background functional $Z_H(J,B)$ does not translate itself into a local symmetry. To prove this point, let us subject the integrand in (6.21) to an infinitesimal local change of variables with a unitary matrix $V=V(\xi)$, $\xi=\xi(x)$,

$$\Phi \xrightarrow{V} \Phi', \quad B_\mu \xrightarrow{V} B'_\mu = VB_\mu V^{-1} + g^{-1}V\left(\partial_\mu V^{-1}\right), \tag{6.27}$$

which implies a unit Jacobian and induces a variation $\delta_\xi S_{\mathrm{GZ}}(\Phi,B) = \delta_\xi S_K(\Phi,B)$ in (6.22),

$$S_K(\Phi,B) \equiv \int d^Dx\, d^Dy\, \mathrm{Tr}\left[-\bar{\varphi}^\mu(x)\, K_B(x,y)\, \varphi_\mu^{\mathrm{T}}(y) + \bar{\omega}^\mu(x)\, K_B(x,y)\, \omega_\mu^{\mathrm{T}}(y)\right],$$

due to the parameterization term $S_\gamma(\Phi,B)$ in (6.19), (6.20),

$$S_{\mathrm{GZ}}(\Phi',B') = S_{\mathrm{FP}}(\phi,B) - H(0,B) + S_\gamma(\Phi',B').$$

Using the explicit form of $K_B(x,y)$ given by (6.17), we find[8]

$$S_K(\Phi,B) = \int d^Dx\, \mathrm{Tr}\left[-\bar{\varphi}^\mu D_\nu(A+B)\, D^\nu(B)\, \varphi_\mu^{\mathrm{T}} + \bar{\omega}^\mu D_\nu(A+B)\, D^\nu(B)\, \omega_\mu^{\mathrm{T}}\right], \tag{6.28}$$

so that the presence of extra derivatives $\partial_\mu V^{-1}$ and $\partial^2 V^{-1}$ in the transformed expression

$$\begin{aligned} S_K(\Phi',B') &= \int d^Dx\, \mathrm{Tr}\left[-V\bar{\varphi}^\mu D_\nu(A+B)\, D^\nu(B)\, \varphi_\mu^{\mathrm{T}} V^{-1} + V\bar{\omega}^\mu D_\nu(A+B)\, D^\nu(B)\, \omega_\mu^{\mathrm{T}} V^{-1}\right] \\ &\neq \int d^Dx\, \mathrm{Tr}\left[-V^{-1}V\bar{\varphi}^\mu D_\nu(A+B)\, D^\nu(B)\, \varphi_\mu^{\mathrm{T}} + V^{-1}V\bar{\omega}^\mu D_\nu(A+B)\, D^\nu(B)\, \omega_\mu^{\mathrm{T}}\right], \end{aligned}$$

leads to

$$S_K(\Phi',B') \neq S_K(\Phi,B),$$

which also implies a local non-invariance of the background horizon functional,

$$H(A,B) \xrightarrow{V} H(A',B') \neq H(A,B).$$

As a consequence, we conclude that the background functional $Z_H(B,J)$ is not left invariant by the gauge transformations of the background field B_μ combined with the local $SU(N)$ transformations of the sources J_A, since the latter do not compensate the variation $\delta_\xi S_{\mathrm{GZ}}(\Phi,B)\neq 0$, due to $\delta_\xi\left(J_A\phi^A\right) = 0$. In other words, the global invariance of $Z_H(J)$ is not inherited by a related local symmetry of $Z_H(B,J)$ in the theory (6.14), (6.15), (6.17).

6.2 Locally Invariant Horizon Term

The local non-invariance of the functional $Z_H(B,J)$ can be traced back to the fact that the background B_μ has been incorporated directly into the non-local horizon term via (6.17), whereas the emergence of the auxiliary fields $(\bar{\varphi}_\mu, \varphi_\mu)$ and $(\bar{\omega}_\mu, \omega_\mu)$ as a means of parameterizing the term $H(A,B)$ does not provide them with a covariant derivative in the form (2.16), as one can observe from (6.28). To resolve this issue, it is natural to examine an alternative way of introducing a background, namely, by using a local parameterization of the original term $H(A)$ prior to the point the background has been

[8]One uses a repeated integration by parts and the antisymmetry of f^{pqr} to remove the delta-function $\delta(x-y)$ absorbed in $K_B(x,y)$ and to recast $S_K(\Phi,B)$ in the form (6.28).

incorporated. To do so, we consider the expressions (6.19), (6.20), (6.21), (6.22) restricted to $B_\mu = 0$ and present the functional $Z_H(J)$ in (6.5) as follows:

$$Z_H(J) = \int d\Phi \ \exp\left\{-\hbar^{-1}\left[S_{\mathrm{GZ}}(\Phi) + J_A\phi^A\right]\right\}, \ S_{\mathrm{GZ}}(\Phi) = S_{\mathrm{GZ}}(\Phi, 0), \ \Phi = (\phi, \bar{\varphi}, \ \varphi, \bar{\omega}, \omega).$$

For a treatment of the auxiliary fields $(\bar{\varphi}, \bar{\omega})$ and $\left(\varphi^{\mathrm{T}}, \omega^{\mathrm{T}}\right)$ on equal footing, notice that the action $S_{\mathrm{GZ}}(\Phi)$ in the above integrand, with the Landau gauge condition $\partial_\mu A^\mu = 0$ absorbed in the factor $\exp\left[-\hbar S_{\mathrm{FP}}(\phi)\right]$, is equivalent to an action $\mathcal{S}_{\mathrm{GZ}}(\Phi)$ arising from the replacement of $K(x,y)$ by $\mathcal{K}(x,y)$, defined as a symmetrization:

$$\mathcal{K}(x,y) \equiv \frac{1}{2}\left[K(x,y) + \tilde{K}(x,y)\right], \quad \tilde{K}(x,y) = K(x,y) + g\left[\partial_\mu, A^\mu\right]\delta(x-y). \qquad (6.29)$$

The action $\mathcal{S}_{\mathrm{GZ}}(\Phi)$ reads (see Appendix B.2)

$$\begin{aligned} \mathcal{S}_{\mathrm{GZ}}(\Phi) &= S_{\mathrm{FP}}(\phi) - H(0) + S_{\mathcal{K}}(\Phi) - i\gamma g \int d^Dx \ \mathrm{Tr}\ A^\mu\left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right), \qquad (6.30)\\ S_{\mathcal{K}}(\Phi) &\equiv \frac{1}{2}\int d^Dx \ \mathrm{Tr}\left\{\left[D^\nu(A), \bar{\varphi}^\mu\right]\partial_\nu\varphi^{\mathrm{T}}_\mu + \left(\partial^\nu\bar{\varphi}^\mu\right)\left[D_\nu(A), \varphi^{\mathrm{T}}_\mu\right]\right\} - \left(\bar{\varphi}, \varphi \to \bar{\omega}, \omega\right), \end{aligned}$$

and implies a natural introduction of a background, $\mathcal{S}_{\mathrm{GZ}}(\Phi) \to \mathcal{S}_{\mathrm{GZ}}(\Phi, B)$, in the form (2.16),

$$\begin{aligned} \mathcal{S}_{\mathrm{GZ}}(\Phi, B) &= S_{\mathrm{FP}}(\phi, B) - H(0) + S_{\mathcal{K}}(\Phi, B) - i\gamma g \int d^Dx \ \mathrm{Tr}\ A^\mu\left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right), \qquad (6.31)\\ S_{\mathcal{K}}(\Phi, B) &= \frac{1}{2}\int d^Dx \ \mathrm{Tr}\left\{\left[D^\nu(A+B), \bar{\varphi}^\mu\right]\left[D_\nu(B), \varphi^{\mathrm{T}}_\mu\right] + \left[D^\nu(B), \bar{\varphi}^\mu\right]\left[D_\nu(A+B), \varphi^{\mathrm{T}}_\mu\right]\right\}\\ &\quad - \left(\bar{\varphi}, \varphi \to \bar{\omega}, \omega\right) \equiv \int d^Dx \ \left(-\bar{\varphi}^{pq}_\mu \mathcal{K}^{pq|rs}_B \varphi^{rs|\mu} + \bar{\omega}^{pq}_\mu \mathcal{K}^{pq|rs}_B \omega^{rs|\mu}\right). \end{aligned}$$

Using the notation

$$\int d^Dx \ F^{pq}\mathcal{K}^{pq|rs}_B G^{rs} \equiv \int d^Dx \ d^Dy \ F^{pq}(x)\,\mathcal{K}^{pq|rs}_B(x;y)\,G^{rs}(y), \quad \epsilon(F) = \epsilon(G),$$

for the expression

$$-\frac{1}{2}\int d^Dx \ \mathrm{Tr}\left\{\left[D_\mu(A+B), F\right]\left[D^\mu(B), G\right] + \left[D_\mu(B), F\right]\left[D^\mu(A+B), G\right]\right\},$$

we find, due to the (anti)symmetry of the latter under $F \leftrightarrow G$, the following property:

$$\mathcal{K}^{pq|rs}_B(x;y) = \mathcal{K}^{rs|pq}_B(y;x).$$

Thereby, we interpret $\mathcal{S}_{\mathrm{GZ}}(\Phi, B)$ in (6.31) as an alternative local Gribov–Zwanziger action on the background B_μ, with the corresponding background horizon functional $\mathcal{H}(A, B)$ given by

$$\exp\left\{\hbar^{-1}\left[\mathcal{H}(A,B) - \mathcal{H}(0,B)\right]\right\} = \int d\bar{\varphi}\ d\varphi\ d\bar{\omega}\ d\omega \exp\left[-\hbar^{-1}\mathcal{S}_\gamma(\Phi, B)\right], \quad \mathcal{H}(0,B) \equiv H(0), \quad (6.32)$$

where

$$\begin{aligned} \mathcal{S}_\gamma(\Phi, B) &= S_{\mathcal{K}}(\Phi, B) - i\gamma g \int d^Dx \ \mathrm{Tr}\ A^\mu\left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right)\\ &= \int d^Dx \ \left[-\bar{\varphi}^{pq}_\mu \mathcal{K}^{pq|rs}_B \varphi^{rs|\mu} + \bar{\omega}^{pq}_\mu \mathcal{K}^{pq|rs}_B \omega^{rs|\mu} + i\gamma g \ f^{prq}A^{r|\mu}\left(\bar{\varphi}^{pq}_\mu + \varphi^{pq}_\mu\right)\right]. \end{aligned}$$

The action $\mathcal{S}_{\mathrm{GZ}}(\Phi, B)$ in (6.31) is manifestly invariant with respect to the local transformations (6.27), which produces a unit Jacobian in the infinitesimal case and implies an invariance of the background generating functional

$$Z_{\mathcal{H}}(B, J) = \int d\Phi \ \exp\left\{-\hbar^{-1}\left[\mathcal{S}_{\mathrm{GZ}}(\Phi, B) + J_A \Phi^A\right]\right\}, \quad Z_{\mathcal{H}}(0, J) = Z_H(J), \tag{6.33}$$

under the following local transformations of the sources and the background field:

$$\delta_\xi B^p_\mu = D^{pq}_\mu(B)\,\xi^q, \quad \delta_\xi (J_{\mu(A)}, J_{(b)}, J_{(\bar{c})}, J_{(c)})^p = g f^{prq} (J_{\mu(A)}, J_{(b)}, J_{(\bar{c})}, J_{(c)})^r \xi^q. \tag{6.34}$$

This also means a local invariance of the alternative horizon functional $\mathcal{H}(A, B)$ in (6.32),

$$\delta_\xi \mathcal{H}(A, B) = 0, \quad \delta_\xi B^p_\mu = D^{pq}_\mu(B)\,\xi^q, \quad \delta_\xi A^p_\mu = g f^{prq} A^r_\mu \xi^q.$$

As we introduce the generating functionals of connected $W_{\mathcal{H}}(B, J)$ and vertex $\Gamma_{\mathcal{H}}(B, \phi)$ Green's functions on a background,

$$Z_{\mathcal{H}} = \exp\left(-\hbar^{-1} W_{\mathcal{H}}\right), \quad \Gamma_{\mathcal{H}}(B, \phi) = W_{\mathcal{H}}(B, J) - J_A \phi^A, \quad \phi^A = \frac{\overrightarrow{\delta}}{\delta J_A} W_{\mathcal{H}}, \quad J_A = -\Gamma_{\mathcal{H}} \frac{\overleftarrow{\delta}}{\delta \phi^A}, \tag{6.35}$$

the invariance of $Z_{\mathcal{H}}(B, J)$ with respect to (6.34) can be recast as the invariance of $\Gamma_{\mathcal{H}}(B, \phi)$ under the following local transformations, $\delta_\xi \Gamma_{\mathcal{H}} = 0$ (see Appendix B.3),

$$\delta_\xi B^p_\mu = D^{pq}_\mu(B)\,\xi^q, \quad \delta_\xi (A_\mu, b, \bar{c}, c)^p = g f^{prq} (A_\mu, b, \bar{c}, c)^r \xi^q, \tag{6.36}$$

which consist of the gauge transformations for the background field B_μ and of the local $SU(N)$ transformations for the quantum fields ϕ^A. These symmetry properties are readily generalized to the case of extended functionals $Z_{\mathcal{H}}(B, \mathfrak{J})$, $W_{\mathcal{H}}(B, \mathfrak{J})$, $\Gamma_{\mathcal{H}}(B, \Phi)$, where $\mathfrak{J}$ are sources to the fields Φ, with the invariance $\delta_\xi Z_{\mathcal{H}} = \delta_\xi W_{\mathcal{H}} = \delta_\xi \Gamma_{\mathcal{H}} = 0$ under the gauge transformations of B_μ combined with the local $SU(N)$ transformations of $\mathfrak{J}$ or Φ, so that the background effective action $\Gamma_{\text{eff}}(B)$ for the Gribov–Zwanziger model defined as

$$\Gamma_{\text{eff}}(B) = \Gamma_{\mathcal{H}}(B, \Phi)\big|_{\Phi=0} \tag{6.37}$$

is invariant, $\delta_\xi \Gamma_{\text{eff}} = 0$, under the gauge transformations of the background field B_μ.

6.3 Local BRST Invariant Horizon Term

By considering a gauge-invariant horizon $H\left(A^{\mathrm{h}}\right) = H(A)\big|_{A=A^{\mathrm{h}}}$ of [53], involving non-local transverse fields $A^{\mathrm{h}}_\mu = (A^{\mathrm{h}})^p_\mu T^p$, the case of the background term becomes simplified using gauge- and BRST-invariant fields A^{h}_μ, defined[9] according to [54], $A_\mu = A^{\mathrm{h}}_\mu + A^{\mathrm{L}}_\mu$,

$$A^{\mathrm{h}}_\mu = \left(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\right)\left(A^\nu - g\left[\frac{\partial A}{\partial^2}, A^\nu - \frac{1}{2}\partial^\nu \frac{\partial A}{\partial^2}\right]\right) + O(A^3), \quad A^{\mathrm{h}}_\mu \overleftarrow{s} = 0, \tag{6.38}$$

$$H(A) = H(A^{\mathrm{h}}) + \gamma^2 \int d^D x\, d^D y\ R^p(x, y) \partial^\mu A^p_\mu(y), \quad H(A^{\mathrm{h}}) \overleftarrow{s} = 0, \tag{6.39}$$

[9]In (6.38), we maintain the notation for the interaction constant g consistent with (2.7). The same is implied in (6.43) below.

with a non-local function $R^p(x,y)$ of [53]. Due to this structure, the second term in $H(A)$ can be added to the gauge-fixing term $b^p\partial^\mu A^p_\mu$ of the Faddeev–Popov action $S_0+\Psi\overleftarrow{s}$ in a way reduced to a change of variables in $Z_{H,\Psi}$ given by the shift $b^p\to b^p+\gamma^2R^p$ with a unit Jacobian, which entirely removes the dependence on the BRST symmetry breaking term entering $Z_{H,\Psi}$. Thereby, the action

$$\tilde{S}_{\mathrm{GZ}}(\phi)=S_0+\int d^Dx\ (\bar{c}^p\partial^\mu A^p_\mu)\overleftarrow{s}+H(A^{\mathrm{h}}),\quad \phi^A\overleftarrow{s}=(D^{pq}_\mu c^q,0,b^p,g/2f^{pqr}c^qc^r), \tag{6.40}$$

provides independence under R_ξ-gauges in the YM theory and the Standard Model [47, 55], with the Faddeev–Popov operator $(K)^{pq}(x,y)$ being unaltered and the BRST symmetry unaffected, in which case one may expect the theory to be unitary in the framework of the Faddeev–Popov quantization rules [60]. The same results concerning the issues of unitarity and gauge-independence can be presented using the above fields A^{h}_μ in a description of the Gribov–Zwanziger theory when the horizon functional is localized [38, 70] using a quartet of auxiliary fields $\phi_{\mathrm{aux}}=(\bar{\varphi}^{pq}_\mu,\varphi^{pq}_\mu,\bar{\omega}^{pq}_\mu,\omega^{pq}_\mu)$ having opposite Grassmann parities, $\epsilon(\varphi,\bar{\varphi})=\epsilon(\omega,\bar{\omega})+1=0$. Using the previously employed parameterization [70] of the gauge-invariant horizon $H\left(A^{\mathrm{h}}\right)$ in terms of ϕ_{aux}, namely, by setting $B_\mu=0$ and replacing $K^{pq}_B(A)\to K^{pq}(A^{\mathrm{h}})$ in (6.19), (6.20), we have

$$\begin{aligned} S_{\mathrm{GZ}}(\phi,\phi_{\mathrm{aux}}) &= S_0(A)+\int d^Dx(\bar{c}^p\partial^\mu A^p_\mu)\overleftarrow{s}+\tilde{S}_\gamma(A^{\mathrm{h}},\phi_{\mathrm{aux}}), &(6.41)\\ \tilde{S}_\gamma(A^{\mathrm{h}},\phi_{\mathrm{aux}}) &= \int d^Dx\left[-\bar{\varphi}^{rp}_\mu K^{pq}(A^{\mathrm{h}})\varphi^{rq|\mu}+\bar{\omega}^{rp}_\mu K^{pq}(A^{\mathrm{h}})\omega^{rq|\mu}\right.\\ &\qquad \left.+\,i\gamma g\ f^{prq}(A^{\mathrm{h}})^{r|\mu}\left(\bar{\varphi}^{pq}_\mu+\varphi^{pq}_\mu\right)+\gamma^2D(N^2-1)\right]. &(6.42)\end{aligned}$$

The part $\tilde{S}_\gamma$ additional to the Faddeev–Popov action is manifestly invariant under the BRST transformations (2.4) combined with a trivial form of BRST transformations for the auxiliary fields, $\phi_{\mathrm{aux}}\overleftarrow{s}=0$, suggested for the first time in [55].

Despite a formally localized description, the Gribov–Zwanziger (GZ) action $S_{\mathrm{GZ}}(\phi,\phi_{\mathrm{aux}})$ in (6.41) remains in fact non-local due to the presence of the non-local field A^{h}_μ. To render the action local, we use a parameterization in terms of a Stueckelberg-like field ζ^p introduced in [56] with the help of a matrix-valued field h^{pq} defined by

$$A^{\mathrm{h}}_\mu=g^{-1}hD_\mu(A)h^{-1}=hA_\mu h^{-1}+g^{-1}h\partial_\mu h^{-1},\quad h=\exp\left(-g\zeta^pT^p\right), \tag{6.43}$$

and subject to the transversality condition, implying (6.38),

$$\partial^\mu A^{\mathrm{h}}_\mu=0. \tag{6.44}$$

Given this, a completely local and BRST-invariant GZ action can be determined in an extended configuration space parameterized by the fields

$$\mathbf{\Phi}^{\mathcal{A}}=\left(A^{p|\mu},b^p,\bar{c}^p,c^p;\bar{\varphi}^{pq|\mu},\varphi^{pq|\mu},\bar{\omega}^{pq|\mu},\omega^{pq|\mu};h^{pq}(\zeta),\tau^p,\bar{\eta}^p,\eta^p\right), \tag{6.45}$$

where $\epsilon\left(\bar{\eta}\right)=\epsilon\left(\eta\right)=\epsilon\left(\tau\right)+1=1$, and has the form

$$\begin{aligned} S_{\mathrm{loc,GZ}}(\phi,\phi_{\mathrm{aux}}) &= S_{\mathrm{FP}}(\phi)-H(0)+\bar{S}_\gamma(A^{\mathrm{h}},\phi_{\mathrm{aux}},h(\zeta),\tau,\bar{\eta},\eta), &(6.46)\\ \bar{S}_\gamma &= \tilde{S}_\gamma+\int d^Dx\left[\tau^p\partial^\mu(A^{\mathrm{h}})^p_\mu-\bar{\eta}^pK^{pq}(A^{\mathrm{h}})\eta^q\right]. &(6.47)\end{aligned}$$

A generating functional for the local BRST-invariant horizon is then given by

$$Z_{\mathrm{loc},H}\left(J\right) = \int d\Phi\ d\Phi_{\mathrm{loc}}\ \exp\left\{-\hbar^{-1}\left[S_{\mathrm{loc,GZ}}\left(\Phi\right) + J_A\phi^A\right]\right\}, \quad d\Phi_{\mathrm{loc}} = \left(d\zeta, d\tau, d\bar{\eta}, d\eta\right) \tag{6.48}$$

(which is readily extended, along the lines of (6.14), to a generating functional $Z_{\mathrm{loc},H}\left(J, L\right)$ with local composite fields), with the integrand being invariant under the following BRST transformations:

$$\mathbf{\Phi}^{\mathcal{A}}\overleftarrow{s} = \left(D^{pq|\mu}\left(A\right)c^q, 0, b^p, g/2 f^{pqr}c^q c^r; 0, 0, 0, 0; gc^s (T^s)^{pr} h^{rq}, 0, 0, 0\right). \tag{6.49}$$

In a matrix form, the transformation $\delta h = h(\zeta)\overleftarrow{s}\lambda$ can be presented in terms of the field ζ as follows:

$$g\delta\zeta = -gc\lambda + \mathcal{Z}|_{\delta\zeta=-c\lambda} \equiv gj(\zeta)c\lambda, \quad \delta\zeta = \zeta\overleftarrow{s}\lambda, \tag{6.50}$$

where $\mathcal{Z}$ is given by the Baker–Campbell–Hausdorff formula

$$\mathcal{Z} = \frac{1}{2}\left[\mathcal{X}, \mathcal{Y}\right] + \frac{1}{12}\left\{\left[\mathcal{X}, \left[\mathcal{X}, \mathcal{Y}\right]\right] + \left[\mathcal{Y}, \left[\mathcal{Y}, \mathcal{X}\right]\right]\right\} + \cdots, \quad \exp\mathcal{X}\exp\mathcal{Y} = \exp\left(\mathcal{X} + \mathcal{Y} + \mathcal{Z}\right),$$

corresponding to the explicit values

$$\mathcal{X} = -g\left(\zeta + \delta\zeta\right), \quad \mathcal{Y} = g\zeta, \quad \left[\mathcal{X}, \mathcal{Y}\right] = g^2\left[\zeta, \delta\zeta\right].$$

The expression (6.50) for $\delta\zeta$ presents $\zeta^p\overleftarrow{s}$ as an expansion in powers of g, which is also an explicit power series in ζ^p. For instance, the linear approximation has the form

$$\zeta^p\overleftarrow{s} = j^{pq}(\zeta)c^q, \quad j^{pq} = -\delta^{pq} + \frac{g}{2}f^{prq}\zeta^r + O(g^2).$$

The BRST-invariant GZ theory with the action (6.46) can be naturally extended to a background-dependent GZ action $\mathcal{S}_{\mathrm{loc,GZ}}\left(\mathbf{\Phi}, B\right)$, along the lines of the representation (6.31),

$$\begin{aligned}
\mathcal{S}_{\mathrm{loc,GZ}}\left(\mathbf{\Phi}, B\right) &= S_{\mathrm{FP}}\left(\phi, B\right) - H\left(0\right) + S_{\mathrm{loc},\mathcal{K}}\left(\mathbf{\Phi}, B\right) - i\gamma g\int d^Dx\ \mathrm{Tr}\ A^{\mathrm{h}}_\mu\left(\bar{\varphi}^\mu - \varphi^{\mu\mathrm{T}}\right), \qquad (6.51)\\
S_{\mathrm{loc},\mathcal{K}}\left(\mathbf{\Phi}, B\right) &= \frac{1}{2}\int d^Dx\ \mathrm{Tr}\left\{\left[D^\nu\left(A^{\mathrm{h}} + B\right), \bar{\varphi}^\mu\right]\left[D_\nu\left(B\right), \varphi^{\mathrm{T}}_\mu\right] + \left[D^\nu\left(B\right), \bar{\varphi}^\mu\right]\left[D_\nu\left(A^{\mathrm{h}} + B\right), \varphi^{\mathrm{T}}_\mu\right]\right\}\\
&\quad - \left(\bar{\varphi}, \varphi^{\mathrm{T}} \to \bar{\omega}, \omega^{\mathrm{T}}\right) + 2\left(\bar{\varphi}, \varphi^{\mathrm{T}} \to \bar{\eta}, \eta\right) + 2\int d^Dx\ \mathrm{Tr}\ \tau\left[D^\mu(B), A^{\mathrm{h}}_\mu\right]\\
&\equiv \int d^Dx\left[-\bar{\varphi}^{pq}_\mu\mathcal{K}^{pq|rs}_{\mathrm{loc},B}\varphi^{rs|\mu} + \bar{\omega}^{pq}_\mu\mathcal{K}^{pq|rs}_{\mathrm{loc},B}\omega^{rs|\mu} - \bar{\eta}^p\mathcal{K}^{pq}_{\mathrm{loc},B}\eta^q + \tau^p D^{pq}_\mu(B)(A^{\mathrm{h}})^{q|\mu}\right].
\end{aligned}$$

This action is background-invariant, including the corresponding generating functional of Green's functions,

$$\begin{aligned}
Z_{\mathrm{loc},H}\left(B, \mathbf{J}, L, \mathbf{\Phi}^*\right) &= \int d\Phi\ d\Phi_{\mathrm{loc}}\exp\left\{-\hbar^{-1}\left[S_{\mathrm{loc,GZ}}\left(\mathbf{\Phi}, B\right) + \zeta^{*p}j^{pq}(\zeta)c^q\right.\right.\\
&\quad\left.\left. + \mathbf{J}_{\mathcal{A}}\mathbf{\Phi}^{\mathcal{A}} + L_m\sigma^m\left(\Phi, B\right)\right]\right\}, \quad Z_{\mathrm{loc},H}\left(0, \mathbf{J}, 0, 0\right)|_{\mathbf{J}=J} = Z_{\mathrm{loc},H}\left(J\right), \qquad (6.52)
\end{aligned}$$

so that an effective action $\Gamma_{\mathrm{loc,eff}}\left(B, \Sigma\right)$ for the GZ theory featuring a local BRST-invariant horizon with background and composite fields,

$$\Gamma_{\mathrm{loc,eff}}\left(B, \Sigma\right) = \Gamma_{\mathrm{loc}}\left(B, \mathbf{\Phi}, \Sigma, \mathbf{\Phi}^*\right)|_{\mathbf{\Phi}=\mathbf{\Phi}^*=0}, \tag{6.53}$$

proves to be invariant under the local transformations (3.26). This is a first main result of the present subsection.

For the generating functionals of Green's functions $Z_{\mathrm{loc},H}$ and $W_{\mathrm{loc},H}$, related by $Z_{\mathrm{loc},H} = e^{-\hbar^{-1}W_{\mathrm{loc},H}}$ and depending on $(B, \mathbf{J}, L, \boldsymbol{\Phi}^*)$ in (6.52), as well as for the effective action $\Gamma_{\mathrm{loc}}(B, \boldsymbol{\Phi}, \Sigma, \boldsymbol{\Phi}^*)$ in (6.53) obtained by a Legendre transform of $W_{\mathrm{loc},H}$ along the lines of (3.18), we can derive modified Ward identities in the respective forms (3.8), (3.13), (3.19), as well as the usual Ward identities (3.12), (3.17), (3.22), with appropriate Grassmann-odd operators $\hat{\omega}_H$, $\hat{\Omega}_H$, $\hat{\omega}_{\Gamma,H}$. These identities are deduced starting from the FD BRST transformations (6.49), $\Delta\Phi^{\mathcal{A}} = \Phi^{\mathcal{A}}\overleftarrow{s}\lambda(\Phi)$, with a Grassmann-odd FD functional $\lambda(\Phi)$, further background-extended as $\overleftarrow{s} \to \overleftarrow{s}_{\mathrm{q}}$, $\lambda(\Phi) \to \lambda(\Phi, B)$. The operators $\hat{\omega}_H$, $\hat{\Omega}_H$, $\hat{\omega}_{\Gamma,H}$ are constructed as their counterparts $\hat{\omega}$, $\hat{\Omega}$, $\hat{\omega}_\Gamma$ of (3.9), (3.14), (3.20), albeit with a GZ action $S_{\mathrm{loc,GZ}}$ defined in a space of variables which is larger than that for the Faddeev–Popov action S_{FP}. For instance, the operator $\hat{\omega}_H$ is given by

$$\hat{\omega}_H = \left[\mathbf{J}_{\mathcal{A}} + \delta_{\mathcal{A}\mathcal{A}} L_m \sigma^m_{,\mathcal{A}}\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J}, B\right)\right]\frac{\overrightarrow{\delta}}{\delta \boldsymbol{\Phi}^*_{\mathcal{A}}}, \quad \hat{\omega}_H^2 = 0. \tag{6.54}$$

A study of the gauge-dependence problem following the receipt of Section 4 leads to the representations (4.1), (4.2), (4.3) for the respective finite variations $\Delta Z^{\mathrm{loc}}_{\Psi} \equiv \Delta Z_{\mathrm{loc},H}(B, \mathbf{J}, L, \boldsymbol{\Phi}^*)$, $\Delta W^{\mathrm{loc}}_{\Psi} \equiv \Delta W_{\mathrm{loc},H}(B, \mathbf{J}, L, \boldsymbol{\Phi}^*)$, $\Delta\Gamma^{\mathrm{loc}}_{\Psi} \equiv \Delta\Gamma_{\mathrm{loc},H}(B, \boldsymbol{\Phi}, L, \boldsymbol{\Phi}^*)$ generated by finite variations of the gauge Fermion $\Delta\Psi$, so that $\Delta Z^{\mathrm{loc}}_{\Psi} = Z^{\mathrm{loc}}_{\Psi+\Delta\Psi} - Z^{\mathrm{loc}}_{\Psi}$,

$$\begin{aligned}
\Delta Z^{\mathrm{loc}}_{\Psi} &= \hat{\omega}_H \lambda\left(\frac{\hbar\overrightarrow{\delta}}{i\delta\mathbf{J}}, B\,|-\Delta\Psi\right) Z^{\mathrm{loc}}_{\Psi} = \frac{i}{\hbar}\hat{\omega}_H \Delta\Psi\left(\frac{\hbar\overrightarrow{\delta}}{i\delta\mathbf{J}}, B\right) Z^{\mathrm{loc}}_{\Psi} + o(\Delta\Psi), && (6.55)\\
\Delta W^{\mathrm{loc}}_{\Psi} &= \frac{\hbar}{i}\hat{\Omega}_H \lambda\left(\frac{\hbar\overrightarrow{\delta}}{i\delta\mathbf{J}} + \frac{\overrightarrow{\delta} W^{\mathrm{loc}}}{\delta\mathbf{J}}, B\,|-\Delta\Psi\right) = \hat{\Omega}_H \Delta\Psi\left(\frac{\hbar\overrightarrow{\delta}}{i\delta\mathbf{J}} + \frac{\overrightarrow{\delta} W^{\mathrm{loc}}}{\delta\mathbf{J}}, B\right) + o(\Delta\Psi), && (6.56)\\
\Delta\Gamma^{\mathrm{loc}}_{\Psi} &= \frac{\hbar}{i}\hat{\omega}_\Gamma\langle\langle\lambda(B| - \Delta\Psi)\rangle\rangle = \delta\Gamma_{\mathrm{loc},H} + o(\langle\langle\Delta\Psi\rangle\rangle). && (6.57)
\end{aligned}$$

As a result, the EA with composite and background fields for the GZ action $S_{\mathrm{loc,GZ}}$ determined by the local BRST-invariant horizon does not depend on a variation of the gauge condition on the extremals $\delta\Gamma_{\mathrm{loc},H}/\delta\boldsymbol{\Phi}^{\mathcal{A}} = \delta\Gamma_{\mathrm{loc},H}/\delta\Sigma^m = 0$. Thereby, we can state that the Gribov horizon defined using a composite field (added to the Faddeev–Popov quantum action) and the horizon defined without such a field lead to different forms of the mass shell for the respective EA. This is a second main result of the present subsection.

By choosing local BRST-invariant composite fields $\sigma^m = \sigma^m\left(A, A^{\mathrm{h}}, B\right)$, related in the case of $D = 4$ to an emergence of dimension-two condensates, with $Z_{\mathrm{loc},H}(B, J, L)$ defined along the lines of (6.14) according to

$$\left(\sigma^1, \sigma^2\right) = \frac{1}{2}\left(\mathrm{Tr}\ A^{\mathrm{h}}_\mu A^{\mathrm{h}\mu}, 2\mathrm{Tr}\left[\bar{\varphi}^\mu \varphi^{\mathrm{T}}_\mu - \bar{\omega}^\mu \omega^{\mathrm{T}}_\mu\right]\right), \quad \left(\sigma^1, \sigma^2\right) = \left(\sigma^1, \sigma^2\right)(x), \tag{6.58}$$

we arrive (for $L_1(x) = m^2$ and $L_2(x) = -M^2$) at a refined GZ action S_{RGZ} in Landau gauge. Using FD BRST transformations relating the integrands of generating functionals of Green's functions in Landau gauge and arbitrary R_ξ-gauges, we obtain from (6.51) a refined GZ action $S^{\mathrm{LCG}}_{\mathrm{RGZ}}$ in covariant gauges; see [71], Eq. (34). Thereby, one can extend the related study of renormalizability [71] for the resulting quantum action and generating functional $Z_{\mathrm{loc},H}(B, J, L)$ in all orders of perturbation theory to the case of arbitrary local composite fields. This is a third main result of the present subsection.

7 Two Dimensional Gravity with Dynamical Torsion

Consider a theory of two-dimensional gravity with dynamical torsion described in terms of a zweibein e^i_μ and a Lorentz connection ω_μ by the action [62]

$$S_0(e,\omega) = \int d^2x \; e \left(\frac{1}{16\alpha} R_{\mu\nu}{}^{ij} R^{\mu\nu}{}_{ij} - \frac{1}{8\beta} T_{\mu\nu}{}^{i} T^{\mu\nu}{}_{i} - \gamma \right), \tag{7.1}$$

where α, β, γ are constant parameters. For indices of quantities transforming by the local Lorentz group, we use Latin characters: $i,\ j,\ k\ldots$ ($i = 0,1$); ε^{ij} is a constant antisymmetric second-rank pseudo-tensor subject to the normalization condition $\varepsilon^{01} = 1$. Greek characters stand for indices of quantities transforming as (pseudo-)tensors under the general coordinate transformations: $\lambda,\ \mu,\ \nu\ldots$ ($\lambda = 0,1$). The Latin indices are raised and lowered by the Minkowski metric η_{ij} $(+,-)$ and the Greek indices, by the metric tensor $g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu$. Besides, the following notation is used:

$$\begin{aligned} e &= \det e^i_\mu, \\ R_{\mu\nu}{}^{ij} &= \varepsilon^{ij} R_{\mu\nu}, \quad R_{\mu\nu} = \partial_\mu \omega_\nu - (\mu \leftrightarrow \nu), \\ T_{\mu\nu}{}^{i} &= \partial_\mu e^i_\nu + \varepsilon^{ij} \omega_\mu e_{\nu j} - (\mu \leftrightarrow \nu). \end{aligned} \tag{7.2}$$

The action (7.1) is invariant under the local Lorentz transformations $e^i_\mu \to e'^i_\mu$, $\omega_\mu \to \omega'_\mu$

$$\begin{aligned} e'^i_\mu &= (\Lambda e_\mu)^i, \\ (\Omega'_\mu)^i_j &= (\Lambda \Omega_\mu \Lambda^{-1})^i_j + (\Lambda \partial_\mu \Lambda^{-1})^i_j, \quad (\Omega_\mu)^i_j \equiv \varepsilon^{ik} \eta_{kj} \omega_\mu, \end{aligned} \tag{7.3}$$

or, infinitesimally, with a parameter ζ,

$$\delta e^i_\mu = \varepsilon^{ij} e_{\mu j} \zeta, \quad \delta \omega_\mu = -\partial_\mu \zeta, \tag{7.4}$$

as well as under the general coordinate transformations, $x \to x' = x'(x)$,

$$\begin{aligned} e^i_\mu &\to e'^i_\mu(x') = \frac{\partial x^\lambda}{\partial x'^\mu} e^i_\lambda(x), \\ \omega_\mu &\to \omega'_\mu(x') = \frac{\partial x^\lambda}{\partial x'^\mu} \omega_\lambda(x), \end{aligned} \tag{7.5}$$

implying the infinitesimal field variations, with some parameters ξ^μ,

$$\delta e^i_\mu = e^i_\nu \partial_\mu \xi^\nu + (\partial_\nu e^i_\mu) \xi^\nu, \quad \delta \omega_\mu = \omega_\nu \partial_\mu \xi^\nu + (\partial_\nu \omega_\mu) \xi^\nu. \tag{7.6}$$

The gauge transformations (7.4), (7.6) form a closed algebra:

$$\begin{aligned} [\delta_{\zeta(1)},\ \delta_{\zeta(2)}] &= 0, \\ [\delta_{\xi(1)},\ \delta_{\xi(2)}] &= \delta_{\xi(1,2)}, \\ [\delta_\zeta,\ \delta_\xi] &= \delta_{\zeta'}, \end{aligned} \tag{7.7}$$

where

$$\xi^\mu{}_{(1,2)} = \xi^\nu{}_{(1)} \partial_\nu \xi^\mu{}_{(2)} - (\partial_\nu \xi^\mu{}_{(1)}) \xi^\nu{}_{(2)}, \quad \zeta' = (\partial_\mu \zeta) \xi^\mu.$$

so that the Faddeev–Popov method applies to the given theory, with the total configuration space ϕ^A given by the classical fields (e^i_μ, ω_μ), as well as by the Faddeev–Popov ghosts ($\bar{c}$, c, $\bar{c}^\mu$, c^μ) and the Nakanishi–Lautrup fields $(b,\, b^\mu)$, according to the respective number of gauge parameters $\zeta,\ \xi^\mu$ in (7.4), (7.6). The fields $\phi^A = (e^i_\mu, \omega_\mu; b, b^\mu; \bar{c}, \bar{c}^\mu, c, c^\mu)$ possess the following Grassmann parity and ghost number:

	(e^i_μ, ω_μ)	(b, b^μ)	$(\bar{c}, \bar{c}^\mu)$	(c, c^μ)
ϵ	0	0	1	1
gh	0	0	-1	1

Let us present a quantum theory for (7.1), (7.4), (7.6), (7.7) in the background field method by following the treatment [61], based on an ansatz for the vacuum functional (see also [3]) which corresponds to $Z\,(B)$ of (2.27) in the case of Yang–Mills theories with the Nakanishi–Lautrup fields eliminated using some background gauges. Namely, we assign to the initial classical fields the sets of quantum Q and background B fields, which, in view of further convenience, we denote by $Q = (q^i_\mu, q_\mu)$ and $B = (e^i_\mu, \omega_\mu)$, with the associated metric tensor $g_{\mu\nu}$ and the notation e, $(\Omega_\mu)^i_j$ in (7.2), (7.3) being related to the background fields alone. Let us also associate the gauge transformations (7.4), (7.6) with two kinds of infinitesimal transformations, namely, background δ_{b} and quantum δ_{q}, introduced by analogy with 2.28, so that the action $S_0(Q+B)$ in (7.1) should be left invariant under both kinds of these transformations:

$$\begin{aligned} &\delta_{\mathrm{b}} e^i_\mu = \varepsilon^{ij} e_{\mu j}\zeta + e^i_\nu \partial_\mu \xi^\nu + (\partial_\nu e^i_\mu)\xi^\nu, \quad \delta_{\mathrm{b}} q^i_\mu = \varepsilon^{ij} q_{\mu j}\zeta + q^i_\nu \partial_\mu \xi^\nu + (\partial_\nu q^a_\mu)\xi^\nu, \\ &\delta_{\mathrm{b}} \omega_\mu = -\partial_\mu \zeta + \omega_\nu \partial_\mu \xi^\nu + (\partial_\nu \omega_\mu)\xi^\nu, \quad \delta_{\mathrm{b}} q_\mu = q_\nu \partial_\mu \xi^\nu + (\partial_\nu q_\mu)\xi^\nu, \end{aligned} \tag{7.8}$$

$$\begin{aligned} &\delta_{\mathrm{q}} e^i_\mu = 0, \quad \delta_{\mathrm{q}} q^i_\mu = \varepsilon^{ij}(e_{\mu j} + q_{\mu j})\zeta + (e^i_\nu + q^i_\nu)\partial_\mu \xi^\nu + (\partial_\nu e^i_\mu + \partial_\nu q^i_\mu)\xi^\nu, \\ &\delta_{\mathrm{q}} \omega_\mu = 0, \quad \delta_{\mathrm{q}} q_\mu = -\partial_\mu \zeta + (\omega_\nu + q_\nu)\partial_\mu \xi^\nu + (\partial_\nu \omega_\mu + \partial_\nu q_\mu)\xi^\nu. \end{aligned} \tag{7.9}$$

Following [61], we introduce an analogue [3] of the generating functional of Green's functions, as we denote $(\bar{c}, \bar{c}^\mu, c, c^\mu) = (\overline{C}, C)$,

$$Z(B,J) = \int dQ\ d\overline{C}\ dC\ \exp\left\{\frac{i}{\hbar}\left[S_0(Q+B) + S_{\mathrm{gf}}(Q,B) + S_{\mathrm{gh}}(Q,B;\overline{C},C) + JQ\right]\right\}, \tag{7.10}$$

where $J = (J^\mu_i, J^\mu)$ are sources to the quantum fields $Q = (q^i_\mu, q_\mu)$, and the functional $S_{\mathrm{gf}} = S_{\mathrm{gf}}(Q,B)$ is determined by some background gauge functions χ, χ_μ for the respective gauge parameters ζ, ξ^μ, according to the condition of invariance under the background transformations, $\delta_{\mathrm{b}} S_{\mathrm{gf}} = 0$, with the ghost term $S_{\mathrm{gh}} = S_{\mathrm{gh}}(Q,B;\overline{C},C)$ given by the rule

$$S_{\mathrm{gh}} = \int d^2x\ \left(\bar{c}\delta_{\mathrm{q}}\chi + \bar{c}^\mu \delta_{\mathrm{q}}\chi_\mu\right)\big|_{(\zeta,\xi^\mu)\to(c,c^\mu)}\,. \tag{7.11}$$

The background gauge functions $\chi = \chi(Q,B)$ and $\chi_\mu = \chi_\mu(Q,B)$ will be chosen, according to [61], as linear in the quantum fields $Q = (q^i_\mu, q_\mu)$,

$$\chi = e g^{\mu\nu}\nabla_\mu q_\nu\ , \quad \chi_\mu = e g^{\lambda\nu} e_{\mu i}\nabla_\lambda q^i_\nu\ , \tag{7.12}$$

where e, $g^{\mu\nu}$ are determined by the background fields e^i_μ ($g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$, $g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu$, $e = \det\, e^i_\mu$), and ∇_μ is a covariant derivative, whose action on an arbitrary (psedo-)tensor field $T^{\nu_1\ldots\nu_l\,j_1\ldots j_n}_{\mu_1\ldots\mu_k\,i_1\ldots i_m}$ is given in terms of $(\Omega_\mu)^i_j = \varepsilon^{ik}\eta_{kj}\omega_\mu$ and the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{7.13}$$

by the rule

$$\begin{aligned}\nabla_\mu T^{\nu_1\ldots\nu_l\,j_1\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots i_m} &= \partial_\mu T^{\nu_1\ldots\nu_l\,j_1\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots i_m} - \Gamma^{\hat\lambda}_{\mu\{\mu\}} T^{\nu_1\ldots\nu_l \quad j_1\ldots j_n}_{\mu_1\ldots\hat\lambda\ldots\mu_k\,i_1\ldots i_m} + \Gamma^{\{\nu\}}_{\mu\hat\lambda} T^{\nu_1\ldots\hat\lambda\ldots\nu_l j_1\ldots j_n}_{\mu_1\ldots\mu_k \quad i_1\ldots i_m} \\ &+ (\Omega_\mu)^{\{j\}}_{\hat p} T^{\nu_1\ldots\nu_l\,j_1\ldots\hat p\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots i_m} - (\Omega_\mu)^{\hat p}_{\{i\}} T^{\nu_1\ldots\nu_l\,j_1\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots\hat p\ldots i_m}\,, \end{aligned} \tag{7.14}$$

with the notation (C.3), so that the covariant derivative ∇_μ has the usual properties (F, G are arbitrary (psedo-)tensor fields)

$$\nabla_\sigma g_{\mu\nu} = \nabla_\sigma g^{\mu\nu} = 0, \quad \nabla_\mu(FG) = F\nabla_\mu G + (\nabla_\mu F)G. \tag{7.15}$$

The above ingredients allow one to construct the gauge-fixing term $S_{\rm gf}$ as a functional being quadratic in χ, χ^μ (with certain numeric parameters a, b)

$$S_{\rm gf} = \frac{1}{2}\int d^2x\; e^{-1}\left(a\chi^2 + b\chi_\mu\chi^\mu\right) \tag{7.16}$$

and invariant under the local Lorentz transformations

$$\begin{aligned} e^{'i}_\mu &= (\Lambda e_\mu)^i, \;\; q^{'i}_\mu = (\Lambda q_\mu)^i, \\ (\Omega'_\mu)^i_b &= (\Lambda\Omega_\mu\Lambda^{-1})^i_j + (\Lambda\partial_\mu\Lambda^{-1})^i_j, \;\; q'_\mu = q_\mu, \end{aligned} \tag{7.17}$$

as well as under the general coordinate transformations, $x \to x' = x'(x)$,

$$\begin{aligned} e^{'i}_\mu(x') &= \frac{\partial x^\lambda}{\partial x'^\mu} e^i_\lambda(x), \quad \omega'_\mu(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\omega_\lambda(x), \\ q^{'i}_\mu(x') &= \frac{\partial x^\lambda}{\partial x'^\mu} q^i_\lambda(x), \quad q'_\mu(x') = \frac{\partial x^\lambda}{\partial x'^\mu} q_\lambda(x). \end{aligned} \tag{7.18}$$

Indeed, the infinitesimal form of field transformations implied by (7.17) and (7.18) is identical with the background transformations (7.8), which satisfies the requirement $\delta_{\rm b} S_{\rm gf} = 0$. Given this and the fact that the non-vanishing quantum transformations (7.9), with allowance for (7.14), can be represented as ($\zeta \to c$, $\xi^\mu \to c^\mu$)

$$\begin{aligned} \delta_{\rm q} q^i_\mu &= \varepsilon^{ij}(e_{\mu j} + q_{\mu j})c + (e^i_\nu + q^i_\nu)\nabla_\mu c^\nu + (\nabla_\nu e^i_\mu + \nabla_\nu q^i_\mu)c^\nu - \varepsilon^{ij}\omega_\nu(e_{\mu j} + q_{\mu b})c^\nu, \\ \delta_{\rm q} q_\mu &= -\nabla_\mu c + (\omega_\nu + q_\nu)\nabla_\mu c^\nu + (\nabla_\nu\omega_\mu + \nabla_\nu q_\mu)c^\nu, \end{aligned}$$

the ghost contribution $S_{\rm gh}$ in (7.11) acquires the form

$$\begin{aligned} S_{\rm gh} &= \int d^2x\; e\,\{-\bar c\nabla_\mu\nabla^\mu c + \bar c\nabla^\mu[(\nabla_\nu\omega_\mu + \nabla_\nu q_\mu)c^\nu + (\omega_\nu + q_\nu)\nabla_\mu c^\nu] \\ &\quad + \varepsilon^{ij}\bar c^\mu e_{\mu i}\nabla^\nu[(e_{\nu j} + q_{\nu j})(c - \omega_\lambda c^\lambda)] \\ &\quad + \bar c^\mu e_{\mu i}\nabla^\nu[(\nabla_\lambda e^i_\nu + \nabla_\lambda q^i_\nu)c^\lambda + (e^i_\lambda + q^i_\lambda)\nabla_\nu c^\lambda]\}\,. \end{aligned} \tag{7.19}$$

The quantum action in (7.10), determined by (7.1), (7.12), (7.16), (7.19), proves to be invariant (as well as the integrand in (7.10) at the vanishing sources $J = 0$, within the usual assumption $\delta(x) = \partial_\mu\delta(x)|_{x=0} = 0$) under the background transformations (7.8), combined with a set of compensating local transformations for the ghost fields [61],

$$\begin{aligned} \delta\bar c &= (\partial_\mu\bar c)\xi^\mu, & \delta\bar c^\mu &= -\bar c^\nu\partial_\nu\xi^\mu + (\partial_\nu\bar c^\mu)\xi^\nu, \\ \delta c &= -c^\mu\partial_\mu\zeta + (\partial_\mu c)\xi^\mu, & \delta c^\mu &= -c^\nu\partial_\nu\xi^\mu + (\partial_\nu c^\mu)\xi^\nu. \end{aligned} \tag{7.20}$$

As a consequence of (7.8), (7.20), the generating functional $Z(B,J)$ in (7.10) is invariant [61] under the initial gauge transformations (7.4), (7.6) of the background fields $B=(e^i_\mu,\omega_\mu)$, combined with the following local transformations of the sources $J=(J^\mu_i,J^\mu)$:

$$\delta J^\mu_i=-\varepsilon^k_i J^\mu_k\zeta-J^\nu_i\partial_\nu\xi^\mu+\partial_\nu(J^\mu_i\xi^\nu),\quad \delta J^\mu=-J^\nu\partial_\nu\xi^\mu+\partial_\nu(J^\mu\xi^\nu),\quad \varepsilon^i_j\equiv\varepsilon^{ik}\eta_{kj}\ . \tag{7.21}$$

On the one hand, this ensures the property

$$\delta\left(JQ\right)=\int d^2x\ \partial_\mu F^\mu\ ,\quad F^\mu\left(x\right)\equiv\left(J^\nu_i q^i_\nu+J^\nu q_\nu\right)\xi^\mu$$

for the source term JQ in (7.10), and, on the other hand, this extends a tensor transformation law for the sources J^μ_i and J^μ, at the infinitesimal level, by including the respective contributions $J^\mu_i\partial_\nu\xi^\nu$ and $J^\mu\partial_\nu\xi^\nu$,

$$\delta J^\mu_i=\left[-\varepsilon^k_i J^\mu_k\zeta-J^\nu_i\partial_\nu\xi^\mu+(\partial_\nu J^\mu_i)\xi^\nu\right]+J^\mu_i\partial_\nu\xi^\nu,\quad \delta J^\mu=\left[-J^\nu\partial_\nu\xi^\mu+(\partial_\nu J^\mu)\xi^\nu\right]+J^\mu\partial_\nu\xi^\nu.$$

Due to the invariance of $Z(B,J)=\exp\{(i/h)\,W(B,J)\}$ under (7.4), (7.6), (7.21), one achieves an invariance [61] of the functional $\Gamma=\Gamma(B,Q)$ given by

$$\Gamma(B,Q)=W(B,J)-JQ,\quad Q=\frac{\delta W}{\delta J},\quad J=-\frac{\delta\Gamma}{\delta Q},\quad Q=(q^i_\mu,q_\mu) \tag{7.22}$$

under the background transformations (7.8) of the fields B and Q (see Appendix C.1)

$$\delta_{\rm b}\Gamma=\int d^2x\left[\frac{\delta\Gamma}{\delta B\left(x\right)}\delta_{\rm b}B\left(x\right)+\frac{\delta\Gamma}{\delta Q\left(x\right)}\delta_{\rm b}Q\left(x\right)\right]=0, \tag{7.23}$$

which implies that the effective action $\Gamma_{\rm eff}(B)$ of the background field method defined by

$$\Gamma_{\rm eff}(B)=\Gamma(B,Q)|_{Q=0}\ ,$$

is invariant under the gauge transformations (7.4), (7.6) of the background fields $B=(e^i_\mu,\omega_\mu)$.

Let us proceed to extend the generating functional (7.10), suggested in [61], with the entire quantum action now denoted by $S(Q,B;\overline{C},C)$, to a functional $Z(B,J,L)$, as we introduce some background-dependent composite fields $\sigma^m\left(Q,B\right)$ with sources L_m,

$$\sigma^m\left(Q,B\right)=\sigma^{i_1\cdots i_k}_{\mu_1\cdots\mu_l}\left(Q\left(x\right),B\left(x\right)\right),\quad L_m=L^{\mu_1\cdots\mu_l}_{i_1\cdots i_k}\ \left(x\right),\quad m=\left(x,i_1,\ldots,i_k,\mu_1,\ldots,\mu_l\right),$$

namely,

$$Z(B,J,L)=\int dQ\ d\overline{C}\ dC\ \exp\left\{\frac{i}{\hbar}\left[S(Q,B;\overline{C},C)+JQ+L_m\sigma^m\left(Q,B\right)\right]\right\}. \tag{7.24}$$

In doing so, we require that the extended functional $Z(B,J,L)$ should inherit the local symmetry of $Z(B,J)$ under the transformations (7.4), (7.6), (7.21) of the background fields $B=(e^i_\mu,\omega_\mu)$ and the sources $J=(J^\mu_i,J^\mu)$. To this end, we demand that the composite fields $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(x)=\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(Q(x),B(x))$ transform as tensors with respect to the Lorentz (7.17) and general coordinate (7.18) transformations of the quantum Q and background B fields,

$$\begin{aligned}\sigma'^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(x)&=\Lambda^{i_1}_{j_1}\cdots\Lambda^{i_m}_{j_m}\sigma^{j_1\cdots j_m}_{\mu_1\cdots\mu_n}(x)\ ,\\ \sigma'^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(x')&=\frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\cdots\frac{\partial x^{\nu_n}}{\partial x'^{\mu_n}}\sigma^{i_1\cdots i_m}_{\nu_1\cdots\nu_n}(x)\ ,\quad x'=x'(x)\ .\end{aligned} \tag{7.25}$$

In general, a composite field $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(Q,B)$ subject to (7.25) is multiplicative with respect to the quantum fields $Q=(q^i_\mu,q_\mu)$ and the background field ingredients e^i_μ, $g_{\mu\nu}$, $R_{\mu\nu}{}^{ij}$, $T_{\mu\nu}{}^i$, see (7.2); besides, it may contain a number of background covariant derivatives ∇_μ acting according to (7.14), (7.13), with the properties (7.15). It is obvious, however, that the composite fields subject to the restriction $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(Q,0)\neq 0$ are allowed to contain the background fields B only via covariant derivatives ∇_μ, given in terms of $\Gamma^\lambda_{\mu\nu}$, $(\Omega_\mu)^i_j$ and acting on q^i_μ, q_μ, namely,

$$\nabla_\mu q^i_\nu=\partial_\mu q^i_\nu-\Gamma^\lambda_{\mu\nu}q^i_\lambda+(\Omega_\mu)^i_j q^j_\nu\ ,\quad \nabla_\mu q_\nu=\partial_\mu q_\nu-\Gamma^\lambda_{\mu\nu}q_\lambda\ .$$

Infinitesimally, the transformations (7.25) correspond to local tensor variations $\delta\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$ with parameters ζ and ξ^μ,

$$\delta\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}=\varepsilon^{\{i\}}_{\hat p}\sigma^{i_1\cdots\hat p\cdots i_m}_{\mu_1\cdots\mu_n}\zeta+\sigma^{i_1\cdots i_m}_{\mu_1\cdots\hat\nu\cdots\mu_n}\partial_{\{\mu\}}\xi^{\hat\nu}+(\partial_\nu\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n})\xi^\nu, \tag{7.26}$$

Given this assumption and the invariance of the vacuum functional in (7.24) under the background transformations (7.8) combined with the compensating local transformations (7.20) of the ghost fields, the extended generating functional $Z(B,J,L)$ in (7.24) proves to be invariant under the initial gauge transformations (7.4), (7.6) of the background fields $B=(e^i_\mu,\omega_\mu)$ combined with the local transformations (7.21) of the sources $J=(J^\mu_i,J^\mu)$ and some local transformations of the sources $L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}$, namely,

$$\delta L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}=-\varepsilon^{\hat p}_{\{i\}}L^{\mu_1\cdots\mu_n}_{i_1\cdots\hat p\cdots i_m}\zeta-L^{\mu_1\cdots\hat\nu\cdots\mu_n}_{i_1\cdots i_m}\partial_{\hat\nu}\xi^{\{\mu\}}+\partial_\nu(L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\xi^\nu)\ , \tag{7.27}$$

which differs from the (infinitesimal) tensor transformation law by the contribution $L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\partial_\nu\xi^\nu$,

$$\delta L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}=\left[-\varepsilon^{\hat p}_{\{i\}}L^{\mu_1\cdots\mu_n}_{i_1\cdots\hat p\cdots i_m}\zeta-L^{\mu_1\cdots\hat\nu\cdots\mu_n}_{i_1\cdots i_m}\partial_{\hat\nu}\xi^{\{\mu\}}+(\partial_\nu L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m})\xi^\nu\right]+L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\partial_\nu\xi^\nu\ ,$$

and provides for the source term $L_m\sigma^m$ in (7.24) the corresponding property

$$\delta\left(L_m\sigma^m\right)=\int d^2x\ \partial_\mu G^\mu\ ,\quad G^\mu\equiv L^{\nu_1\cdots\nu_n}_{i_1\cdots i_m}\sigma^{i_1\cdots i_m}_{\nu_1\cdots\nu_n}\xi^\mu\ .$$

The invariance of $Z(B,J,L)$ and the subsequent invariance of $W(B,J,L)=(h/i)\ln Z(B,J,L)$ can be recast, with the corresponding variations δB, δJ, δL given by (7.4), (7.6), (7.21), (7.27), in the form, $Y=\{Z,W\}$,

$$\int d^2x\ \left[\delta B\left(x\right)\frac{\delta}{\delta B\left(x\right)}+\delta J\left(x\right)\frac{\delta}{\delta J\left(x\right)}+\delta L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\left(x\right)\frac{\delta}{\delta L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\left(x\right)}\right]Y\left(B,J,L\right)=0. \tag{7.28}$$

Let us consider a functional $\Gamma=\Gamma(B,Q,\Sigma)$ given by the double Legendre transformation

$$\Gamma\left(B,Q,\Sigma\right)=W\left(B,J,L\right)-JQ-L_m\left[\sigma^m\left(Q,B\right)+\Sigma^m\right] \tag{7.29}$$

in terms of additional fields $\Sigma^m=\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(x)$,

$$Q=\frac{\delta W}{\delta J},\quad \Sigma^m=\frac{\delta W}{\delta L_m}-\sigma^m\left(\frac{\delta W}{\delta J},B\right),\quad -J=\frac{\delta\Gamma}{\delta Q}+L_m\frac{\delta\sigma^m}{\delta Q},\quad -L_m=\frac{\delta\Gamma}{\delta\Sigma^m}\ .$$

Then, the effective action $\Gamma_{\text{eff}}\left(B,\Sigma\right)$ with composite and background fields,

$$\Gamma_{\text{eff}}\left(B,\Sigma\right)=\Gamma\left(B,Q,\Sigma\right)\big|_{Q=0}\ , \tag{7.30}$$

is invariant, as a consequence of (7.28), under a set of local transformations given by the gauge transformations (7.4), (7.6) of the background fields $B=(e^i_\mu,\omega_\mu)$ combined with the infinitesimal local tensor transformations (see Appendix C.2)

$$\delta\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}=\varepsilon^{\{i\}}_{\hat p}\Sigma^{i_1\cdots\hat p\cdots i_m}_{\mu_1\cdots\mu_n}\zeta+\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\hat\nu\cdots\mu_n}\partial_{\{\mu\}}\xi^{\hat\nu}+(\partial_\nu\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n})\xi^\nu \tag{7.31}$$

of the additional fields $\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$, cf. (7.25), (7.26).

8 Summary

The present work has been devoted to quantum non-Abelian gauge models with composite and background fields. According to the principal research issues listed in Introduction, the following tasks have been completed:

1. A generating functional (2.18), (3.1) of Green's functions has been introduced for composite and background fields in Yang–Mills theories. The corresponding symmetry properties have been investigated, as well as the properties of a generating functional (3.18), (3.24) of vertex Green's functions (effective action). These properties can be expressed in a differential form as the relations (2.20), (3.12), (3.22), (3.25), where (2.20), (3.25) reflect the gauge transformations (2.19), (3.26), which consist of local $SU(N)$ transformations accompanied by gauge transformations of a background field B_μ, whereas (3.12), (3.22) are related to the BRST symmetry transformations (2.24), (2.26) with a modified Slavnov variation $\overleftarrow{s}_{\rm q}$ depending on B_μ.

2. On the basis of the above BRST transformations, we have proposed, for the first time, a set of finite field-dependent BRST (FD BRST) transformations (3.2), including a background field dependence, and have studied their properties; see Appendix A.1.

3. Using the finite FD BRST transformations, we have investigated the related (modified) Ward identities (3.6), (3.10), (3.13), (3.19), depending on an FD parameter, as well as the gauge dependence (4.1), (4.2), (4.3) of the above generating functionals with composite and background fields. It should be noticed that the modified Ward identities for a constant anticommuting parameter are reduced to the familiar identities (3.12), (3.17), (3.22) of [66, 67]. A gauge variation of the effective action has been found in terms of a nilpotent operator (3.20) depending on the composite and background fields, and the conditions (4.4) of on-shell gauge-independence have been established. A procedure of loop expansion for the effective action with composite and background fields has been examined to determine the representation (5.9) for a one-loop effective action.

4. The Gribov–Zwanziger theory [38] has been examined, being a quantum Yang–Mills theory incorporating the presence of a Gribov horizon [37] in terms of a non-local composite field. The theory [38] has been extended (6.33) by introducing a background field B_μ and shown to provide an effective action (6.35), (6.37) invariant under the gauge transformations of B_μ. The same result is shown to hold for the effective action (6.53) of a GZ theory having a local BRST-invariant horizon with background and additional local composite fields. A quantum action has been suggested, having a local BRST-invariant horizon (6.51) with background and composite fields. The corresponding generating functional of Green's functions extends the scope of the study [71], devoted to renormalizability in the presence of local BRST-invariant quadratic composite fields (6.58), to the case of arbitrary local composite fields in the background formalism. The problem of gauge independence has been studied for the effective action (EA) with composite and background fields, starting from the Gribov–Zwanziger action (6.51). It has been shown (6.57) that the EA does not depend on a variation of the gauge condition on the extremals. This makes it possible to conclude that the Gribov horizon, when defined with a composite field (added to the Faddeev–Popov quantum action) and without such a field, leads to different mass-shell conditions. The only representation using the Gribov–Zwanziger quantum action that is physically relevant is the one with an on-shell non-vanishing Gribov mass parameter γ.

5. The model of two-dimensional gravity with dynamical torsion by Volovich and Katanaev [62] has been considered, being quantized according to the background field method in [61] and featuring a gauge-invariant effective action, due to (7.23). The quantized two-dimensional gravity [61] has been generalized to the presence of composite fields (7.24), and the corresponding effective action (7.29), (7.30) with composite and background fields has been found to be gauge-invariant under (7.4), (7.6),

(7.31), in a way similar to the Yang–Mills case, cf. (3.26).

Possible applications of the approach developed in the present work can be the following. The present study of Yang–Mills theories can be employed to include the QCD gauge theory of strong interactions with the $SU(3)$ gauge group for the purpose of describing hadron particles (mesons and baryons) as composite fields. The part related to the two-dimensional gravity with dynamical torsion can be turned to the advantage of dealing with the so-called Generalized Lagrange space (for metric fields), so as to exploit its properties of curvature, torsion and deflection in order to take into account the asymmetries and anisotropies emerging in physical phenomena mostly at the cosmological level. The suggested background gauge-invariant effective action for the Gribov–Zwanziger theory appears to be promising as a next point in a renormalization analyzis of the Gribov–Zwanziger model, as one accounts for both the non-local, and localized BRST-invariant Gribov horizon in the background formalism, while extending the scope of [71]. Finally, the general approach to Yang–Mills theories with composite and background fields can be extended to the case of field-dependent BRST–anti-BRST symmetry along the lines of [45, 46, 47].

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A Yang–Mills Theory

A.1 Background FD BRST transformations

The transformations of the fields $(A_\mu, b, \bar{c}, c)$ induced by a Slavnov generator $\overleftarrow{s}_{\rm q}$, which depends on a background field B^μ in (2.24), and parameterized by a finite FD Grassmann-odd functional $\lambda(\phi, B)$ in (3.2) represent finite FD BRST transformations with a background field in Yang–Mills theories. The Jacobian Sdet $\|\delta\phi'/\delta\phi\|$ for a change of variables induced by the transformations (3.2) in the path integral (3.1) can be expressed according to the recipe [42, 40],

$$\begin{aligned} \mathrm{Sdet}\,\|\delta\phi'/\delta\phi\| &= \exp\left[\mathrm{Str}\ln\left(\delta^A_B + \frac{\overleftarrow{\delta}\left(\phi^A\overleftarrow{s}_{\rm q}\lambda\right)}{\delta\phi^B}\right)\right] = \exp\left[-\mathrm{Str}\sum_{n=1}\frac{(-1)^n}{n}\left(\frac{\delta(\phi^A\overleftarrow{s}_{\rm q}\lambda)}{\delta\phi^B}\right)^n\right] \\ &= \exp\left[-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Str}\left(\phi^A\overleftarrow{s}_{\rm q}\lambda\frac{\overleftarrow{\delta}}{\delta\phi^B}\right)^n\right] = \left[1+\lambda(\phi,B)\overleftarrow{s}_{\rm q}\right]^{-1}. \end{aligned} \tag{A.1}$$

In calculating the Jacobian, we have used the properties $\frac{\overleftarrow{\delta}\,(\phi^A\overleftarrow{s}_{\rm q})}{\delta\phi^B}(\phi^B\overleftarrow{s}_{\rm q}) = 0$ and $\frac{\overleftarrow{\delta}\,(\phi^A\overleftarrow{s}_{\rm q})}{\delta\phi^A}\lambda = 0$, due to the antisymmetry of the structure constants f^{pqr}. For a vanishing background field B^μ, the Jacobian (A.1) assumes the usual form [42].

The invariance of the quantum action $S_{\rm ext}(\phi, \phi^*, B)$ in (3.1) with respect to the finite FD BRST transformations (3.2) implies that a change $\phi^A \to \phi'^A = \phi^A[1 + \overleftarrow{s}_{\rm q}\lambda(\phi, B)]$ induces in (3.1) a transformation of the integrand $\mathcal{I}^\Psi_{\phi,\phi^*,B}$, namely,

$$\begin{aligned} \mathcal{I}^\Psi_{\phi+\phi\overleftarrow{s}_{\rm q}\lambda,\phi^*,B} &= d\phi\ \exp\left(\ln\mathrm{Sdet}\,\|\delta\phi'/\delta\phi\|\right)\exp\left[(i/\hbar)\,S_{\rm ext}\left(\phi+\phi\overleftarrow{s}_{\rm q}\lambda,\phi^*,B\right)\right] \\ &= d\phi\ \exp\left\{(i/\hbar)\left[S_{\rm ext}\left(\phi+\phi\overleftarrow{s}_{\rm q}\lambda,\phi^*,B\right) - i\hbar\ln\mathrm{Sdet}\,\|\delta\phi'/\delta\phi\|\right]\right\}, \end{aligned} \tag{A.2}$$

and therefore

$$\mathcal{I}^{\Psi}_{\phi+\phi\overleftarrow{s}_{\mathrm{q}}\lambda,\phi^{*},B} = d\phi \ \exp\left\{(i/\hbar)\left[S_{\mathrm{ext}}\left(\phi,\phi^{*},B\right) + i\hbar\,\ln\left(1+\lambda\overleftarrow{s}_{\mathrm{q}}\right)\right]\right\} \ . \tag{A.3}$$

Due to the explicit form of the initial quantum action $S^{\Psi}_{\mathrm{ext}} = S_0(A) + \Psi\left(\phi,B\right)\overleftarrow{s}_{\mathrm{q}} + \phi^{*}_{A}(\phi^{A}\overleftarrow{s}_{\mathrm{q}})$ in (2.3), (3.1), the BRST-exact contribution $i\hbar\ln\left(1+\lambda(\phi,B)\overleftarrow{s}_{\mathrm{q}}\right)$ to S^{Ψ}_{ext} can then be interpreted as a change of the gauge-fixing Fermion made in the original integrand $\mathcal{I}^{\Psi}_{\phi,\phi^{*},B}$:

$$i\hbar\,\ln\left(1+\lambda(\phi,B)\overleftarrow{s}_{\mathrm{q}}\right) = \left(\Delta\Psi\right)\overleftarrow{s}_{\mathrm{q}} \tag{A.4}$$

$$\Longrightarrow \mathcal{I}^{\Psi}_{\phi+\phi\overleftarrow{s}_{\mathrm{q}}\lambda(\phi,B),\phi^{*},B} = d\phi \ \exp\left\{(i/\hbar)\left[S_0 + \left(\Psi+\Delta\Psi\right)\overleftarrow{s}_{\mathrm{q}} + \phi^{*}_{A}(\phi^{A}\overleftarrow{s}_{\mathrm{q}})\right]\right\} = \mathcal{I}^{\Psi+\Delta\Psi}_{\phi,\phi^{*},B}, \tag{A.5}$$

with a certain $\Delta\Psi\left(\phi,B|\lambda\right)$, whose correspondence to $\lambda\left(\phi,B\right)$ is established by the relation (A.4), which is a familiar compensation equation for an unknown parameter $\lambda(\phi,B)$ now including a certain background B^{μ}, and which implies the gauge-independence of the vacuum functional, $Z_{\Psi}(B,\phi^{*}) = Z_{\Psi+\Delta\Psi}(B,\phi^{*})$, in (3.5). An explicit solution of (A.4) satisfying the solvability condition due to the BRST exactness of both sides (up to BRST exact terms) is given by (3.4).

A.2 Legendre transformation. Differential consequences

The operator $\hat{\omega}_{\Gamma}$ in (3.20) is obtained from $\hat{\Omega}$ in (3.13) with the help of a Legendre transformation and differential consequences of the usual Ward identities (3.12), (3.17) for Z, W, by using differentiation with respect to J_A, L_m, namely,

$$\hat{\Omega}\,\phi^{A}\big|_{J,L} = \frac{\overrightarrow{\delta}\Gamma}{\delta\phi^{*}_{A}}(-1)^{\epsilon_A} - \frac{i}{\hbar}\left[\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^{m}}\left(\sigma^{m}_{,C}(\hat{\phi},B)\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^{*}_{C}}\right),\phi^{A}\right\}, \tag{A.6}$$

$$\begin{aligned}\hat{\Omega}\,\Sigma^{m}\big|_{J,L} = {} & \left[\sigma^{m}_{,A}(\hat{\phi},B) - \sigma^{m}_{,A}(\phi,B)\right]\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^{*}_{A}} - \frac{i}{\hbar}\left[\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^{m}}\left(\sigma^{m}_{,C}(\hat{\phi},B)\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^{*}_{C}}\right),\Sigma^{m}\right\}\\ & -\frac{i}{\hbar}\left[\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^{n}}\left(\sigma^{n}_{,C}(\hat{\phi},B)\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^{*}_{C}}\right),\sigma^{m}(\phi,B)\right\}\\ & +\frac{i}{\hbar}(-1)^{\epsilon(\sigma^{m})+\epsilon(\phi^{D})}\sigma^{m}_{,D}(\phi,B)\left[\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^{n}}\left(\sigma^{n}_{,C}(\hat{\phi},B)\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^{*}_{C}}\right),\phi^{D}\right\}\\ & +(-1)^{\epsilon(\sigma^{m})+\epsilon(\phi^{D})\epsilon(\phi^{A})}\left[\sigma^{m}_{,D}(\phi,B),\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^{n}}\sigma^{n}_{,A}(\hat{\phi},B)\right\}\left(G''^{-1}\right)^{A\mathsf{a}}\left(\frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}}}\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^{*}_{D}}\right),\end{aligned} \tag{A.7}$$

as we calculate the variational derivatives according to the definitions (3.18),

$$\frac{\overrightarrow{\delta}}{\delta\phi^{*}_{A}}\bigg|_{J,L} = \frac{\overrightarrow{\delta}}{\delta\phi^{*}_{A}}\bigg|_{\phi,\Sigma} + \frac{\overrightarrow{\delta}\,\phi^{B}}{\delta\phi^{*}_{A}}\frac{\overrightarrow{\delta}}{\delta\phi^{B}}\bigg|_{\phi^{*},\Sigma} + \frac{\overrightarrow{\delta}\,\Sigma^{m}}{\delta\phi^{*}_{A}}\frac{\overrightarrow{\delta}}{\delta\Sigma^{m}}\bigg|_{\phi^{*},\phi}, \tag{A.8}$$

with allowance for (3.21).

A.3 Background effective action. Gauge invariance

The invariance of $Z\left(B,J,L,\phi^{*}\right)$ in (3.1) and that of the related functional $W\left(B,J,L,\phi^{*}\right)$ under the set of local transformations (2.19), (3.23) translates itself into an extension of (2.20), as we denote

$Y=\{Z,W\}$,

$$\int d^{D}x\ \left\{\left[D_{\mu}^{pq}\left(B\right)\xi^{q}\right]\frac{\overrightarrow{\delta}}{\delta B_{\mu}^{p}}+g\xi^{q}f^{\{p\}\hat{r}q}L_{\mu_{1}\cdots\mu_{j}}^{p_{1}\cdots\hat{r}\cdots p_{i}}\frac{\overrightarrow{\delta}}{\delta L_{\mu_{1}\cdots\mu_{j}}^{p_{1}\cdots p_{i}}}\right.$$
$$+g\xi^{q}f^{prq}\left(J_{(A)}^{r|\mu}\frac{\overrightarrow{\delta}}{\delta J_{(A)}^{p|\mu}}+J_{(b)}^{r}\frac{\overrightarrow{\delta}}{\delta J_{(b)}^{p}}+J_{(\bar{c})}^{r}\frac{\overrightarrow{\delta}}{\delta J_{(\bar{c})}^{p}}+J_{(c)}^{r}\frac{\overrightarrow{\delta}}{\delta J_{(c)}^{p}}\right)$$
$$\left.+\ g\xi^{q}f^{prq}\left(A_{\mu}^{*r}\frac{\overrightarrow{\delta}}{\delta A_{\mu}^{*p}}+b^{*r}\frac{\overrightarrow{\delta}}{\delta b^{*p}}+\bar{c}^{*r}\frac{\overrightarrow{\delta}}{\delta\bar{c}^{*p}}+c^{*r}\frac{\overrightarrow{\delta}}{\delta c^{*p}}\right)\right\}Y\left(B,J,L,\phi^{*}\right)=0. \tag{A.9}$$

The functional $\Gamma\left(B,\phi,\Sigma,\phi^{*}\right)$ in (3.18) satisfies the subsequent identity

$$\int d^{D}x\ \left\{\left[D_{\mu}^{pq}\left(B\right)\xi^{q}\right]\frac{\overrightarrow{\delta}}{\delta B_{\mu}^{p}}-g\xi^{q}f^{\{p\}\hat{r}q}\left[\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{k}}+\sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{k}}\left(\phi,B\right)\right]\frac{\overrightarrow{\delta}}{\delta\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots\hat{r}\cdots p_{k}}}\right.$$
$$-g\xi^{q}f^{prq}\left(A^{p|\mu}\frac{\overrightarrow{\delta}}{\delta A^{r|\mu}}+b^{p}\frac{\overrightarrow{\delta}}{\delta b^{r}}+\bar{c}^{p}\frac{\overrightarrow{\delta}}{\delta\bar{c}^{r}}+c^{p}\frac{\overrightarrow{\delta}}{\delta c^{r}}\right)$$
$$+g\xi^{q}f^{prq}\left(A_{\mu}^{*r}\frac{\overrightarrow{\delta}}{\delta A_{\mu}^{*p}}+b^{*r}\frac{\overrightarrow{\delta}}{\delta b^{*p}}+\bar{c}^{*r}\frac{\overrightarrow{\delta}}{\delta\bar{c}^{*p}}+c^{*r}\frac{\overrightarrow{\delta}}{\delta c^{*p}}\right)$$
$$\left.+\ g\xi^{q}f^{prq}\left(A^{p|\mu}\frac{\overrightarrow{\delta}}{\delta A^{r|\mu}}+b^{p}\frac{\overrightarrow{\delta}}{\delta b^{r}}+\bar{c}^{p}\frac{\overrightarrow{\delta}}{\delta\bar{c}^{r}}+c^{p}\frac{\overrightarrow{\delta}}{\delta c^{r}}\right)\sigma^{m}\left(\phi,B\right)\frac{\overrightarrow{\delta}}{\delta\Sigma^{m}}\right\}\Gamma\left(B,\phi,\Sigma,\phi^{*}\right)=0, \tag{A.10}$$

where the terms containing the derivatives of $\sigma^{m}=\sigma^{m}\left(\phi,B\right)$ over $\phi^{A}=\left(A^{r|\mu},b^{r},\bar{c}^{r},c^{r}\right)(x)$ are understood in the form

$$\frac{\overrightarrow{\delta}\sigma^{m}}{\delta\phi^{A}}\frac{\overrightarrow{\delta}}{\delta\Sigma^{m}}=\int d^{D}y\ \frac{\overrightarrow{\delta}}{\delta\phi^{A}}\sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{k}}\left(y\right)\frac{\overrightarrow{\delta}}{\delta\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{k}}\left(y\right)}.$$

Then the functional $\Gamma_{\text{eff}}\left(B,\Sigma\right)$ defined by (3.24) satisfies the relation (3.25) as a consequence of the equality

$$f^{\{p\}\hat{r}q}\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{k}}\frac{\overrightarrow{\delta}}{\delta\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots\hat{r}\cdots p_{k}}}=-f^{\{p\}\hat{r}q}\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots\hat{r}\cdots p_{k}}\frac{\overrightarrow{\delta}}{\delta\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{k}}}, \tag{A.11}$$

implied by the notation $f^{\{p\}\hat{r}q}$ in (2.10) and by the antisymmetry of the structure constants:

$$f^{p_{s}r_{s}q}\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{s}\cdots p_{k}}\frac{\overrightarrow{\delta}}{\delta\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots r_{s}\cdots p_{k}}}=-f^{r_{s}p_{s}q}\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{s}\cdots p_{k}}\frac{\overrightarrow{\delta}}{\delta\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots r_{s}\cdots p_{k}}}=-f^{p_{s}r_{s}q}\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots r_{s}\cdots p_{k}}\frac{\overrightarrow{\delta}}{\delta\Sigma_{\mu_{1}\cdots\mu_{l}}^{p_{1}\cdots p_{s}\cdots p_{k}}}. \tag{A.12}$$

B Gribov–Zwanziger Theory

B.1 Background-dependent Faddeev–Popov operator

From the expression (6.13) for $(\tilde{K})_{B}^{pq}\left(x;y\right)$ written in the matrix form

$$\tilde{K}_{B}\left(x;y\right)=D_{\mu}\left(B\right)D^{\mu}\left(A+B\right)\delta\left(x-y\right) \tag{B.1}$$

it follows that

$$\tilde{K}_B\left(x;y\right)=\left[\partial^2+g\left(\partial_\mu A^\mu\right)+g\left(\partial_\mu B^\mu\right)+g\left(A^\mu+2B^\mu\right)\partial_\mu+g^2B_\mu\left(A_\mu+B_\mu\right)\right]\delta\left(x-y\right).$$

Subtracting from this matrix the expression for $K_B\left(x;y\right)$ given by (6.16), we find

$$\tilde{K}_B\left(x;y\right)-K_B\left(x;y\right)=g\left\{\left(\partial_\mu A^\mu\right)+g\left[B_\mu,A^\mu\right]\right\}\delta\left(x-y\right)=g\left[D_\mu\left(B\right),A^\mu\right]\delta\left(x-y\right). \tag{B.2}$$

where, in virtue of the Jacobi identity for the structure constants f^{pqr},

$$\left[D_\mu\left(B\right),A^\mu\right]^{pq}=\left(\partial_\mu A^\mu+g\left[B_\mu,A^\mu\right]\right)^{pq}=f^{prq}D_\mu^{rs}\left(B\right)A^{s|\mu}.$$

which proves that the matrix $\tilde{K}_B\left(x;y\right)$ defined by (6.13) does indeed reduce to $K_B\left(x;y\right)$ in (6.16) under the background gauge condition $D_\mu^{pq}\left(B\right)A^{q|\mu}=0$, and also that the two matrices are related by (6.17).

B.2 Gribov–Zwanziger local action. Alternative representation

Consider the integral expression

$$\left\langle F,G\right\rangle\equiv\int d^Dx\ d^Dy\ \mathrm{Tr}\ F\left(x\right)\mathcal{K}\left(x,y\right)G\left(y\right), \tag{B.3}$$

constructed using the matrix $\mathcal{K}\left(x,y\right)$ in (6.29),

$$\mathcal{K}\left(x,y\right)=\frac{1}{2}\left[K\left(x,y\right)+\tilde{K}\left(x,y\right)\right]\delta\left(x-y\right),$$

where the matrix elements of $K\left(x,y\right)$ and $\tilde{K}\left(x,y\right)$ are given by (6.3), (6.9), which implies

$$\mathcal{K}\left(x,y\right)=\frac{1}{2}\left[D^\nu\left(A\right)\partial_\nu+\partial_\nu D^\nu\left(A\right)\right]\delta\left(x-y\right). \tag{B.4}$$

The expression (B.3) then transforms into

$$\left\langle F,G\right\rangle=\frac{1}{2}\int d^Dx\ \mathrm{Tr}\ F\left[D^\nu\left(A\right)\partial_\nu+\partial_\nu D^\nu\left(A\right)\right]G, \tag{B.5}$$

and integration by parts results in

$$\left\langle F,G\right\rangle=\frac{1}{2}\int d^Dx\ \mathrm{Tr}\left[-\left(\partial^\nu F\right)\partial_\nu G-\left(\partial_\nu F\right)\partial^\nu G+gFA^\nu\partial_\nu G-g\left(\partial_\nu F\right)A^\nu G\right]. \tag{B.6}$$

As we rewrite $FA^\nu=-\left[A^\nu,F\right]+A^\nu F$, $A^\nu G=\left[A^\nu,G\right]-GA^\nu$ and use a permutation under the sign of Tr, the expression (B.6) transforms into

$$\left\langle F,G\right\rangle=-\frac{1}{2}\int d^Dx\ \mathrm{Tr}\left\{\left[D^\nu\left(A\right),F\right]\partial_\nu G+\left(\partial_\nu F\right)\left[D^\nu\left(A\right),G\right]-gA^\nu\left[F\partial_\nu G+\left(\partial_\nu F\right)G\right]\right\}.$$

Applying this result to the settings

$$\left(F,G\right)=\left(\bar{\varphi}^\mu,\varphi_\mu^{\mathrm{T}}\right),\quad\left(F,G\right)=\left(\bar{\omega}^\mu,\omega_\mu^{\mathrm{T}}\right)$$

made in the path integral (6.21), (6.22) restricted to the case $B_\mu=0$, where $K\left(x,y\right)\to\mathcal{K}\left(x,y\right)$ due to the Landau gauge condition $\partial_\nu A^\nu=0$, which implies (after integrating by parts)

$$\int d^Dx\ \mathrm{Tr}\ A^\nu\left[F\partial_\nu G+\left(\partial_\nu F\right)G\right]=0,$$

we find that the action $S_{\mathrm{GZ}}\left(\Phi\right)$ in (6.5) is indeed equivalent to $\mathcal{S}_{\mathrm{GZ}}\left(\Phi\right)$ given by (6.30).

B.3 Background gauge invariance of W, Γ

The invariance of the functional $Z_{\mathcal{H}} = Z_{\mathcal{H}}(B,J)$ in (6.33) under the local transformations (6.34) in terms of the related functional $W_{\mathcal{H}} = W_{\mathcal{H}}(B,J)$ reads

$$\int d^D x\ \left\{ \left[D^{pq}_{\mu}(B)\,\xi^q \right] \frac{\overrightarrow{\delta}}{\delta B^p_{\mu}} + g\xi^q f^{prq} \left(J^{r|\mu}_{(A)} \frac{\overrightarrow{\delta}}{\delta J^{p|\mu}_{(A)}} + J^r_{(b)} \frac{\overrightarrow{\delta}}{\delta J^p_{(b)}} + J^r_{(\bar{c})} \frac{\overrightarrow{\delta}}{\delta J^p_{(\bar{c})}} + J^r_{(c)} \frac{\overrightarrow{\delta}}{\delta J^p_{(c)}} \right) \right\} W_{\mathcal{H}} = 0. \tag{B.7}$$

and translates itself for the functional $\Gamma_{\mathcal{H}} = \Gamma_{\mathcal{H}}(B,\phi)$ in (6.35) as follows:

$$\int d^D x\ \left\{ \left[D^{pq}_{\mu}(B)\,\xi^q \right] \frac{\overrightarrow{\delta}}{\delta B^p_{\mu}} + g\xi^q f^{prq} \left(A^r_{\mu} \frac{\overrightarrow{\delta}}{\delta A^p_{\mu}} + b^r \frac{\overrightarrow{\delta}}{\delta b^p} + \bar{c}^r \frac{\overrightarrow{\delta}}{\delta \bar{c}^p} + c^r \frac{\overrightarrow{\delta}}{\delta c^p} \right) \right\} \Gamma_{\mathcal{H}} = 0, \tag{B.8}$$

which implies the invariance of $\Gamma_{\mathcal{H}}(B,\phi)$ under the local transformations (6.36).

C Volovich–Katanaev Model

C.1 Background effective action. Gauge invariance

The invariance of $W = W(B,J)$ under (7.4), (7.6), (7.21) implies

$$\begin{aligned} \int d^2x \Bigg\{ & \left[\varepsilon^{ij} e_{\mu j}\zeta + e^i_{\nu}\partial_{\mu}\xi^{\nu} + (\partial_{\nu} e^i_{\mu})\xi^{\nu} \right] \frac{\delta W}{\delta e^i_{\mu}} + \left[-\partial_{\mu}\zeta + \omega_{\nu}\partial_{\mu}\xi^{\nu} + (\partial_{\nu}\omega_{\mu})\xi^{\nu} \right] \frac{\delta W}{\delta \omega_{\mu}} \\ & - \left[\varepsilon^k_i J^{\mu}_k \zeta + J^{\nu}_i \partial_{\nu}\xi^{\mu} - \partial_{\nu}(J^{\mu}_i \xi^{\nu}) \right] \frac{\delta W}{\delta J^{\mu}_i} - \left[J^{\nu}\partial_{\nu}\xi^{\mu} - \partial_{\nu}(J^{\mu}\xi^{\nu}) \right] \frac{\delta W}{\delta J^{\mu}} \Bigg\} = 0 \end{aligned} \tag{C.1}$$

and transforms into a relation for $\Gamma = \Gamma(B,Q)$ defined by (7.22),

$$\begin{aligned} \int d^2x \Bigg\{ & \left[\varepsilon^{ij} e_{\mu j}\zeta + e^i_{\nu}(\partial_{\mu}\xi^{\nu}) + (\partial_{\nu} e^i_{\mu})\xi^{\nu} \right] \frac{\delta \Gamma}{\delta e^i_{\mu}} + \left[-\partial_{\mu}\zeta + \omega_{\nu}(\partial_{\mu}\xi^{\nu}) + (\partial_{\nu}\omega_{\mu})\xi^{\nu} \right] \frac{\delta \Gamma}{\delta \omega_{\mu}} \\ & + q^i_{\mu} \left[\varepsilon^k_i \frac{\delta\Gamma}{\delta q^k_{\mu}}\zeta + \frac{\delta\Gamma}{\delta q^i_{\nu}}\partial_{\nu}\xi^{\mu} - \partial_{\nu}\left(\frac{\delta\Gamma}{\delta q^i_{\mu}}\xi^{\nu} \right) \right] + q^{\mu} \left[\frac{\delta\Gamma}{\delta q_{\nu}}\partial_{\nu}\xi^{\mu} - \partial_{\nu}\left(\frac{\delta\Gamma}{\delta q_{\mu}}\xi^{\nu} \right) \right] \Bigg\} = 0, \end{aligned} \tag{C.2}$$

which is integrated by parts to result in (7.23).

C.2 Background gauge invariance of W, Γ with composite fields

Let us introduce the notation, $p_k \in \{p_1, \ldots, p_m\}$, $\nu_k \in \{\nu_1, \ldots, \nu_n\}$,

$$\begin{aligned} F^{\hat{p}}_{\{i\}} T^{\mu_1\cdots\mu_n}_{i_1\cdots\hat{p}\cdots i_m} &= \sum_{p_k} F^{p_k}_{i_k} T^{\mu_1\cdots\mu_n}_{i_1\cdots p_k\cdots i_m}\ , \quad G^{\{\mu\}}_{\hat{\nu}} T^{\mu_1\cdots\hat{\nu}\cdots\mu_n}_{i_1\cdots i_m} = \sum_{\nu_k} G^{\mu_k}_{\nu_k} T^{\mu_1\cdots\nu_k\cdots\mu_n}_{i_1\cdots i_m}\ , \\ F^{\{i\}}_{\hat{p}} T^{i_1\cdots\hat{p}\cdots i_m}_{\mu_1\cdots\mu_n} &= \sum_{p_k} F^{i_k}_{p_k} T^{i_1\cdots p_k\cdots i_m}_{\mu_1\cdots\mu_n}\ , \quad G^{\hat{\nu}}_{\{\mu\}} T^{i_1\cdots i_m}_{\mu_1\cdots\hat{\nu}\cdots\mu_n} = \sum_{\nu_k} G^{\nu_k}_{\mu_k} T^{i_1\cdots i_m}_{\mu_1\cdots\nu_k\cdots\mu_n}\ . \end{aligned} \tag{C.3}$$

Then the invariance of $W = W(B, J, L)$ under (7.4), (7.6), (7.21), (7.27) implies

$$\int d^2x \left\{ \left[\varepsilon^{ij} e_{\mu j}\zeta + e^i_\nu(\partial_\mu \xi^\nu) + (\partial_\nu e^i_\mu)\xi^\nu\right] \frac{\delta W}{\delta e^i_\mu} + \left[-\partial_\mu \zeta + \omega_\nu(\partial_\mu \xi^\nu) + (\partial_\nu \omega_\mu)\xi^\nu\right] \frac{\delta W}{\delta \omega_\mu} \right.$$
$$- \left[\varepsilon^k_i J^\mu_k \zeta + J^\nu_i(\partial_\nu \xi^\mu) - \partial_\nu(J^\mu_i \xi^\nu)\right] \frac{\delta W}{\delta J^\mu_i} - \left[J^\nu(\partial_\nu \xi^\mu) - \partial_\nu(J^\mu \xi^\nu)\right] \frac{\delta W}{\delta J^\mu}$$
$$\left. - \left[\varepsilon^{\hat{p}}_{\{i\}} L^{\mu_1 \cdots \mu_n}_{i_1 \cdots \hat{p} \cdots i_m} \zeta + L^{\mu_1 \cdots \hat{\nu} \cdots \mu_n}_{i_1 \cdots i_m} (\partial_{\hat{\nu}} \xi^{\{\mu\}}) - \partial_\nu (L^{\mu_1 \cdots \mu_n}_{i_1 \cdots i_m} \xi^\nu)\right] \frac{\delta W}{\delta L^{\mu_1 \cdots \mu_n}_{i_1 \cdots i_m}} \right\} = 0 \tag{C.4}$$

and transforms into a relation for $\Gamma = \Gamma(B, Q, \Sigma)$ defined by (7.29),

$$\int d^2x \left\{ \left[\varepsilon^{ij} e_{\mu j}\zeta + e^i_\nu(\partial_\mu \xi^\nu) + (\partial_\nu e^i_\mu)\xi^\nu\right] \frac{\delta \Gamma}{\delta e^i_\mu} + \left[-\partial_\mu \zeta + \omega_\nu(\partial_\mu \xi^\nu) + (\partial_\nu \omega_\mu)\xi^\nu\right] \frac{\delta \Gamma}{\delta \omega_\mu} \right.$$
$$+ \left(\Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n} + \sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}\right) \left[\varepsilon^{\hat{p}}_{\{i\}} \frac{\delta \Gamma}{\delta \Sigma^{i_1 \cdots \hat{p} \cdots i_m}_{\mu_1 \cdots \mu_n}} \zeta + \frac{\delta \Gamma}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \hat{\nu} \cdots \mu_n}} \partial_{\hat{\nu}} \xi^{\{\mu\}} - \partial_\nu \left(\frac{\delta \Gamma}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}} \xi^\nu \right) \right]$$
$$+ q^\mu_i \left[\varepsilon^k_i \left(\frac{\delta \Gamma}{\delta q^k_\mu} - \frac{\delta \Gamma}{\delta \Sigma^m} \frac{\delta \sigma^m}{\delta q^k_\mu} \right) \zeta + \left(\frac{\delta \Gamma}{\delta q^i_\nu} - \frac{\delta \Gamma}{\delta \Sigma^m} \frac{\delta \sigma^m}{\delta q^i_\nu} \right) \partial_\nu \xi^\mu - \partial_\nu \left(\frac{\delta \Gamma}{\delta q^i_\mu} \xi^\nu - \frac{\delta \Gamma}{\delta \Sigma^m} \frac{\delta \sigma^m}{\delta q^i_\mu} \xi^\nu \right) \right]$$
$$\left. + \; q^\mu \left[\left(\frac{\delta \Gamma}{\delta q_\nu} - \frac{\delta \Gamma}{\delta \Sigma^m} \frac{\delta \sigma^m}{\delta q_\nu} \right) \partial_\nu \xi^\mu - \partial_\nu \left(\frac{\delta \Gamma}{\delta q_\mu} \xi^\nu - \frac{\delta \Gamma}{\delta \Sigma^m} \frac{\delta \sigma^m}{\delta q_\mu} \xi^\nu \right) \right] \right\} = 0, \tag{C.5}$$

where the terms containing the derivatives $\frac{\delta \sigma^m}{\delta q^i_\mu}$, $\frac{\delta \sigma^m}{\delta q_\mu}$ of the composite fields $\sigma^m(Q, B)$ are understood in the form

$$\frac{\delta \Gamma}{\delta \Sigma^m} \frac{\delta \sigma^m}{\delta Q} = \int d^2y \; \frac{\delta \Gamma}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}(y)} \frac{\delta \sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}(y)}{\delta Q(x)} \;, \quad Q(x) = (q^i_\mu, q_\mu)(x) \,.$$

The functional $\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(B, \Sigma)$ defined by (7.30) satisfies the identity

$$\int d^2x \left\{ \left[\varepsilon^{ij} e_{\mu j}\zeta + e^i_\nu(\partial_\mu \xi^\nu) + (\partial_\nu e^i_\mu)\xi^\nu\right] \frac{\delta}{\delta e^i_\mu} + \left[-\partial_\mu \zeta + \omega_\nu(\partial_\mu \xi^\nu) + (\partial_\nu \omega_\mu)\xi^\nu\right] \frac{\delta}{\delta \omega_\mu} \right.$$
$$\left. + \left[\varepsilon^{\hat{p}}_{\{i\}} \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n} \zeta \frac{\delta}{\delta \Sigma^{i_1 \cdots \hat{p} \cdots i_m}_{\mu_1 \cdots \mu_n}} + \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n} (\partial_{\hat{\nu}} \xi^{\{\mu\}}) \frac{\delta}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \hat{\nu} \cdots \mu_n}} + (\partial_\nu \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}) \xi^\nu \frac{\delta}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}} \right] \right\} \Gamma_{\text{eff}} = 0, \tag{C.6}$$

obtained by setting $\sigma^m(0, B) = 0$ and integrating by parts in (C.5). Using the latter property and the following consequences, cf. (A.11), (A.12), of the notation (C.3),

$$\begin{aligned} \varepsilon^{\hat{p}}_{\{i\}} \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n} \frac{\delta}{\delta \Sigma^{i_1 \cdots \hat{p} \cdots i_m}_{\mu_1 \cdots \mu_n}} &= \varepsilon^{\{i\}}_{\hat{p}} \Sigma^{i_1 \cdots \hat{p} \cdots i_m}_{\mu_1 \cdots \mu_n} \frac{\delta}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}} \;, \\ \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n} (\partial_{\hat{\nu}} \xi^{\{\mu\}}) \frac{\delta}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \hat{\nu} \cdots \mu_n}} &= \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \hat{\nu} \cdots \mu_n} (\partial_{\{\mu\}} \xi^{\hat{\nu}}) \frac{\delta}{\delta \Sigma^{i_1 \cdots i_m}_{\mu_1 \cdots \mu_n}} \;, \end{aligned}$$

by virtue of $\varepsilon^j_k = \varepsilon^k_j$ ($\varepsilon^0_1 = \varepsilon^1_0 = -1$ and $\varepsilon^0_0 = \varepsilon^1_1 = 0$ due to $\varepsilon^j_k = \varepsilon^{ji}\eta_{ik}$), one arrives at the invariance of $\Gamma_{\text{eff}}(B, \Sigma)$ under (7.4), (7.6) and (7.31).

References

[1] B.S. DeWitt, Quantum theory of gravity. II. The manifestly covariant theory, Phys. Rev. **162**, 1195–1239 (1967).

[2] I.Ya. Aref'eva, L.D. Faddeev, A.A. Slavnov, Generating functional for the S matrix in gauge theories, Theor. Math. Phys. **21**, 1165–1172 (1975); in Russian: Teor. Mat. Fiz. **21**, 311–321 (1974).

[3] L.F. Abbott, The background field method beyond one loop, Nucl. Phys. B **185**, 189–203 (1981).

[4] R.W. Haymaker, Variational methods for composite operators, Rivista Nuovo Cim. **14**, 1–89 (1991).

[5] J.M. Cornwall, R. Jackiw, E. Tomboulis, Effective action for composite operators, Phys. Rev. D **10**, 2428–2445 (1974).

[6] L.F. Abbott, Introduction to the background field method, Acta Phys. Polon. B **13**, 33–50 (1982).

[7] S. Weinberg, *The Quantum Theory of Fields. Vol. II Modern Applications* (Cambridge University Press, 1996).

[8] G. 't Hooft, An algorithm for the poles at dimension four in the dimensional regularization procedure, Nucl. Phys. B **62**, 444–460 (1973).

[9] H. Kluberg-Stern, J.B. Zuber, Renormalization of non-Abelian gauge theories in a background-field gauge. I. Green's Functions, Phys. Rev. D **12**, 482–488 (1975).

[10] M.T. Grisaru, P. van Nieuwenhuizen, C.C. Wu, Background-field method versus normal field theory in explicit examples: one loop divergences in S matrix and Green's functions for Yang–Mills and gravitational fields, Phys. Rev. D **12**, 3203–3213 (1975).

[11] D.M. Capper, A. MacLean, The background field method at two loops: a general gauge Yang–Mills calculation, Nucl. Phys. B **203**, 413–422 (1982).

[12] S. Ichinose, M. Omote, Renormalization using the background-field method, Nucl. Phys. B **203**, 221–267 (1982).

[13] M.H. Goroff, A. Sagnotti, The ultraviolet behavior of Einstein gravity, Nucl. Phys. B **266**, 709–736 (1986).

[14] A.E.M. van de Ven, Two-loop quantum gravity. Nucl. Phys. B **378**, 309–366 (1992).

[15] P.A. Grassi, Algebraic renormalization of Yang–Mills theory with background field method, Nucl. Phys. B **462**, 524–550 (1996).

[16] C. Becchi, R. Collina, Further comments on the background field method and gauge invariant effective actions, Nucl. Phys. B **562**, 412–430 (1999); [arXiv:hep-th/9907092].

[17] R. Ferrari, M. Picariello, A. Quadri, Algebraic aspects of the background field method, Annals Phys. **294**, 165–181 (2001); [arXiv:hep-th/0012090].

[18] D. Binosi, A. Quadri, The background field method as a canonical transformation, Phys. Rev. D **85**, 121702 (2012).

[19] A.O. Barvinsky, D. Blas, M. Herrero-Valea, S.M. Sibiryakov, C.F. Steinwachs, Renormalization of gauge theories in the background-field approach, JHEP **1807**, 035 (2018); [arXiv:1705.03480 [hep-th]].

[20] J. Frenkel, J.C. Taylor, Background gauge renormalization and BRST identities, Annals Phys. **389**, 234–238 (2018).

[21] I.A. Batalin, P.M. Lavrov, I.V. Tyutin, Multiplicative renormalization of Yang–Mills theories in the background-field formalism, Eur. Phys. J. C **78**, 570 (2018); [arXiv:1806.02552 [hep-th]].

[22] F.T. Brandt, J. Frenkel, D.G.C. McKeon, Renormalization of six-dimensional Yang–Mills theory in a background gauge field, Phys. Rev. D **99**, 025003 (2019).

[23] Y. Nambu, G. Jona-Lasinio, Dynamical model of elementary particles based on an analogy with superconductivity, Phys. Rev. **122**, 345–358 (1961); D. Gross, A. Neveu, Dynamical symmetry breaking in asymptotically free field theories, Phys. Rev. D **10**, 3235–3253 (1974).

[24] S.W. Hawking, I.G. Moss, Fluctuations in the inflationary Universe, Nucl. Phys. B **224**, 180–192 (1983); A.A. Tseytlin, Conformal anomaly in two-dimensional σ-model on curved background and strings, Phys. Lett. B **178**, 34–40 (1986).

[25] R. Casalbuoni, S. DeCurtis, R. Gatto, Composite operator calculation of chiral symmetry breaking in color gauge theory, Phys. Lett. B **140**, 357–362 (1984).

[26] W. Bardeen, C. Hill, M. Lindner, Minimal dynamical symmetry breaking of the Standard Model, Phys. Rev. D **41**, 1647–1660 (1990).

[27] R.P. Grigorian, I.V. Tyutin, Renormalization group equation for composite fields, Sov. J. Nucl. Phys. **26**, 593 (1977); in Russian: Yad. Fiz. **26**, 1121–1129 (1977); V.P. Gusynin, V.A. Miransky, Nonperturbative scale anomaly and composite operators in gauge field theories, Phys. Lett. B **198**, 362–366 (1987).

[28] N. Seiberg, The power of duality – exact results in 4D SUSY field theory, Int. J. Mod. Phys. A **16**, 4365–4376 (2001); [arXiv:hep-th/9506077].

[29] M. Shifman, K. Stepanyantz, Exact Adler function in supersymmetric QCD, Phys. Rev. Lett. **114**, 051601 (2015); [arXiv:1412.3382 [hep-th]].

[30] M. Shifman, K.V. Stepanyantz, Derivation of the exact expression for the D function in $N = 1$ SQCD, Phys. Rev. D **91**, 105008 (2015); [arXiv:1502.06655 [hep-th]].

[31] M.A.L. Capri, D.M. van Egmond, M.S. Guimarães, O. Holanda, S.P. Sorella, R.C. Terin, H.C. Toledo, Renormalizability of $N = 1$ super Yang–Mills theory in Landau gauge with a Stueckelberg-like field, Eur. Phys. J. C **78**, 797 (2018); [arXiv:1803.03077 [hep-th]].

[32] C. Wetterich, Average action and the renormalization group equations, Nucl. Phys. B **352**, 529–584 (1991).

[33] M. Reuter, C. Wetterich, Average action for the Higgs model with Abelian gauge symmetry, Nucl. Phys. B **391**, 147–175 (1993).

[34] M. Reuter, C.Wetterich, Effective average action for gauge theories and exact evolution equations, Nucl. Phys. B **417**, 181–214 (1994).

[35] P. Lavrov, I. Shapiro, On the functional renormalization group approach for Yang–Mills Fields, JHEP **1306**, 086 (2013); [arXiv:1212.2577 [hep-th]].

[36] V.F. Barra, P.M. Lavrov, E.A. dos Reis, T. de Paula Netto, I.L. Shapiro, Functional renormalization group approach and gauge dependence in gravity theories, Phys. Rev. D **101**, 065001 (2020); [arXiv:1910.06068[hep-th]].

[37] V.N. Gribov, Quantization of nonabelian gauge theories, Nucl. Phys. B **139**, 1–19 (1978).

[38] D. Zwanziger, Action from the Gribov horizon, Nucl. Phys. B **321**, 591–604 (1989); Local and renormalizable action from the Gribov horizon, Nucl. Phys. B **323**, 513–544 (1989).

[39] P. Lavrov, O. Lechtenfeld, A. Reshetnyak, Is soft breaking of BRST symmetry consistent? JHEP **1110**, 043 (2011); [arXiv:1108.4820 [hep-th]].

[40] A. Reshetnyak, On gauge independence for gauge models with soft breaking of BRST symmetry, Int. J. Mod. Phys. A **29**, 1450184 (2014); [arXiv:1312.2092 [hep-th]].

[41] P. Lavrov, O. Radchenko, A. Reshetnyak, Soft breaking of BRST symmetry and gauge dependence, Mod. Phys. Lett. A **27**, 1250067 (2012); [arXiv:1201.4720 [hep-th]].

[42] P. Lavrov, O. Lechtenfeld, Field-dependent BRST transformations in Yang–Mills theory. Phys. Lett. B **725**, 382–385 (2013); [arXiv:1305.0712[hep-th]].

[43] S.D. Joglekar, B.P. Mandal, Finite field dependent BRS transformations. Phys. Rev. D **51**, 1919–1927 (1995).

[44] S. Upadhyay, B.P. Mandal, Field dependent nilpotent symmetry for gauge theories Eur. Phys. J. C **72**, 2065 (2012); [arXiv:1201.0084[hep-th]].

[45] P.Yu. Moshin, A.A. Reshetnyak, Field-dependent BRST-antiBRST transformations in Yang–Mills and Gribov–Zwanziger theories, Nucl. Phys. B **888**, 92–128 (2014); [arXiv:1405.0790[hep-th]].

[46] P.Yu. Moshin, A.A. Reshetnyak, Finite BRST-antiBRST transformations in Lagrangian formalism, Phys. Lett. B **739**, 110–116 (2014); [arXiv:1406.0179[hep-th]]; Field-dependent BRST-antiBRST Lagrangian transformations. Int. J. Mod. Phys. A **30**, 1550021 (2015); [arXiv:1406.5086[hep-th]].

[47] P.Yu. Moshin, A.A. Reshetnyak, Finite field-dependent BRST-antiBRST transformations: Jacobians and application to the Standard Model. Int. J. Mod. Phys. A **31**, 1650111 (2016); [arXiv:1506.04660[hep-th]]

[48] S. Upadhyay, B.P. Mandal, A.A. Reshetnyak, Comments on interactions in the SUSY models, Eur. Phys. J. C **6**, 391 (2016); [arXiv:1605.02973[physics.gen-ph]].

[49] A.A. Reshetnyak, Generalization of Faddeev–Popov rules in Yang–Mills theories: $N = 3, 4$ BRST symmetries, Int. J. Mod. Phys. A **33**, 1850006 (2018); [arXiv:1701.00009 [hep-th]].

[50] P. Lavrov, O. Lechtenfeld, Gribov horizon beyond the Landau gauge, Phys. Lett. B **725**, 386–388 (2013); [arXiv:1305.2931 [hep-th]].

[51] I.A. Batalin, P.M. Lavrov, I.V. Tyutin, A systematic study of finite BRST–BV transformations in field-antifield formalism, Int. J. Mod. Phys. A **29**, 1450166 (2014); [arXiv:1405.2621 [hep-th]].

[52] A. Reshetnyak, On composite fields approach to Gribov copies elimination in Yang–Mills theories, Phys. Part. Nucl. Lett. **11**, 964–967 (2014); [arXiv:1402.3060 [hep-th]].

[53] M.A.L. Capri, D. Fiorentini, M.S. Guimaraes, B.W. Mintz, L.F. Palhares, S.P. Sorella, D. Dudal, I.F. Justo, A.D. Pereira, R. F. Sobreiro, An exact nilpotent nonperturbative BRST symmetry for the Gribov–Zwanziger action in the linear covariant gauge, Phys. Rev. D **92**, 045039 (2015).

[54] M. Semenov-Tyan-Shanskii, V. Franke, Zap. Nauch. Sem. LOMI im. V.A. Steklov, AN SSSR 120 (1982) 159 (English translation: New York: Plenum Press 1986).

[55] A.A. Reshetnyak, P.Yu. Moshin, On the finite BRST transformations: the Jacobians and the Standard Model with the gauge-invariant Gribov horizon, Russian Physics Journal, **59** (2017) 1921, [arXiv:1607.07253[hep-th]].

[56] M.A.L. Capri, D. Dudal, D. Fiorentini, M.S. Guimaraes, et al, A local and BRST-invariant Yang–Mills theory within the Gribov horizon, Phys. Rev. D 94, no. 2, 025035 (2016), [arXiv:1605.02610 [hep-th]].

[57] M.A.L. Capri, D. Fiorentini, M.S. Guimaraes, B.W. Mintz, L.F. Palhares, S.P. Sorella, A local and renormalizable framework for the gauge-invariant operator $A^2_{\rm min}$ in Euclidean Yang–Mills theories in linear covariant gauges, Phys. Rev. D **94**, 065009 (2016); [arXiv:1606.06601 [hep-th]].

[58] M.A.L. Capri, S.P. Sorella, R.C. Terin, Study of a gauge invariant local composite fermionic field, Annals Phys. **414**, 168077 (2020); [arXiv:1909.07927 [hep-th]].

[59] F.E. Canfora, D. Hidalgo, P. Pais, The Gribov problem in presence of background field for $SU(2)$ Yang–Mills theory, Phys. Lett. B **763**, 94–101 (2016); [arXiv:1610.08067].

[60] L.D. Faddeev, V.N. Popov, Feynman diagrams for the Yang–Mills field, Phys. Lett. B **25**, 29–30 (1967).

[61] P.M. Lavrov, P.Yu. Moshin, Quantization of two-dimensional gravity with dynamical torsion, Class. Quant. Grav. **16**, 2247–2258 (1999); [arXiv:hep-th/9710100].

[62] M.O. Katanaev, I.V. Volovich, Two-dimensional gravity with dynamical torsion and strings, Ann. Phys. **197**, 1–32 (1990).

[63] B.S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[64] C. Becchi, A. Rouet, R. Stora, The Abelian Higgs–Kibble model, unitarity of the S-operator, Phys. Lett. B **52**, 344–346 (1974); Renormalization of Gauge Theories, Ann. Phys. (N.Y.) **98**, 287–321 (1976).

[65] I.V. Tyutin, Gauge invariance in field theory and statistical mechanics, Lebedev Inst. preprint No. 39 (1975); [arXiv:0812.0580 [hep-th]].

[66] P.M. Lavrov, S.D. Odintsov, The gauge dependence of the effective action of composite fields in general gauge theories, Int. J. Mod. Phys. A **4**, 5205–5212 (1989).

[67] P.M. Lavrov, S.D. Odintsov, A.A. Reshetnyak, Effective action of composite fields for general gauge theories in BLT-covariant formalism, J. Math. Phys. **38**, 3466–3477 (1997); Physics of Atomic Nuclei **60**, 1020–1027 (1997); [arXiv:hep-th/9604061].

[68] P.M. Lavrov, B.S. Merzlikin, Loop expansion of the average effective action in the functional renormalization group approach, Phys. Rev. D **92**, 085038 (2015) [arXiv:1506.04941[hep-th]].

[69] P.M. Lavrov, B.S. Merzlikin, Legendre transformations and Clairaut-type equations, Phys. Lett. B **756**, 188–193 (2016); [arXiv:1602.04911[hep-th]].

[70] K.-I. Kondo, The nilpotent "BRST" symmetry for the Gribov–Zwanziger theory, Report number: CHIBA-EP-176; [arXiv:0905.1899 [hep-th]].

[71] M.A.L. Capri, D. Fiorentini, A.D. Pereira, S.P. Sorella, Renormalizability of the refined Gribov–Zwanziger action in the linear covariant gauges, Phys. Rev. D **96**, 054022 (2017); [arXiv:1708.01543[hep-th]].