STOCHASTIC AND CHAOTIC PHENOMENA

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Abstract.

In our Demonstration Laboratory of Physics (EWHL) we intend to show, in a simple way, to students of Physics, the difference between **stochastic** and **chaotic** phenomena. As examples of **stochastic** processes will be shown two cases: (1) the historical analysis performed by Langevin in 1908 to describe the motion of a particle with mass m and radius a immersed in a viscous fluid ("**Brownian motion").**[1,2] (2)Motion of small balls submitted to the gravitational field in a inclined plate colliding periodically with nails displayed along the plate **("Galton Board").** As an example of **chaotic process** we present the motion of a **double pendulum** in a gravitational field submitted to a driven force. *Key words: particles in viscous fluids; collisions of beads with pegs; pendulum in gravitation field .*

(I) Introduction.

There is a growing field of mathematics, physics and engineering that has been applied to study a large number of phenomena generically named ***chaotic.*** These are present in many areas in science and engineering,[1,2] including astronomy, plasma physics, statistical physics hydrodynamics and biology. As in Greek the word chaos (χαος) means confusion, random, stochastic and turbulent processes may be misleading associated with chaos. However, rigorously they are different in the framework of physics and mathematics, as will be seen. This article analyses only two kinds of chaos theory, *stochastic* and *chaotic*, using simple mathematical approaches. In **Section 1** is shown the process named **stochastic**. Is seen the general case of particles diffusion in fluids submitted to a well known external force and to forces defined as *stochastic*. Is also seen the case of the **Galton board** when beads move in vertical board, submitted to a gravitational force and colliding with interleaved rows of pegs. In **Section 2** is shown the process named **chaotic** analyzing the **double pendulum** motion in a gravitational field.

(2) Stochastic Processes.

(2.1)Diffusion of Particles in Viscous Fluids.

As an example of stochastic process we show the historical analysis performed by Langevin in 1908 to describe the "**Brownian motion".** That is**,** the motion of a particle with mass **m** and radius **a** immersed in a viscous fluid.[1,2] Assuming that the particle is also submitted to a known external force F, along the x-axis it would obey the differential equation,

$$ma = m\,(d^2x/dt^2) = F - \eta(dx/dt) + f_s(t) \qquad \textbf{(2.1.1),}$$

where $-\eta(dx/dt) = -\eta v$ ("Stokes law") is the dissipative **viscous force** on the particle, where η is the viscous coefficient, and $f_s(t)$ is force due to *unknown* interactions ("collisions") of the particle with the molecules of the fluid. The force $f_s(t)$, named *noise term*, which is interpreted as been created by **"stochastic forces"** obeys the following average values

$$< f_s(t) > = 0 \quad \text{and} \quad < f_s(t)f_s(t') > = B\,\delta(t-t'), \text{ where } B = \text{constant.}$$

When the effects of the external force F can be neglected, following Langevin,[1,2] defining $\zeta = d(x^2)/dt$ we get $d\zeta/dt = 2v^2 + 2x(dv/dt)$. In this way, Eq.(2.1.1) can be written as

$$m d\zeta/dt = 2mv^2 - \eta\zeta + 2xf_s(t) \qquad \textbf{(2.1.2).}$$

Assuming that the time average $< xf_s(t) > = 0$ we get from Eq.(2.1.2),

$$m\, d < \zeta >/dt = 2m < v^2 > - \eta < \zeta > \qquad \textbf{(2.1.3).}$$

In the stationary state ("equilibrium") $< \zeta(t) >$ = constant and the average kinetic energy of the particle is $(1/2)m< v^2 > = (1/2)k_BT$, where k_B is the Boltzmann constant. In this case, from Eq.(2.1.3) $< \zeta > = 2$ D, where $\mathbf{D = 2k_BT/\eta}$ is the "*diffusion coefficient*".

If instead of $\zeta = d(x^2)/dt$, defining distance as $d(t) = x(t) - x(0)$, taking $x(0) = 0$, we get

$$< d^2 > = < (x(t) - x(0))^2 > = < x^2(t) > = 2Dt \qquad \textbf{(2.1.4).}$$

showing that in the time interval t the average distance *d* covered by the particle from the initial deposited position would be given by

$$d = \sqrt{< d^2 >} = \sqrt{2Dt} \qquad \textbf{(2.1.5).}$$

In Figures 1a,1b and 1c are shown one plastic bottle containing water and a syringe containing blue ink. The needle is stuck laterally inside the bottle and the ink is injected into the bottle that becomes to diffuse into the water (**Fig.(1.b)**). After 1 h, the blue ink is entirely diffused throughout the bottle (**Fig.(1c)**). The effect of the gravitational F = mg is very small.


Fig.(1a)


Fig.(1b)


Fig.(1c).

(2.2)Galton Board.

The Galton board[3] consists of a vertical board with interleaved rows of pegs seen in **Figure (2.1).**

Figure (2.1). Galton board has N horizontal rows of pegs and k columns.

The boxes are along an horizontal line which center O is in the intersection of a vertical line that passes by the **beads** source. The leftmost box, at right or left side of O, is the ***1-box***, the next is the ***2-box,***.......

The beads that are put,one by one, at the upper side of the board, falling due to the gravitational field and deviated due to stochastic collisions with pegs, are deposited at the bottom of the board**.** Beads deviated, either to left or right sides of the pegs, are deposited at the bottom of the board, forming columns as seen in **Figure (2.2).**

The beads trajectories **r**(t), obey by the differential equation,

$$m\,(d^2\mathbf{r}/dt^2) = mg\,\mathbf{k} + \mathbf{f}_s(t) \qquad \textbf{(2.2.1),}$$

where m is bead mass, g the gravity constant, $\mathbf{r(t)} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ and $\mathbf{f}_s(t)$ are **stochastic** ("**random")** forces responsible for deviations of vertical beads trajectories. They would be distributed in the Galton board according the **Random walk** hypothesis, seen below.

Figure (2.2). Columns of beads deposited at the bottom of the board

Random walk

Let us imagine a bead with an irregular motion, according to a process that is called *random walk*.[4] At the end of each time interval **τ** it either has moved a distance **δ** to the **right** or a distance **δ** to the **left**.

Suppose that the direction of each successive "step"("collision") is independent of the direction of the previous one. Let us assume that the probability that the step is in the right direction is **p** and that the probability that it is in the left direction is **q = 1- p**. Then, it can be shown [4] that the probability $P_k(N)$ after N periods that the bead, after **k positive steps** and **(N - k) negative steps,** ends up in the **k**-box is given by the *binomial distribution*,

$$P_k(N) = \{N!/(k!(N-k)!)\}p^k(1-p)^{N-k} \qquad \textbf{(2.2.2).}$$

When N is very large, that is, when $N \to \infty$, the discrete **binomial distribution** $P_k(N)$ approximates the ***normal*** or "bell shaped" ***Gaussian*** continuous distribution[4] given by

$$F(x - X) = \{1/\sigma\sqrt{2\pi}\} \exp\{(x - X)^2/2\sigma^2\} \qquad \textbf{(2.2.3),}$$

when $p = q = 1/2$ and $\delta = \sigma/\sqrt{N} << 1$. The *standard deviation* σ and the average value $X = < x >$ are shown in **Figure (2.3).** This is the typical behavior of a system subject to a large number of very small independent random effects.

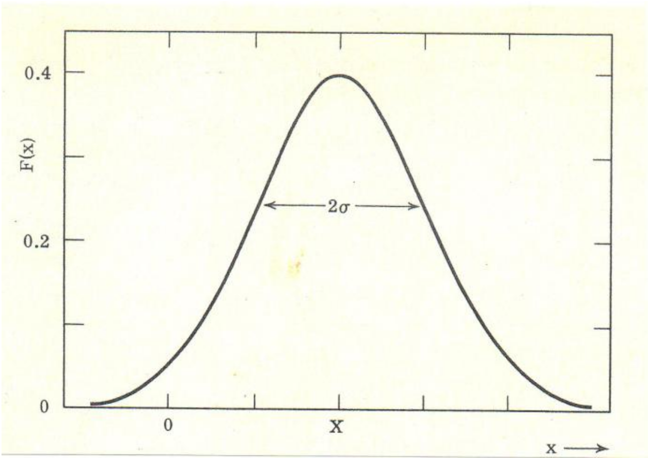

Figure (2.3). Gaussian or normal distribution F(x- X) as a function of x.

(3) Chaotic Phenomena.

In physics basics courses we learn that all physical laws are described by differential equations. So, integrating, that is, solving analytically or numerically, these equations knowing the initial and boundary conditions we would know the future of a physical system for all times. This is the **deterministic** view of nature. That is, physics systems are deterministic because they obey **deterministic differential equations**. They can be conservative or dissipative. If the initial states of deterministic systems were exactly known, future states could be theoretically predicted.

The **deterministic theory** survived till the 19.th beginning to be questioned after the famous visionary works of Henri Poincaré on Celestial Mechanics[5] performed at the end of the 19th. According to Poincaré, it is not always so: it may happen that **small differences in the initial conditions** produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Exact predictions becomes impossible, and we have fortuitous phenomena. In practice, as observed for many systems, knowledge about the future state is limited by the precision with which the initial state can be measured. That is, the exact knowledge of the laws of nature is not enough to an exact prediction of the future. There are deterministic systems whose time evolution has a very strong dependence on initial conditions. That is, the differential equations that govern the evolution of the system are very sensitive to initial conditions. In these cases we say that the phenomena are **"chaotic"**. *"Even a tiny effect, such as a **butterfly** flying nearby, may be enough to vary the conditions such that the future is entirely different than what it might have been, not just a tiny bit different".* **[6]**

Many examples of chaotic system can be seen in literature.[5,6] In this paper will be only analyzed the chaotic motion of the ***double pendulum*** [7]

In next **Section (3.1)** is seen the **non chaotic** motion of the **physical pendulum.**[8,9...]

(3.1)Physical Pendulum.

Let us briefly present the **non chaotic** motion of the physical pendulum (**Figure(3.1.1)**) which is studied in basic physics courses[8,9]

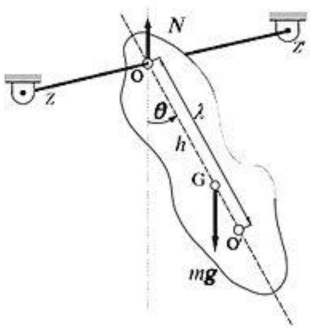

Figure (3.1). Physical pendulum rotating around a fixed horizontal axis ZZ´.

In **Figure (3.1)** is seen a **physical pendulum** or **simple pendulum** with mass **m** and moment of inertia **I** around the ZZ´axis submitted to a gravitational field **g**. In the general case, the angle θ(t) is given by[8,9.]

$$d^2\theta/dt^2 + (Mgh/I)\sin\theta = 0 \qquad (3.1.1),$$

which is difficult to solve exactly, that is, to obtain θ = θ(t).

For small deflection angles, that is, θ << 1 we get[8,9]

$$\theta(t) = A\cos(\omega t + \theta_o) \qquad (3.1.2),$$

which describes the simple pendulum**,** where A is the amplitude of motion, θ_o is initial deviation angle and $\omega = (mgh/I)^{1/2}$ is the frequency of the oscillations. If the initial values of A and θ_o are given, the θ(t) values will be **well known** with the time evolution. This shows the **non chaotic** motion of the simple pendulum. Pendulum clocks are used to measure (*up to now days*) with great precision the time in our houses. In the general case solving **Eq.(3.1),** we verify that the oscillations are **non chaotic.** That is, we have always an oscillatory motion.[10]

(3.1.2)Double Pendulum.

A double pendulum rotating around a fixed horizontal axis,[11] shown in **Figure (3.2),**

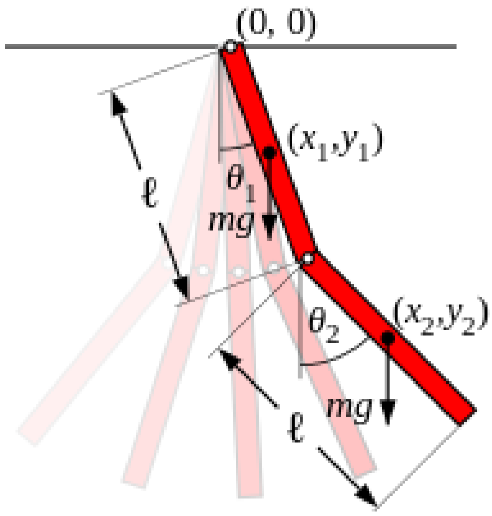

Figure (3.2).Rotating axis passing by (0,0) and perpendicular to the plane (x,y).

Equations of motion for the centers of massed **(x_1,y_1)** and **(x_2,y_2)** and for the angles **θ_1** and **θ_2** are shown in reference [11]. Solving numerically these equations (*very complicate*!) we verify that the motion is **chaotic**, that is, the trajectories of the centers of mass and angles are very sensitivity to their initial values. This chaotic motion can be seen in **Figure (3.3)** where is shown the trajectories of the end point**(obtained by numerical integration of the equations of motion)** of double pendulum.

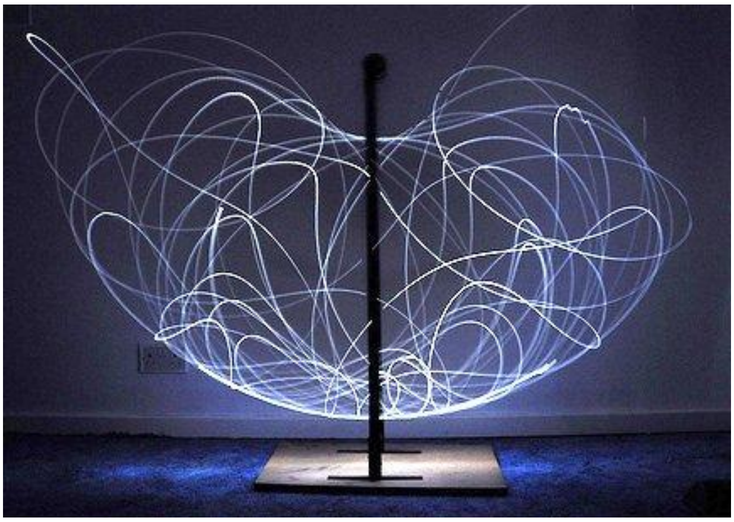

Figure (3.3). Trajectories of the end point of the double pendulum.

Physical and double pendulums in our laboratory are seen below.

Physical pendulum **Double pendulum.**

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REFERENCES

[1]T.Tomé e M.J.de Oliveira."Dinâmica Estocástica e Irreversibilidade". EDUSP (2001).
[2]M.Cattani. "Diffusion Process and Brownian Motion."
http://publica-sbi.if.usp.br/PDFs/pd1714.pdf
https://zenodo.org/record/3435003
[3] https://en.wikipedia.org/wiki/Galton_board
[4]P.M.Morse ."Thermal Physics". W.A.Benjamin,Inc.(1961).
[5]H.Poincaré. Acta Mathematica.13.1(1890).
[6]M.Cattani et al.Revista Brasileira.de Ensino de Física,vol.39,no.1,e1309(2017).
[7]https://pt.wikipedia.org/wiki/Duplo_P%C3%AAndulo
[8]A.Sommerfeld."Mechanics". Lectures on Theoretical Physics,vol.1(1952).
[9]R.Resnick and D.Halliday. Física 1(vol.2).Livros Técnicos e Científicos (1976).
[10]Iberê Caldas.
http://portal.if.usp.br/controle/sites/portal.if.usp.br.ifusp/files/5.%20Integrabilidade.pdf
[11]https://pt.wikipedia.org/wiki/Duplo_P%C3%AAndulo