TIME DILATION AND CONTRACTION IN GENERAL RELATIVITY

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Abstract. This paper was written to graduate and post-graduate students of Physics. In General Relativity context we have estimated differences of time that would be observed, in gravitational field, between travelling, fixed and rotating clocks.
Key words: *general relativity; gravitational effects; time dilation and contraction.*

INTRODUCTION.

In **Section I** are pointed some features of Special and General Relativity. It is also seen in General Relativity the *fundamental connection* between *time* and *spatial coordinates*. In **Section II** are taken into account clocks submitted to a weak gravitation field within the General Relativity context. In **Section III** is proposed an hypothetical experiment to estimate the time dilation between two clocks: one fixed at an spherical body with mass M and the other making a round trip to M. In **Section IV** is analyzed the time **dilation** measured between two clocks near the Earth surface. In **Section V** is estimated the time **contraction** when a clock is rotating around a spherical body.

(I) Special and General Relativity.

Special Relativity.

According to **Special Relativity** the space-time of the physical events, is a 4-dim Riemannian space named Minkowski space[1,2] with a pseudo-Euclidean metric. Light velocity is taken as constant in inertial and non-inertial reference systems. Let us consider systems with 4 coordinates, $x_1= x$, $x_2 = y$, $x_3 = z$ e $x_4 = ict$. In this space-time the *trajectory* of a particle, named *world line,* obeys the condition,

$$ds^2 = - dx^2- dy^2- dz^2- dx_4^2 = c^2dt^2 - dx^2- dy^2- dz^2 \qquad \textbf{(I.1),}$$

where the *invariant* ds is the *line element* in the 4-dim space. Where *invariant* means that ds is independent of the referential system.

Let us assume that in a time interval dt_o a clock Co fixed in S_o covers a distance $[dx_o^2 + dy_o^2 +dz_o^2]^{1/2}$ in S_o. What will be the time dt indicated by a watch C fixed in another system S ? As C is fixed in S, dx = dy = dz = 0 . Thus, as ds is invariant we get,

$$ds_o^2 = c^2dt_o^2 - dx_o^2 - dy_o^2 - dz_o^2 = c^2 dt^2 = ds^2,$$

giving $\quad dt = ds/c = (1/c)[c^2dt_o^2 - dx_o^2 - dy_o^2 - dz_o^2]^{1/2}$, that is,

$$dt = ds/c = dt_o[1- (dx_o^2 - dy_o^2 - dz_o^2)/c^2dt^2]^{1/2} = ds/c = dt_o [1-(v_o/c)^2]^{1/2} \qquad \textbf{(I.2).}$$

Putting $ds/c = d\tau$, defining $d\tau$ as the *proper time interval* of the clock C. That is, the time interval measured by a clock that moves attached to S. That is,

$$d\tau = [1 - (v_o/c)^2]^{1/2/}\, dt_o \qquad \textbf{(I.3).}$$

If an object is attached to the clock C, τ will be the *proper time* of this object.[3]

Consider, for instance, the case of an unstable elementary particle. Let us assume that it is at rest in an inertial system S_o where its lifetime is T_o. If S_o is moving with velocity V relative to a system S, its lifetime T measured in S would $T = \gamma T_o$, where $\gamma = 1/[1 - (V/c)^2]^{1/2}$.[1-3] As $\gamma \geq 1$ the particle **lifetime increases** when it is in motion.

General Relativity.

Einstein generalized the Newtonian gravitation theory assuming that the Minkowsky geometry is modified by the gravitational field.[3,4]Proposed that the new spacetime would be a 4-dim Riemannian space[3.4.]where the line element ds is given by

$$ds^2 = - c^2 d\tau^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu} \ (\mu,\nu = 1,2,3,4) \qquad \textbf{(I.4),}$$

where $g_{\mu\nu}(x)$ are *metric tensors*, determined by the mass distribution which creates the gravitational field.[3,5]

In General Relativity the choice for the reference system is arbitrary. The *space coordinates* x_1, x_2 and x_3, that define the position of the masses in the *space,* can be arbitrary quantities. The fourth component that would be the *temporal coordinate* x_o can be determined by a watch that marks its *proper time*.[3] The problem is to know, using these variables, how to describe **measured** distances and time intervals.

A particle traces in this 4-dim spacetime its *trajectory* ds (or *story)* defined as its *world line* and living its *proper time* or *real time* $d\tau$.

Let us show now how to determine the *fundamental connection* between the real time τ and the coordinate x_o. To do this let us take into account two simultaneous events happening in the same point in the space. That is,obeying the condition $\mathbf{dx = dx_2 = dx_3 = 0}$. So, the interval **ds** between these two events is *surely* $\mathbf{c d\tau}$, where $\mathbf{d\tau}$ is the interval of time (real) between the two events. As these events occurs in the same point of the space we have, according to **Eq.(I.4)**[3]

$$ds^2 = - c^2 d\tau^2 = g_{oo} dx_o^2 \ ,$$

that is,

$$d\tau = (1/c)(-g_{oo})^{1/2}\, dx_o, \qquad \textbf{(I.5),}$$

showing that the time $\boldsymbol{\tau}$ elapsed between two arbitrary events happening in the same point of the space is given by

$$\tau = (1/c)\int [-g_{oo}(x_o)]^{1/2} dx_o \qquad \textbf{(I.6).}$$

These relations determine the *real times* $\boldsymbol{\tau}$ (or, as we say, the *proper time* in a given point of the space) as a function of the coordinate $\mathbf{x_o}$. Note that to have *real* times it is necessary that $\mathbf{g_{oo} < 0}$. If this condition is not satisfied the coordinate system do not represent real events.

(II)Clocks in Gravitational Field.

Let us analyze time intervals marked by clocks located in the vicinity of a spherical body with mass M and radius R neglecting rotational and relativistic effects .

It will be assumed that the body is an inertial reference frame and that its gravitational field $\varphi(r) = - GM/r$ is *weak*. In these conditions $g_{oo}(x_o)$ becomes given by,[3]

$$g_{oo}(x_o) = - 1 - 2\varphi(r)/c^2. \qquad \textbf{(II.5).}$$

In this way, from **Eq.(I.5):**

$$d\tau = (1/c)[1 + 2\varphi(r)/c^2]^{1/2}dx_o = [1 + 2\varphi(r)/c^2]^{1/2}dt \qquad \textbf{(II.6)}$$

When $\varphi(r) = - GM/r$, using Schwarzschild metric(see **Appendix)** and **Eq.(II.6)** for one clock at r_1 we have

$$d\tau = 1 - (2GM/r)c^2]^{1/2}\, dt \qquad \textbf{(II.7),}$$

showing that as r increases clock runs faster.

If $\varphi(r) = - GM/r$ is *weak*, using **Eq.(II.6)** we obtain

$$d\tau \approx (1 - R_s/2r)dt \qquad \textbf{(II.8),}$$

where $R_s = 2GM/c^2$ is the Schwarzschild radius of the body with mass M.

(III)Time Dilation in Weak Gravitational Field.

Let us take two identical clocks. Clock (1) fixed at the Earth surface, that is, at the sea level at r = R. Clock (2) will perform a **round trip**, that is, going from r = R, up to r = R + h and coming back to the sea level, that is, at r = R. If during this trip clock (1) runs a time $\boldsymbol{\tau_1} = 2T$ what will be the time $\boldsymbol{\tau_2}$ measured by clock (2)?

In our **ideal experiment** clock (2) will be transported through the gravitation field with an *average* velocity $dr/dt = V << c$. V will be assumed very small; unable to create meaningful kinematic relativistic effects and accelerations during the trip and at the points r = R and r = R + h. Time $\boldsymbol{\tau_2}$ will be estimated taking into account the round trip using **Eqs.(I.6)** and **(II.8):**

$$\tau_2 = (1/c) \oint [-g_{oo}(x_o)]^{1/2}dx_o = \oint [1 - R_s/2r(t)]dt \qquad \textbf{(III.1),}$$

In the first step of the trip, putting $dt/dr \approx dr/V$, when t goes from 0 to T and r(t) from R to R+ h, we get from **Eqs.(II.8)** and **(III.1)**,

$$\int_0^T \left(1 - \frac{Rs}{2r}\right) dt \approx T - (Rs/2V)\ln\left\{\frac{R + h}{R}\right\} \qquad \textbf{(III.2).}$$

In the second step, that is, that goes from r = R + h up to r = R we obtain, putting $dr/dt = - V$, and following similar calculations done before,

$$T - (R_s/2V)\ln\{(R + h)/R\} \qquad \textbf{(III.3).}$$

So, taking into account Eqs.**(III.2)** and **(III.3),** $\boldsymbol{\tau}_2$ measured by the clock (2), along the round trip $r = R \rightarrow R + d \rightarrow R$, is given by

$$\boldsymbol{\tau}_2 \approx 2T - (R_s/2V)\ln\{(R + VT)/R\} \qquad \textbf{(III.4).}$$

So , the time difference between clocks 1 and 2 will be

$$\Delta\boldsymbol{\tau} = \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \approx (R_s/V)\ln\{(R + VT)/R\} \qquad \textbf{(III.5),}$$

showing that clock 2, which is in motion, **runs more slowly** than clock 1**.**

Let us estimate $\Delta\boldsymbol{\tau}$ for **Earth** and **Neutron Stars.**

(a) Earth.

$R_s = 8.9$ mm $\sim 10^{-6}$ Km and radius $R \sim 10^4$ Km. Let us assume V = 100 Km/h and that the travel time is T ~ 10 years $\sim 10^5$ hours. From **Eq.(III.5)** we see that

$$\Delta\boldsymbol{\tau} \sim 10^{-8}\ln(10^7/10^4) \sim \mathbf{10^{-7}\ hours}!!!!$$

Assuming that the **time dilation** $\Delta\boldsymbol{\tau}^*$ would be created only by *kinematic relativistic* effects, predicted by the Special Relativity, we obtain

$$\Delta\boldsymbol{\tau}^* = (2T)\gamma \qquad \textbf{(III.6),}$$

where $\gamma = 1/[1 - (V/c)^2]^{1/2}$. Thus, as V = 100 km/h and $c = 10^9$ Km/h we have $\gamma = 1/[1 - (V/c)^2]^{1/2} \sim 1 + (V/c)^2 \sim 1 + 10^{-14}$. This, would imply that

$$\Delta\tau^* \approx 10^5 * 10^{-14} \text{ hours} = 10^{-9} \text{ hours} \qquad \textbf{(III.7),}$$

showing that the **kinematic dilation time** would be much smaller than that created by gravitational effects, $\Delta\boldsymbol{\tau} \sim \mathbf{10^{-7}\ hours}$.

(b)Neutron Stars.

$R_s \sim 10^3$ Km and radius $R \sim 10^4$ Km. Assuming, as before, V =100 Km/h and the travel time T ~10 years $\sim 10^5$ hours we obtain from **Eq.(III.5),**

$$\Delta\boldsymbol{\tau} \sim 10^2\ln(10^7/10^4) \sim \mathbf{10\ hours.}$$

(IV)Time Dilation near Earth.

First measurements of gravitational *time dilation* near the Earth were done by Pound and Rebka[6] and Pound and Snider.[7] Recently these measurements have also been performed by American undergraduate students.[8] These were done by comparing the signals generated by a GPS frequency standard (sea level time τ) to a Cs-beam frequency standard at different altitude h above sea level. Very small time dilation due Earth´s rotation was neglected.

In these experiments were used two clocks: one at sea level at r = R and another orbiting at r = R + h. When h/R << 1, time intervals predicted by these clocks are given by **Eq.(II.8):**

$$d\tau \approx (1 - R_s/2r)dt \approx 1 - (R_s/2R)(1 + h/R)^{-1}dt, \qquad \text{that is,}$$

$$d\tau \approx 1 - \{(R_s/2R) - (R_s/2R)(h/R)\}\, dt \qquad \textbf{(IV.1),}$$

where $R_s = 2GM/c^2$ is the Earth Schwarzschild radius.

The $d\tau$ measurements[8] were performed comparing signals generated by GPS frequency standard at different altitudes h above sea level. As the contributions of the term $(R_s/2R)$ vanishes in the GPS frame, putting $dt = d\tau_o$, **Eq.(IV.1)** becomes,[8]

$$(d\tau/d\tau_o) = 1 + (R_s/2R)(h/R)$$

or

$$\tau_h = \{1 + (R_s/2R^2)h\}\, \tau_o \qquad \textbf{(IV.2),}$$

showing that the proper time, $\boldsymbol{\tau_h}$,elapsed on a clock at a height **h** above **R** is, therefore, *greater* than the time $\boldsymbol{\tau_o}$ elapsed on the GPS clocks at r = R. That is, we have a "*time dilation*". Putting $\Delta\tau = \tau_h - \tau_o$ and remembering that $GM/R^2 = g$ is the acceleration due to the gravity we have

$$\Delta\tau = (g/c^2)h\, \tau_o \qquad \textbf{(IV.3).}$$

Their $\Delta\tau$ measurements give $\Delta\tau \approx 9.5\ 10^{-9}$ s /day Km.[8]

(V)Rotational Time Contraction.

Let us consider a clock fixed at a sphere with radius R which is rotating around the z-axis with angular velocity Ω . Thus, taking into account the Schwarzschild metric, shown in **Appendix,** putting $dr = d\theta = 0$, $d\varphi/dt = \Omega$ and neglecting gravitational effects,

$$d\tau = [1 - \Omega^2 r^2/c^2\,]^{1/2}\, dt \qquad \textbf{(V.1),}$$

according to **Section II**, since $g_{oo}(x_o) = 1 - \Omega^2 r^2/c^2$.

In the case of a clock fixed at the **Earth** equator, that is, at $r \approx 6.4\ 10^6$ m and remembering that $\Omega \approx 7.3\ 10^{-5}$ s, we obtain with **Eq.(V.1)**

$$\tau(R) = (1 - 2\times 10^{-12})\tau_o \qquad \textbf{(V.2),}$$

letting $\tau(R)$ be the *proper time* measured by a clock at the equator sphere. The time τ_o can be the elapsed time far from the Earth or at the North or South Poles, that is, $\tau(NP) = \tau(SP) = \tau_o$. **Eq.(V.1)** shows a time contraction, that is,

$$\Delta\tau = -\,2\times 10^{-12}\, \tau_o \qquad \textbf{(V.3).}$$

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APPENDIX. Schwarzschild Metric and Metric with Rotation.

The **Schwarzschild metric** that describes the spacetime around a spherically symmetric body with mass M and radius R is given by[2,3,8]

$$-ds^2 = c^2 d\tau^2 = c^2(1 - R_s/r)\, dt^2 - [1 - R_s/r]^{-1}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 \qquad \textbf{(A.1),}$$

where $R_s = 2GM/c^2$. If $R_s/r << 1$ and for a rotating sphere with $dr = d\theta = 0$,we have,

$$(d\tau/dt)^2 = (1 - R_s/r)^2 - \{(r/c)(d\varphi/dt)\}^2 \sin^2\theta = (1 - R_s/r)^2 - (r^2\Omega^2/c^2) \sin^2\theta \qquad \textbf{(A.2),}$$

where $\Omega = d\varphi/dt$ is the angular velocity of the sphere along the z-axis.

When only rotational effects are preponderant we have,[2,3,8]

$$d\tau^2 = -(c^2 - \Omega^2 r^2)dt^2 + 2\Omega r^2 d\varphi dt + dr^2 + r^2 d\varphi^2 + dz^2 \qquad \textbf{(A.3),}$$

from which we see that $g_{oo} = -(c^2 - \Omega^2 r^2)$ and, using **Eq.(I.5)** we obtain,

$$d\tau = [1 - \Omega^2 r^2/c^2]^{1/2}\, dt \qquad \textbf{(A.4).}$$

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