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LEE-NAUENBERG THEOREM AND COULOMB SCATTERING

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ABSTRACT

We extend the Lee-Nauenberg analysis to the case of Coulomb scattering, where the diagonal elements of the interaction Hamiltonian are singular functions.
We then show, using a simple argument, that the leading infrared singularities in the cross-section cancel out.

1. Introduction

It is well known that field theories containing massless particles have infrared singularities associated with the fact that these particle can propagate to arbitrarily large distances. Such is the case of Quantum Eletrodynamics (QED) and, even more importantly, of Quantum Chromodynamics (QCD) which is a non-abelian gauge theory of strong interactions containing massless gluons. The cancellation of the infrared divergences for physical processes like cross-sections e.t.c., is by now well understood in QED[1]. On the other hand, the same cannot be said of QCD where this cancellation has been verified only in lowest orders of perturbation theory for the leading divergences[2]. In this case it results from the consideration of a large number of Feynman diagrams and so it is difficult to have a simple understanding of the reason why the cancellation does occur.

It is therefore important to try to find out another method which hopefully can provide a way for a more direct understanding of this phenomenon. An approach in this direction has been undertaken some time ago by Lee and Nauenberg[3] who have shown that in some cases these cancellations are consequences of an elementary theorem in quantum mechanics. We shall briefly review here their approach mainly in order to explain its limitations and to point out a possible generalization. Consider an arbitrary Hamiltonian (H_0+gH_1) which can be diagonalized by a unitary matrix U:

$$U^{\dagger}(H_0+gH_1)U = E \tag{1}$$

where H_o and E are diagonal matrices and g is the coupling constant. $U=U_-$ or U_+ depending on whether incoming or outgoing scattered waves are used. The S matrix is given by:

$$S = U_-^\dagger U_+ \tag{2}$$

so that the transition probability from a state a to a state b is:

$$|S_{ab}|^2 = \sum_{ij} \left[(U_+)_{ia} (U_+)^*_{ja} \right] \left[(U_-)_{ib} (U_-)^*_{jb} \right]^* \tag{3}$$

Lee and Nauenberg have shown that, provided $(H_1)_{ii}$ is a finite quantity, the infrared divergences are completely cancelled in the power series expansion of the sum:

$$\sum_{D(E_a)} (U_+)_{ia} (U_+)^*_{ja} \equiv \left[T_+ (E_a) \right]_{ij} \tag{4}$$

Here the summation extends over all states degenerated in energy with the state a [of course, the same type of consideration applies for $[T_-(E_b)]_{ij}$].

This theorem is very simple and powerful. Unfortunately, it does not directly apply to the realistic cases where $(H_1)_{ii}$ is not a well behaved function, in general. For instance, the Hamiltonian which describes the familiar Coulomb scattering becomes singular in this case.

We conjecture that in these cases the matrices $T_\pm$ are still relevant quantities in the understanding of the cancellations of the infrared singularities. In fact, we show in the next section that the leading infrared divergences completely cancel in the product:

$$\mathcal{P}_{ij}^{ab} \equiv \left[T_+(E_a) \right]_{ij} \left[T_-(E_b) \right]_{ij}^{*} \qquad (5)$$

in the case of the scattering of a charged particle by an external Coulomb potential.

2. Coulomb scattering in an external potential

We will now consider the interaction Hamiltonian which describes the scattering of a charged particle (a fermion of charge e) by an external potential:

$$V^{ext} = \frac{Ze}{4\pi} \frac{e^{-\lambda r}}{r} \qquad (6)$$

which becomes in the limit $\lambda \to 0$ the familiar Coulomb potential. This procedure has the advantage of providing a regularization for the infrared singularities. In fact, this method has been used by Dalitz (4) whose calcullations in lowest orders indicate that the leading divergences exponentiate in the scattering amplitude.

From (6) we obtain as usual for the Hamiltonian in momentum space:

$$H_{int}(p,p') = \left(\frac{m}{E_{p'}V}\right)^{1/2} \left(\frac{m}{E_p V}\right)^{1/2} \bar{u}(p') \gamma_0 u(p)\, g \frac{1}{(\vec{p}\,' - \vec{p}\,)^2 + \lambda^2} \qquad (7)$$

where $g = Ze^2$ and $E_p \equiv (\vec{p}\,^2 + m^2)^{1/2}$, m being the fermion mass. Note that as $\vec{p} \to \vec{p}\,'$, H_{int} becomes a singular function, as we have mentioned previously.

From equation (4) we see that the infrared singularities of the T matrix are determined by those of the matrices U which are related to H_1 via equation (1).

Now, the matrices $U_\pm$ involve the limit as $t \to \pm\infty$ of $U(t,o)$ over an infinite time interval. To establish the existence of such limits it is useful to multiply the coupling constant g by a slowly varying function of time, say $e^{-\varepsilon |t|}$, where ε is a small positive number. Then, the matrices $U_\pm^{\varepsilon,n}$ defined by the power series expansion of $U_\pm^{\varepsilon}$:

$$U_\pm^{\varepsilon} = \sum_{n=o}^{\infty} g^n \, U_\pm^{\varepsilon,n} \tag{8}$$

will be given recursivelly in terms of the Hamiltonian H_1 by the relation[3]:

$$\left(U_\pm^{\varepsilon,n+1}\right)_{ij} = \frac{1}{E_{p_j} - E_{p_i} \pm i(n+1)\varepsilon} \left\{ \sum_k (H_1)_{ik} \left(U_\pm^{\varepsilon,n}\right)_{kj} + \sum_{m=1}^{n+1} (\Delta^m)_{ii} \left(U_\pm^{\varepsilon,n+1-m}\right)_{ij} \right\} \tag{9}$$

where

$$\Delta \equiv H_o - E = \sum_{n=1}^{\infty} g^n \Delta^n \tag{10}$$

In terms of these quantities, the matrices $T_\pm^n$ defined by the power series expansion:

$$T_\pm(E) = \sum_{n=o}^{\infty} g^n \, T_\pm^n(E) \tag{11}$$

will be given, using equation (4), by the relation:

$$\left[T_{\pm}^{n+1}(E_c)\right]_{ij} = \sum_{m=0}^{n+1} \left(U_{\pm}^{\varepsilon,m}\right)_{ic} \left(U_{\pm}^{\varepsilon,n+1-m}\right)^{*}_{jc} \qquad (12)$$

Here we have used the fact that, in the case of the pure Coulomb scattering we are considering here, there are no other states which become degenerate in energy with the state c in the limit $\lambda \to 0$.

From the structure of equation (9) we see that the infrared singularities arise when the energy denominators vanish and when the interaction Hamiltonian becomes singular in the sum over intermediate states.

We will consider here only the leading infrared singularities which are proportional to $g^2 \ln \lambda$ for each integration over internal momenta. [In the continuum $\sum_k \to V(2\pi)^{-3} \int d^3k$].

Then the leading divergence in the matrix element $U_{\pm}^{n+1}$ will be proportional to $(g^2 \ln \lambda)^n$ and arises from the n integrations implicit in the iterative solution of equation (9).

We can now conclude, using equation (12) that the leading divergence of the matrix element $(T_{\pm}^{n+1})^L$ will be proportional to $(g^2 \ln \lambda)^n$ and comes from only two terms:

$$\left[T_{\pm}^{n+1}(E_c)\right]^L_{ij} = \delta_{jc} \left(U_{\pm}^{\varepsilon,n+1}\right)^L_{ij} + \delta_{ic} \left(U_{\pm}^{\varepsilon,n+1}\right)^{*L}_{ji} \qquad (13)$$

where we used the fact that $(U_0)_{ij} \doteq \delta_{ij}$. All other terms in equation (12) are proportional at most to $(g^2 \ln\lambda)^{n-1}$ and hence are not relevant for our purposes.

We must therefore calculate the leading infrared devergence of the matrix $(U_{\pm}^{\varepsilon, n+1})_{ij}$. Using equation (7) and (9) we obtain:

$$\left(U_{\pm}^{\varepsilon, n+1}\right)_{ij}^{L} = \frac{1}{E_{p_j} - E_{p_i} \pm i\varepsilon} \left(\frac{m}{VE_{p_i}}\right)^{1/2} \bar{u}(p_i) \frac{V}{(2\pi)^3} \int d^3k_1 \cdots \frac{V}{(2\pi)^3} \int d^3k_n \times$$

$$\times\, g \frac{1}{|\vec{p}_j - \vec{p}_i + \vec{k}_1 \ldots + \vec{k}_n|^2 + \lambda^2}\, g \frac{1}{k_n^2 + \lambda^2} \frac{m}{VE_{p_j + k_1 + \ldots k_n}} \gamma_0 \times$$

$$\times \frac{\vec{\gamma}\cdot(\vec{p}_j + \vec{k}_1 + \ldots \vec{k}_n) - \gamma_0 E_{p_j + k_1 + \ldots k_n} + im}{2im} \cdot \frac{1}{E_{p_j + k_1 + \ldots k_n} - E_{p_j} \pm i\varepsilon} \cdots \times$$

$$\times\, g \frac{1}{k_1^2 + \lambda^2} \frac{m}{VE_{p_j + k_1}} \gamma_0 \frac{\vec{\gamma}\cdot(\vec{p}_j + \vec{k}_1) - \gamma_0 E_{p_j + k_1} + im}{2im} \cdot \frac{1}{E_{p_j + k_1} - E_{p_j} \pm i\varepsilon} \times$$

$$\times\, \gamma_0 \left(\frac{m}{VE_{p_j}}\right)^{1/2} u(p_j) \qquad (14)$$

which is represented digramatically in figure 1.

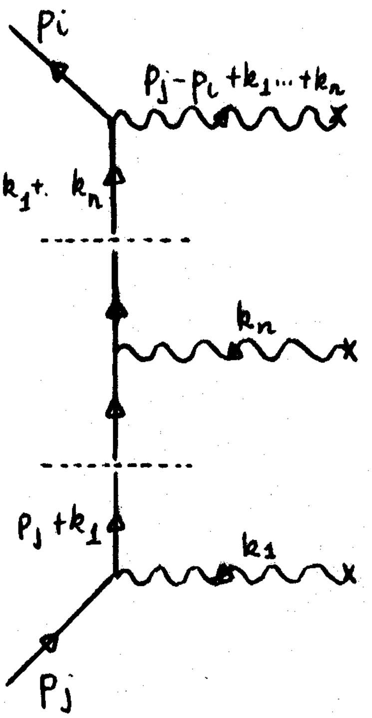

Figure 1. Graphical representation for the matrix element $(U_{\pm}^{\varepsilon, n+1})_{ij}$. The wiggly lines represent the propagator of the Coulomb field and the dotted lines stand for the sum over intermediate states.

Since we are interested only in the leading infrared divergences, we can considerably simplify expression (14) by noting that these occur only when all momenta $\vec{k}_1, \ldots \vec{k}_n$ are simultaneously small quantities. In this case equation (14) reduces to:

$$\left(u_{\pm}^{\varepsilon, n+1}\right)_{ij}^{L} = \frac{1}{E_{p_j} - E_{p_i} \pm i\varepsilon} \left(\frac{m}{E_{p_i} V}\right)^{1/2} \left(\frac{m}{E_{p_j} V}\right)^{1/2} g \, \frac{1}{|\vec{p}_j - \vec{p}_i|^2 + \lambda^2} \times$$

$$\times \, \bar{u}(p_i) \, \gamma_0 \, u(p_j) \, I_{\pm}^{n} \tag{15}$$

where $I_{\pm}^{n}$ is given by the following expression:

$$I_{\pm}^{n} = \left[\frac{g}{(2\pi)^3}\right]^n (i E_{p_j})^n \int \frac{d^3 k_1}{\vec{k}_1^{\,2} + \lambda^2} \cdots\cdots \int \frac{d^3 k_n}{\vec{k}_n^{\,2} + \lambda^2} \times$$

$$\times \frac{1}{\vec{p}_j \cdot \vec{k}_1 \pm i\varepsilon} \cdots \frac{1}{\vec{p}_j \cdot (\vec{k}_1 + \ldots \vec{k}_n) \pm i\varepsilon} \tag{16}$$

In order to obtain this expression we have neglected $\vec{k}^2$ terms compared with $\vec{p}\,\vec{k}$ in the fermion denominators. This is just the eikonal approximation which does not affect the leading divergences.

We now make use of the important identity:

$$\sum_{per} \frac{1}{a_1 \ldots (a_1 + \ldots + a_n)} = \frac{1}{a_1} \cdots\cdots \frac{1}{a_n} \tag{17}$$

where the sum is over all permutations of the indices 1,...n.

Then we can write $I_{\pm}^{n}$ as follows:

$$I_{\pm}^{n} = \frac{1}{n!} \left(I_{\pm}\right)^n \tag{18}$$

where $I_\pm$ represents the lowest order contribuition:

$$I_\pm = \frac{g}{(2\pi)^3} \, i E_{p_j} \int d^3k \, \frac{1}{\vec{k}^2 + \lambda^2} \, \frac{1}{\vec{p}_j \cdot \vec{k} \pm i\varepsilon} \tag{19}$$

The infrared divergent part of this integral is calculated more easily, using the usual Feynman parametrization, by reinstating the factor $\vec{k}^2$ in the fermion denominator:

$$I_\pm^{id} = 2 \, \frac{g}{(2\pi)^3} \, i E_{p_j} \int d^3k \, \frac{1}{\vec{k}^2 + \lambda^2} \cdot \frac{1}{\vec{k}^2 + 2\vec{p}_j \, \vec{k} \pm i\varepsilon}$$

$$= 2 \, \frac{g}{(2\pi)^3} \, i E_{p_j} \left(\pm \frac{i\pi^2}{|\vec{p}_j|} \ln \frac{|\vec{p}_j|^2}{\lambda^2} \right) \tag{20}$$

Substituting equation (20) in the relation (15) and using (18) we obtain for the leading divergences of the matrix element $\left(U_\pm^{\varepsilon, n+1}\right)_{ij}$ the result:

$$\left(U_\pm^{\varepsilon, n+1}\right)_{ij}^L = \frac{1}{E_{p_j} - E_{p_i} \pm i\varepsilon} \left(\frac{m}{E_{p_i} V}\right)^{1/2} \left(\frac{m}{E_{p_j} V}\right)^{1/2} g \, \frac{1}{|\vec{p}_j - \vec{p}_i|^2 + \lambda^2} \times$$

$$\times \, \bar{u}(p_i) \gamma_0 u(p_j) \, \frac{1}{n!} \left(\frac{g}{4\pi} \, \frac{\mp E_{p_i}}{|p_j|} \ln \frac{|p_j|^2}{\lambda^2} \right)^n \tag{21}$$

We are now in a position to calculate the leading infrared behaviour of the matrix $[T_\pm(E_c)]_{ij}$. With the help of the equations (8), (11) and (13) we find:

$$[T_\pm(E_c)]_{ij}^L = \delta_{jc} (U_\pm)_{ij}^L + \delta_{ic} (U_\pm)_{ji}^{*L} \tag{22}$$

where the matrix element $(U_\pm)^L_{ij}$ is given, using equation(21), by the relation:

$$(U_\pm)^L_{ij} = \delta_{ij} + \frac{1}{E_{p_j} - E_{p_i} \pm i\varepsilon} \left(\frac{m}{E_{p_i} V}\right)^{1/2} \left(\frac{m}{E_{p_j} V}\right)^{1/2} g \times$$

$$\times \frac{1}{|\vec{p}_i - \vec{p}_j|^2 + \lambda^2} \bar{u}(p_i) \gamma_0 u(p_j) \exp\left[\mp \frac{g}{4\pi} \frac{E_{p_j}}{|\vec{p}_j|} \ln \frac{|p_j|^2}{\lambda^2}\right] \tag{23}$$

We can finally return to our equation (5). We consider non-forward scattering, so that the state a is different from state b. Of course, $E_a = E_b$ since the energy of the scattered particle is conserved. It is then easy to show, using equations (22) and (23), that the leading infrared singularities do cancell in the product P^{ab}_{ij}.

3. Conclusion

We have seen that, in the case when the diagonal elements of the interaction Hamiltonian are singular functions, the $T_\pm$ matrices do contain infrared singularities. But their infrared behaviour in the leading approximation is very simple: the leading infrared divergences sum into an exponential.
Furthermore these are given by linear combinations of the $U_\pm$ matrices (see equation 22). Consenquently, the leading infrared singularities cancel, in the product $T_+ T_-^*$ (see equation 5), whence it follows, in this approximation, that the cross section is a finite quantity.
The extension of these results to the case of QED is, in principle, rather simple, since in this case only the Hamiltonian describing the Coulomb interaction is a singular function.

However, the generalization to QCD is, in the Coulomb gauge, more complicated, in view of the fact that the interaction Hamiltonian contains an infinite number of such functions[5]. Presumably simplifications will occur in the axial gauge where the Hamiltonian contains only a small number of singular terms[6]. We shall report on these and related matters in a future communication.

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