Time Contraction, Dilation and Twin Paradox in General Relativity

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Abstract.

This paper was written to graduate and undergraduate students of Physics and Engineering. Are analyzed, theoretically and experimentally, contractions and dilations of time predicted by the General Relativity. In this context we also have done a rough estimation of the "twin paradox" in gravitational field.

key words: *General Relativity; time dilation and contraction; twin paradox.*

(I) Special Relativity Approach.

"Of all the supposed paradoxes engendered by relativity theory the *twin paradox* (or *clock paradox*)is most famous and has been the most controversial. It asserts that if one clock remains at rest in an inertial frame, and another, initially agreeing with it, is taken off on any sort of path and finally brought back to its starting point, the second clock will have lost time as compared with the first. In today´s parlance, the astronaut will end up by becoming younger than his twin brother."[1] This result, which was stated by Einstein in his very first relativity paper in 1905, became the subject of a raging controversy in physics literature during many years.[1]

Many papers have been published on this subject inside the Special Relativity context.[2] In our opinion the most simple and convincing analysis about this was done by French.[1]To do this he has taken into account the space-time invariant, where x_4 = ict,

$$ds^2 = - dx^2 - dy^2 - dz^2 - dx_4^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \qquad \textbf{(I.1)},$$

in a 2 dimension space (ict, x), given by

$$ds^2 = c^2dt^2 - dx^2 \qquad \textbf{(I.2)}.$$

In **Figure (I.1)**[1] are shown the two axis and world lines of the two brothers: (**ODF**) and of his twin (**OCF**). The **OCF** remains at rest at **O** in the time interval from 0 up to T. The (ODF) goes in a rocket from **O** to **D** in Δt = T/2 and after from **D** to **F** also in Δt = T/2.

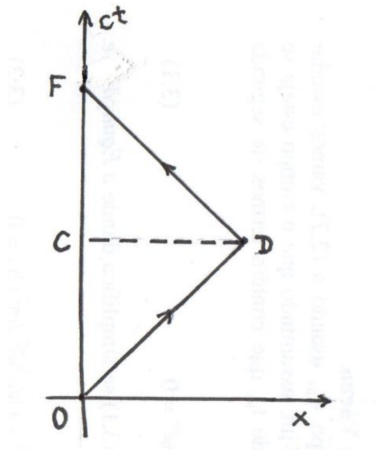

Figure (I.1). Twin Paradox in Special Relativity.

World lines of the astronaut (**ODF**) and his brother (**OCF).**

$$\int_{OCF} ds = c \int_{OCF} dt = cT \qquad \textbf{(I.3).}$$

Assuming that the ODF astronaut moves with a constant velocity v, that is, dx/dt = v, and that along OD we have $ds = dt\,(c^2-v^2)^{1/2}$ we get

$$\int_{OD} ds = c\,(c^2-v^2)^{1/2} \int_{OD} dt = cT/2\gamma \qquad \textbf{(I.4),}$$

where $\gamma = 1/[1-(v/c)^2]^{1/2}$.

Along DF, with dx/dt = -v see that,

$$\int_{DF} ds = c\,(c^2-v^2)^{1/2} \int_{DF} dt = cT/2\gamma \qquad \textbf{(I.5).}$$

Hence, from **Eqs.(I.3)**, **(I.4)** and **(I.5)** we verify that along the path ODF, the expended time is **T/γ**. As $\gamma \geq 1$ the times runs more slowly in the rocket. That is, the twin that arrive at Earth after the trip ODF is younger.

It is important to note, as seen above, in Hermann Bondi´s words, that "**ds^2 is a route - dependent quantity."** ....

(II) General Relativity Approach.

Einstein generalized the Newtonian gravitation theory assuming that Minkowsky geometry is modified by the gravitational field.[3-5] Proposed that the new spacetime would be a 4-dim Riemannian space[3-5]where the line element ds is given by

$$ds^2 = -\,c^2 d\tau^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \;(\mu,\nu = 1,2,3,4) \qquad \textbf{(II.1),}$$

where $g_{\mu\nu}(x)$ are *metric tensors*, determined by the mass distribution which creates the gravitational field.[3-5]

In General Relativity the choice for the reference system is arbitrary. The *space coordinates* x_1, x_2 and x_3, that define the position of the masses in the *space,* can be arbitrary quantities. The fourth component that would be the *temporal coordinate* $x_o = ct$. It is very important to realize[5-6] that this time is not the time that cocks placed in the gravitational field give us. Rather, **t** is the time indicated by a clock placed at large distance from the gravitational body. Note that *real* clocks always show **proper time** τ .In the limit $r \to \infty$, $g_{oo} \to 0$ and then $d\tau$ coincides with dt. A clock placed at a finite distance shows a *time interval dτ that is smaller than dt*. The essential problem of the metric $g_{\mu\nu}(x)$ is to know, using these variables, how to describe **measured** distances and **measured** time intervals.

A particle traces in this 4-dim spacetime its *trajectory* ds (or *story)* defined as its *world line* and living its *proper time* or *real time* $d\tau$.

Let us show now how to determine the *fundamental connection* between the *real time τ* and the *coordinate x_o*. To do this let us take into account two simultaneous events happening in the same point in the space. That is, obeying the condition $\mathbf{dx = dx_2 = dx_3 = 0}$. So, the interval **ds** between these two events is *surely* $\mathbf{cd\tau}$, where $\mathbf{d\tau}$ is the interval of time (real) between the two events. As these events occurs in the same point of the space we have, according to **Eq.(II.1)**[5,6]

$$ds^2 = - c^2 d\tau^2 = g_{oo} dx_o^2 \ ,$$

that is,

$$d\tau = (1/c)(-g_{oo})^{1/2}\, dx_o, \qquad \textbf{(II.2),}$$

showing that the time $\boldsymbol{\tau}$ elapsed between two arbitrary events happening in the same point of the space is given by

$$\tau = (1/c)\int [-g_{oo}(x_o)]^{1/2} dx_o \qquad \textbf{(II.3).}$$

These relations determine the *real times* $\boldsymbol{\tau}$ (or, as we say, the *proper time* in a given point of the space) as a function of the coordinate $\mathbf{x_o}$. Note that to have *real* times it is necessary that $\mathbf{g_{oo} < 0}$. If this condition is not satisfied the coordinate system do not represent real events.

(III)Clocks in Weak Gravitational Field.

Let us analyze time intervals marked by atomic clocks[6] located in the vicinity of a spherical body with mass M and radius R neglecting rotational and relativistic effects. In inertial reference frame with a gravitational field $\varphi(r)$ the $g_{oo}(x_o)$ created by this body is given by [5,6]

$$g_{oo}(x_o) = - [1 + 2\varphi(r)/c^2]^{1/2} \qquad \textbf{(III.1)}$$

In this way, from **Eq.(II.2):**

$$d\tau = [1 + 2\varphi(r)/c^2]^{1/2}dt \qquad \textbf{(III.2).}$$

As $\varphi(r) = - GM/r$, with the Schwarzschild metric (**Appendix A),** two identical clocks, at r_1 and r_2, using (Eq.III.2) obey the relation

$$d\tau_2/d\tau_1 \approx [1 - (2GM/r_2c^2)]^{1/2}/ [1 - (2GM/r_1c^2)]^{1/2} \qquad \textbf{(III.3).}$$

So, if $r_2 > r_1$ we have $d\tau_2 > d\tau_1$, that is, clock 2 will run faster than clock 1.

When gravitational field $\varphi(r) = - GM/r$ is *weak* $g_{oo}(x_o)$ becomes,[5]

$$g_{oo}(x_o) = - 1 - 2\varphi(r)/c^2. \qquad \textbf{(III.4).}$$

and **Eq.(III.2)** can be written as

$$d\tau \approx (1 - R_s/2r)dt \qquad \textbf{(III.5),}$$

where $R_s = 2GM/c^2$ is the Schwarzschild radius of the body.

(IV)Time Dilation near Earth.

Many measurements have been performed[6-8] confirming the gravitational *time dilation* near the Earth. Recently these measurements have also been done by American undergraduate students.[9] These were done by comparing the signals generated by a GPS frequency standard to a Cs-beam frequency standard at different altitude **h** above sea level. Very small time dilation due Earth´s rotation was neglected.

They have used two Cs-clocks: one at sea level at r = R and another orbiting at r = R + h. When h/R << 1, time intervals predicted by these clocks are given by **Eq.(III.5):**

$$\boldsymbol{d\tau} \approx (1 - R_s/2r)dt \approx \{1 - (R_s/2R)(1 + h/R)^{-1}\}dt. \qquad \textbf{(IV.1)}$$

The proper time $\boldsymbol{d\tau}$ intervals are measured by a clock at distance r from the centre of the Earth. Neglecting the contributions of the first term in the parentheses, $R_s/(2R)$,[9] measurements were performed comparing signals generated by GPS frequency standard at different altitudes **h** above sea level. Letting $t = \tau_o$ be the time measured in the GPS frame (which is *far from the Earth*) **Eq.(IV.1)** becomes

$$d\tau_h/d\tau_o = 1 + (R_s/2R)(h/R) \qquad \textbf{(IV.2),}$$

showing that the proper time, $\boldsymbol{\tau_h}$, elapsed on a clock at a height **h** above **R** is, therefore, *greater* than the time $\boldsymbol{\tau_o}$ measured on the set of GPS clocks on the geoid and moving with the Earth, at r = R. From **Eq.(IV.2)** we get the **time dilation:**

$$\boldsymbol{\tau_h - \tau_o = \Delta\tau =} (R_s/2R^2)h\, \boldsymbol{\tau_o} \qquad \textbf{(IV.3),}$$

or, remembering that the gravity acceleration is $g = GM/R^2$:

$$\Delta\boldsymbol{\tau} = (g/c^2)h\, \tau_o \qquad \textbf{(IV.4).}$$

American students measurements[9] give $\Delta\tau \approx 9.5\ 10^{-9}$s/day Km.

(V)Rotational Time Contraction Near Earth.

Let us consider clock fixed at a sphere with radius R which is rotating around the z-axis with angular velocity Ω . Thus, taking into account the Schwarzschild metric, shown in **Appendix,** putting $dr = d\theta = 0$, $d\varphi/dt = \Omega$ and neglecting gravitational effects,

$$d\tau = [1 - \Omega^2 r^2/c^2\,]^{1/2}\, dt \qquad \textbf{(V.1),}$$

according to **Section II**, since $g_{oo}(x_o) = 1 - \Omega^2 r^2/c^2$.

For a clock fixed at the **Earth** equator, that is, at $r = R \approx 6.4\ 10^6$ m and remembering that $\Omega \approx 7.3\ 10^{-5}$ s, we obtain with **Eq.(V.1)**

$$\tau(R) = (1 - 2\ x10^{-12})\tau_o \qquad \textbf{(V.2),}$$

letting $\tau(R)$ be the *proper time* measured by a clock at the equator sphere. The time τ_o can be the elapsed time far from the Earth or at the North or South Poles, that is, $\tau(NP) = \tau(SP) = \tau_o$. **Eq.(V.2)** shows a time contraction, that is, at the equator time flows more slowly than at the poles...

(VI)Twin Paradox in Gravitational Field.

In **Section (VI.A)** the Twin Paradox will be analyzed in a weak gravitational field and in **Section (VI.B)** in a gravitational field that *might* not be weak.

(VI.A)Weak Gravitational Field.

Let us take two identical **atomic clocks**.[6] Clock (**1**) fixed at the *Earth* surface, that is, at the sea level at r = R. Clock (**2)** that will perform a **round trip**, that is, going from r = R, up to r = R + h and coming back to the sea level, that is, at r = R. If, during this process, Clock (**2**) runs a time $\boldsymbol{\tau_1}$ what will be the time $\boldsymbol{\tau_2}$ measured by Clock (**1**) at Earth?

We propose an **ideal experiment** where clock (**2**) will be transported by a rocket through the Earth gravitation field. Time $\boldsymbol{\tau_2}$ will be estimated, taking into account the round trip, using **Eq.(II.3)** and **Eq.(III.5):**

$$\boldsymbol{\tau_2} = (1/c) \oint [-g_{oo}(x_o)]^{1/2} dx_o = \oint [1 - R_s/2r(t)]dt \qquad \textbf{(VI.1),}$$

that can also be written as

$$\boldsymbol{\tau_2} = \oint dt - \oint R_s/2r(t)\, dt \qquad \textbf{(VI.2).}$$

Let us call T an **"ideal time"** measured by a clock in a region where gravitational effects are negligible, that is, T = ∫ dt , where t goes from 0 up to T. In this interval T, the time $\boldsymbol{\tau}$ measured by a clock**,** at rest at the Earth sea level, would be [1,3,4]

$$\boldsymbol{\tau} = T(1 - R_s/R)^{1/2} \approx T(1 - R_s/2R) \qquad \textbf{(VI.3).}$$

So, the time $\boldsymbol{\tau_1}$ measured by clock **(1),** that during the round trip takes a time 2T, would be[4]

$$\boldsymbol{\tau_1} = 2T(1 - R_s/2R) \qquad \textbf{(VI.4).}$$

On the other hand, time $\boldsymbol{\tau_2}$ would be given by,

$$\boldsymbol{\tau_2} = 2T - \oint R_s/2r(t)\, dt \qquad \textbf{(VI.5),}$$

We will perform only a **rough estimation** of the twin paradox. To avoid *extremely* complicate calculations intrinsic to the Riemann geometry[6] we will assume a naïve "technological" approach : "*the rocket has a propulsion engine able to maintain its velocity V constant during the round trip*". In addition, this velocity must be sufficiently small in order to create negligible kinematic relativistic effects and accelerations. In this way gravitation would be the main responsible by the time evolution.

So, in the **first step** of the trip, that is, when the rocket goes from R up R + h, that is, when the time t goes from 0 up to T, putting $dt \approx dr/V$, the expended time $\boldsymbol{\tau_{21}}$ would be given by, according to **Eq.(VI.5):**

$$\boldsymbol{\tau_{21}} = T - \int_o^T [R_s/2r(t)]dt \approx T - (1/V)\int_R^{R+h} (R_s/2V)dr$$

$$= T - (R_s/2V) \ln\{(R + h)/R\} \qquad \textbf{(VI.6),}$$

where $h = VT$.

In the **second step,** the rocket returns to Earth, that is, it comes from $r = R + h$ up to $r = R$. In these conditions putting $dr/dt = - V$, and following similar calculations done in the first step we obtain the expended time $\boldsymbol{\tau_{22}}$

$$\boldsymbol{\tau_{22}} = T - (R_s/2V)\ln\{(R + h)/R\} \qquad \textbf{(VI.7).}$$

Taking into account **Eqs.(VI.6)** and **(VI.7),** the time $\boldsymbol{\tau_2} = \boldsymbol{\tau_{21}} + \boldsymbol{\tau_{22}}$**,** measured by clock (**2**), along the round trip $r = R \rightarrow R + h \rightarrow R$, would be

$$\boldsymbol{\tau_2} \approx 2T - (R_s/V) \ln\{(R + VT)/R\} \qquad \textbf{(VI.8).}$$

So, the age difference $\boldsymbol{\tau_2 - \tau_1}$ between the twins**,** using **Eqs.(VI.4)** and **(VI.8),** is given by

$$\boldsymbol{\tau_2 - \tau_1} \approx (R_s/V) \ln\{(R + VT)/R\} \qquad \textbf{(VI.9).}$$

Earth Twins.

Let us estimate $\boldsymbol{\tau_2}$ and $\boldsymbol{\tau_1}$ in the case of the Earth taking $R \approx 10^4$ km and $R_s \approx 9$ mm~10^{-6} Km. Using **Eq.(VI.4)** the twin age $\boldsymbol{\tau_1}$, after a time T, is given by

$$\boldsymbol{\tau_1} = 2T(1 + R_s/2R) = 2T(1 + 10^{-10}) \approx 2T \qquad \textbf{(VI.10).}$$

After the round trip, the twin age $\boldsymbol{\tau_2}$, given by **Eq.(VI.8)**, will be, assuming that $h = VT >> R$:

$$\boldsymbol{\tau_2} \approx 2T - (R_s/V) \ln(VT/R) \qquad \textbf{(VI.11).}$$

If, for instance, $V = 10^2$ Km/h and $2T = 100$ years ~$2\ 10^5$ hours we get,

$$\boldsymbol{\tau_1 - \tau_2} \approx (R_s/V) \ln\{(R + VT)/R\} \approx 10^{-8} \ln(10^{-2}\ T) \sim \mathbf{10^{-7}\ hours} \qquad \textbf{(VI.12).}$$

Conclusion: the twin at Earth would have 100 years ~ $2\ 10^5$ hours of age and his brother would be only ~10^{-7} hours younger.

Special Relativity Estimation.

Let us estimate the time dilation that would be created only by *kinematic relativistic* effects, according to the Special Relativity(SR).[1,3,4] To do this we take a clock **1** which is at rest in an inertial system S and a clock **2** fixed in another reference system S´. If S´ is moving with velocity V relative to S, according to the SR, in S´ the clock **2** would be indicating a time $T' = \gamma T$, where $\gamma = 1/[1 - (V/c)^2]^{1/2}$. In the small velocity limit, that is, when V << c we verify that $\gamma = 1/[1 - (V/c)^2]^{1/2} \approx\sim 1 + (V/c)^2$. This would imply that $T' \approx T \{1 + (V/c)^2 \}$. Thus, the time difference between S and S´ would be $\Delta\boldsymbol{\tau} \approx T(V/c)^2$. When V = 100 km/h and T = 10^5 hours we see that $(V/c)^2 \sim 10^{-14}$ and $\Delta\boldsymbol{\tau} \approx 10^{-9}$ hours. This shows that the SR time dilation would be much smaller then $\Delta\boldsymbol{\tau} \sim 10^{-7}$ hours that was created by gravitational effects estimated with **Eq.(VI.12)**..

(VII.B) Gravitational Field.

Let us see the Twin Paradox when the gravitational field of the **planet** (or **star**) $\varphi(r) = - GM/r$ cannot be taken as weak. So, instead of **Eq.(III.5)** we have

$$d\tau = [1 - (2GM/rc^2)]^{1/2}\, dt = [1 - (R_S{}^*/r)]^{1/2}\, dt \qquad \textbf{(VII.13),}$$

where $R_S{}^* = 2GM/c^2$. In these conditions the time $\boldsymbol{\tau_2}^*$ expended by the twin in the round trip would be given by

$$(1/c) \oint [-g_{oo}(x_o)]^{1/2} dx_o = \oint \{1 - R_S{}^*/2r(t)\}^{1/2}\, dt \qquad \textbf{(VII.14),}$$

where the condition $r(t) \geq R_S{}^*/2 = GM/c^2 \equiv r_{min}$ must be obeyed.

The twin age $\boldsymbol{\tau_1}^*$ at rest at the surface of the planet is given by [4]

$$\boldsymbol{\tau_1}^* = 2T\ [1 - (R_S{}^*/R)]^{1/2} \qquad \textbf{(VII.15),}$$

where R is the planet (or *aster*) radius.

As before, V = dr/dt is taken as the twin average velocity during the first step of the round trip. The time $\boldsymbol{\tau_{21}}^*$ expended in the **first step** of the trip, that is, when r goes from r_{min} = R up to R + h would be (**Appendix B):**

$$\boldsymbol{\tau_{21}*} \approx T\ \{1- [R_s*/2(R + h)]\}^{1/2}\} \qquad \textbf{(VII.16),}$$

where h = R +VT >> R >> R_s.

When the twin returns to the planet, that is**,** coming back from R + h to R, with velocity dr/dt = - V, the time expended is $\boldsymbol{\tau}_{22}^* = \boldsymbol{\tau}_{21}^*$. So, the time $\boldsymbol{\tau}_2^*$ expended at the round trip R → R + h → R will be,

$$\boldsymbol{\tau}_2^* = 2\boldsymbol{\tau}_{21}^* \approx (2T)\{1 - R_s^*/2(R+h)\}^{1/2}\} \approx (2T) - h^{1/2}(R_s^*)^{1/2}/V \quad \textbf{(VII.16)},$$

assuming that h >> R + VT >> R >> R_s*.

So, using **Eqs.(VII.14)** and **(VII.16),** the age difference $\Delta\boldsymbol{\tau}^*$ between twins would be, putting h = VT,

$$\Delta\boldsymbol{\tau}^* = \boldsymbol{\tau}_1^* - \boldsymbol{\tau}_2^* \approx \{TR_s^*/V\}^{1/2} \quad \textbf{(VII.17).}$$

(a)Earth.

With same values of **Section (VII.A),** for the weak field, that is, T = 100 years ~10^5 hours, R_s*~10^{-6} Km and V = 100 Km/h we get, using Eq.(**VII.17),**

$$\Delta\boldsymbol{\tau}^* = \boldsymbol{\tau}_1^* - \boldsymbol{\tau}_2^* \approx \mathbf{10^{-3}\ hours} \quad \textbf{(VII.18)},$$

instead of **10^{-7} hours** predicted for a weak gravitational field..

(b)Aster.

Considering, again T =100 years ~ 10^5 hours and V = 100 Km/h, using **Eq.(VII.17)** we get,

$$\Delta\boldsymbol{\tau}^* = \boldsymbol{\tau}_1^* - \boldsymbol{\tau}_2^* \approx (R_s^*)^{1/2}\ \text{years.} \quad \textbf{(VII.19)}$$

Eq.(VII.19) shows that the traveling twin would be ~10 years younger only when $(R_s^*)_{aster}$ ~ 100 Km. Since $(R_s^*)_{Earth} \approx 10^{-6}$ Km, the aster Schwarzschild radius ought to be $(R_s^*)_{aster} \sim 10^8 (R_s^*)_{Earth}$ **!!!!**

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APPENDIX A. Schwarzschild Metric and Metric with Rotation.

The **Schwarzschild metric** that describes the spacetime around a spherically symmetric body with mass M and radius R is given by[4]

$$-ds^2 = c^2 d\tau^2 = c^2(1 - R_s/r)\, dt^2 - [1 - R_s/r]^{-1}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 \quad \textbf{(A.1)},$$

where $R_s = 2GM/c^2$. For $R_s/r << 1$, for a rotating body, with $dr = d\theta = 0$:

$(d\tau/dt)^2 = (1- R_s/r)^2 - \{(r/c)(d\varphi/dt)\}^2 \sin^2\theta = (1- R_s/r)^2 -(r^2\Omega^2/c^2) \sin^2\theta$ **(A.2),**

where **$\Omega = d\varphi/dt$** is the angular velocity of the sphere along the z-axis.

When only rotational effects are preponderant we have,[4]

$$d\tau^2 = -(c^2 - \Omega^2r^2)dt^2 + 2\Omega r^2 d\varphi dt + dr^2 + r^2 d\varphi^2 + dz^2 \qquad \textbf{(A.3),}$$

from which we see that $g_{oo} = -(c^2 - \Omega^2r^2)$ and, using **Eq.(V.1)** we obtain,

$$d\tau = [1 - \Omega^2r^2/c^2]^{1/2} dt \qquad \textbf{(A.4).}$$

APPENDIX B. Calculation of the time interval τ^*. .

If $v = dr/dt$ is the twin average velocity during the trip, the time expended τ^* in the first step of the round trip, with t going from 0 up to T, that is, with r going from r up to R + h , is given by

$$\tau^* = \int_R^{R+h} \{1 - R_S^*/2r\}^{1/2} dt = (1/V) \int_R^{R+h} \{(r + b)/(r + a)\}^{1/2} dr \,,$$

where $b = - R_s^*/2$, $a = 0$ and $r^* = r_{min} = R_s^*/2$. In this way,

$$\int_R^{R+h} \{(r + b)/(r + a)\}^{1/2} dr =$$

$$\{(x+a)(x+b)\}^{1/2} + \{(a-b)/2\}\ln\left| \{(x+a)^{1/2} - (x+b)^{1/2}\}/\{(x+a)^{1/2} + (x+b)^{1/2}\} \right|$$

$$= [(r + 0)(r - R_S^*/2)]^{1/2} + (R_s^*/4)\ln\left| [x^{1/2}- (r-R_S^*/2)^{1/2}]/ \right| [x^{1/2} + (r-R_S^*/2)^{1/2}]$$

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