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On Self-Consistent Computation of Spontaneously Generated Mass

by

H. Fleming
Instituto de Física, Universidade de São Paulo, SP
Brasil

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H.Fleming

Instituto de Física - Universidade de São Paulo

A B S T R A C T

Recent results on mass generation in the Gross-Neveu model obtained by the use of self-consistent methods are analysed through renormalization-group methods. The invariance of the mass under a class of renormalizations introduced by T.Muta is proved.

I. INTRODUCTION

The study of mass generation through self-consistency requirements, advocated years ago by Nambu and Jona-Lasinio,[1] has found recently a very elegant application in the study of the Gross-Neveu model[2] made by Abarbanel[3] and Muta[4]. The latter work is particularly interesting as, by introducing an unconventional renormalization procedure, Muta is able to derive a gap equation which gives the fermion mass as a function of the coupling constant λ in terms of finite quantities, containing both Gross-Neveu's and Abarbanel's results.

In this paper we analyse Muta's conclusions by the use of renormalization-group equations adapted to his procedure, proving the renormalization invariance of the self-consistent mass. This is done in Muta's paper only in the particular case in which Abarbanel's results obtain.

In section II we briefly review Muta's calculation. Section III discusses the renormalization group equations for the fermion mass, which are solved in Section IV. A discussion of some consequences of the invariance follows.

II. THE SELF-CONSISTENT METHOD

The Gross-Neveu lagrangean

$$\mathcal{L} = i\bar{\psi}\gamma\cdot\partial\psi + \frac{1}{2}g_0^2(\bar{\psi}\psi)^2 \tag{II.1}$$

is treated in 2 space-time dimensions and large number N of fermionic field components. As it is well-known[2], one can trade it for the lagrangean

$$\mathcal{L} = i\bar{\psi}\gamma\cdot\partial\psi - \frac{1}{2}\sigma^2 + g_0\bar{\psi}\psi\sigma \tag{II.2}$$

In the leading order of the 1/N expansion one has the self-consistent equation for the fermion self-energy part:

$$\Sigma(p) = \frac{Ng_0^2}{(2\pi)^D}\, i\int d^Dq\, \mathrm{Tr}\left[\frac{1}{\gamma\cdot q - \Sigma(q)}\right] \tag{II.3}$$

D being the dimension of space-time. Putting $\Sigma(p) = m$, the fermion mass, one has

$$1 = \frac{Ng_0^2}{(2\pi)^D}\, 2^{\frac{D}{2}}\, i\int \frac{d^Dq}{q^2 - m^2} \tag{II.4}$$

This must be renormalized, so we introduce, following Muta[4], the renormalized dimensionless coupling constant λ through

$$\lambda^2 = Z_\sigma N g_0^2 \mu^{D-2} \tag{II.5}$$

μ being the "renormalization spot", and Z_σ the renormalization constant of the σ field. From (II.4) and (II.5) it follows that

$$Z_\sigma = \frac{\lambda^2}{(2\pi)^{D/2}}\left(\frac{m}{\mu}\right)^{D-2}\Gamma\left(\frac{2-D}{2}\right) \tag{II.6}$$

If Z_σ is known, (II.6) gives a relation between λ and μ, needed to compute the Callan-Symanzik function $\beta(\lambda)$[6]. To evaluate Z_σ we must examine the renormalization of the σ propagator $\Delta(k^2)$. We have, for the σ self-energy part $\Pi(k^2)$,

$$\Pi(k^2) = \frac{Ng_0^2}{(2\pi)^D}(-i)\int d^Dp\, \mathrm{Tr}\left[\frac{1}{\gamma\cdot p - M}\,\frac{1}{\gamma\cdot(p-k) - M}\right] \tag{II.7}$$

where we introduced a new dimensional parameter, M, instead of the mass, in the fermion propagator. This is the new feature introduced by Muta. It is indeed a renormalization, as our analysis in terms of the renormalization group will show. Equation (II.7) can be integrated and gives

$$\Pi(k^2) = \frac{N g_0^2}{(2\pi)^{D/2}} (1-D)\Gamma\left(\frac{2-D}{2}\right)\int_0^1 dx \left[M^2 - x(1-x)k^2\right]^{\frac{D-2}{2}} \tag{II.8}$$

The renormalization prescription which determines Z_σ is

$$\Delta(-\mu^2) = -i \tag{II.9}$$

or, equivalently,

$$Z_\sigma^{-1} = 1 + \Pi(-\mu^2) \quad . \tag{II.10}$$

Using equation (II.10), (II.8) and (II.6), and taking the limit $D \to 2$, one has

$$1 = \frac{\lambda^2}{2\pi}\int_0^1 dx \left\{ \ln\left[\frac{M^2}{m^2} + \frac{\mu^2}{m^2}x(1-x)\right] + 2 \right\} \tag{II.11}$$

which, upon integration, gives

$$-\frac{2\pi}{\lambda^2} = \sqrt{1+4M^2/\mu^2}\, \ln\frac{\sqrt{1+4M^2/\mu^2}-1}{\sqrt{1+4M^2/\mu^2}+1} + \ln\frac{m^2}{M^2} \tag{II.12}$$

This is Muta's gap equation, and this derivation is exactly his own. It contains Abarbanel's results[(3)] as a particular case and has very interesting consequences, like the discovery of a kind of dimensional transmutation at the infrared-stable zero of the β function. We refer the reader to ref.(4).

III. THE RENORMALIZATION-GROUP EQUATIONS

The renormalization-group equations for the mass follow, in a massless theory with two redundant mass-dimensional parameters, from the equations $dm/d\mu = 0$ and $\frac{dm}{dM} = 0$, whose content is the renormalization invariance of the fermionic mass m. Explicitly, they read

$$\left(\mu \frac{\partial}{\partial\mu} + \beta(\lambda) \frac{\partial}{\partial\lambda} + \mu \frac{\partial M}{\partial\mu} \frac{\partial}{\partial M}\right) m = 0 \qquad \text{(III.1)}$$

$$\left(M \frac{\partial}{\partial M} + \bar\beta(\lambda) \frac{\partial}{\partial\lambda} + M \frac{\partial\mu}{\partial M} \frac{\partial}{\partial\mu}\right) m = 0 \qquad \text{(III.2)}$$

where

$$\beta(\lambda) = \mu \frac{\partial\lambda}{\partial\mu} \qquad \text{(III.3)}$$

and

$$\bar\beta(\lambda) = M \frac{\partial\lambda}{\partial M} \; . \qquad \text{(III.4)}$$

We leave open the existence of a functional relation between M and μ, manifest through the presence of terms like $\frac{\partial M}{\partial\mu}$. Muta takes M and μ to be independent parameters and to get his results we will eventually ignore $\frac{\partial M}{\partial\mu}$ at the end. It is, however, instructive to proceed the way we do. We look for solutions of the type

$$m(\lambda, M, \mu) = M\, G(\lambda, \frac{M}{\mu}) \; , \qquad \text{(III.5)}$$

where G is an arbitrary function.

On dimensional grounds, the most general expression is

$$m = M F_1(\lambda, \frac{M}{\mu}) + \mu F_2(\lambda, \frac{M}{\mu})$$

which is contained in (III.5), by taking

$$G(\lambda, \frac{M}{\mu}) = F_1(\lambda, \frac{M}{\mu}) + \frac{\mu}{M} F_2(\lambda, \frac{M}{\mu}) \text{ .}$$

With (III.5), equations (III.1) and (III.2) read

$$\left[\frac{\partial M}{\partial \mu} - \frac{M}{\mu} \right] \frac{\partial G}{\partial (\frac{M}{\mu})} + \beta(\lambda) \frac{\partial G}{\partial \lambda} + \frac{\mu}{M} \frac{\partial M}{\partial \mu} G = 0 \tag{III.6}$$

and

$$\left[\frac{M}{\mu} - \frac{M^2}{\mu^2} \frac{\partial \mu}{\partial M} \right] \frac{\partial G}{\partial (\frac{M}{\mu})} + \bar{\beta}(\lambda) \frac{\partial G}{\partial \lambda} + G = 0 \text{ .} \tag{III.7}$$

If M and μ are related, these equations are not independent. In fact, multiplying (III.6) by $\frac{\mu}{M} \frac{\partial M}{\partial \mu}$ we have the consistency condition between the equations

$$\beta(\lambda) = \frac{\mu}{M} \frac{\partial M}{\partial \mu} \bar{\beta}(\lambda) \tag{III.8}$$

which can be easily verified.

The β and $\bar{\beta}$ functions can be calculated from equation (II.12) reading

$$\beta(\lambda) = - \frac{\lambda^3}{2\pi} \left(2 \frac{M^2}{\mu^2 \sqrt{1 + 4 \frac{M^2}{\mu^2}}} \ln \frac{\sqrt{1 + 4 \frac{M^2}{\mu^2}} - 1}{\sqrt{1 + 4 \frac{M^2}{\mu^2}} + 1} + 1 \right) \tag{III.9}$$

$$\bar{\beta}(\lambda) = \frac{\lambda^3}{\pi} \frac{M^2}{\mu^2 \sqrt{1 + 4 \frac{M^2}{\mu^2}}} \ln \frac{\sqrt{1 + 4 \frac{M^2}{\mu^2}} - 1}{\sqrt{1 + 4 \frac{M^2}{\mu^2}} + 1} \tag{III.10}$$

$$\frac{\partial M}{\partial \mu} = - \frac{M}{\mu} - \frac{\mu \sqrt{1 + 4 \frac{M^2}{\mu^2}}}{2M \ln \frac{\sqrt{1 + 4 \frac{M^2}{\mu^2}} - 1}{\sqrt{1 + 4 \frac{M^2}{\mu^2}} + 1}} \tag{III.11}$$

Let us consider now the case in which M and μ are independent parameters. The equations corresponding to (III.6) and (III.7) are now independent. Introducing the notation

$$x = \frac{M}{\mu}$$

one has

$$-x \frac{\partial G}{\partial x} + \beta(\lambda) \frac{\partial G}{\partial \lambda} = 0 \tag{III.12}$$

and

$$x \frac{\partial G}{\partial x} + \bar{\beta}(\lambda) \frac{\partial G}{\partial \lambda} + G = 0 \tag{III.13}$$

In order that the mass be renormalization-invariant, G must be a solution of both equations.

IV. THE SOLUTIONS

These equations are conveniently solved by the method of the characteristics(5). The general solution of (III.12) is

$$G(\lambda, x) = G_0\left(\frac{2\pi}{\lambda^2} + \sqrt{1+4x^2}\, \ln \frac{\sqrt{1+4x^2}-1}{\sqrt{1+4x^2}+1}\right) . \tag{IV.1}$$

That is, an arbitrary function of the indicated argument. To solve eq. (III.13), we first write the equation of the characteristics,

$$\frac{d\lambda}{dx} = \frac{\lambda^3}{\pi} \frac{x}{\sqrt{1+4x^2}} \ln \frac{\sqrt{1+4x^2}-1}{\sqrt{1+4x^2}+1} \tag{IV.2}$$

which can easily be integrated to

$$C = \ln x^2 - \frac{2\pi}{\lambda^2} - \sqrt{1+4x^2}\,\ln \frac{\sqrt{1+4x^2}-1}{\sqrt{1+4x^2}+1} \tag{IV.3}$$

where C is the integration constant. This slightly unconventional way of writing the solution is convenient to what follows. The general solution of (III.13) can be now found by integrating the equation

$$\frac{dG}{dx} = -\frac{G}{x} \tag{IV.4}$$

and writing the new integration constant as an arbitrary function of the constant combination of λ and x exhibited in (IV.3). One has

$$G(\lambda,x) = \frac{1}{x}\,G_1\left(\ln x^2 - \frac{2\pi}{\lambda^2} - \sqrt{1+4x^2}\,\ln \frac{\sqrt{1+4x^2}-1}{\sqrt{1+4x^2}+1}\right) . \tag{IV.5}$$

To determine the invariant masses one must then look for all functions that satisfy both eqs.(IV.1) and (IV.5). Apart from the trivial vanishing solution, the only way of satisfying both constraints is having

$$G(\lambda,x) = \exp\left(-\frac{\pi}{\lambda^2} - \frac{\sqrt{1+4x^2}}{2}\,\ln \frac{\sqrt{1+4x^2}-1}{\sqrt{1+4x^2}+1}\right) \tag{IV.6}$$

except for a multiplicative constant. Therefore, the invariant mass has the form

$$m = M\exp\left(-\frac{\pi}{\lambda^2} - \frac{\sqrt{1+4x^2}}{2}\,\ln \frac{\sqrt{1+4x^2}-1}{\sqrt{1+4x^2}+1}\right) \tag{IV.7}$$

that is, it coincides with Muta's gap equation, eq. (II.12). The mass computed by Muta is, hence, renormalization invariant. This means the following: let us fix our renormalization, giving values for μ, M and λ. This determines the value of C in equation (IV.3), that is, determines λ as a function of M/μ. Now, change the value of M/μ (i.e., change the renormalization). The value of λ will also change, so that the value of $\underline{m}$ will remain fixed.

A simple consequence of this is that the various masses obtained by divers particular values of M and μ, though looking rather different from one another, are in fact, one and the same.

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