Comments on Gravitational Deflection and Retardation of Light Signals.

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Abstract.

This paper was written to postgraduate students of Physics. In the General Relativity context we present brief comments on gravitational deflection and retardation of light signals. Deflection was analyzed using Lagrange Equations and the retardation effect taking into account the fundamental light interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$.

Key words: general relativity; light motion in gravitational field

Introduction.

In **Section 1** we justify the use of the **Lagrange Equations** to describe particle motion in General Relativity. In **Section 2** is analyzed particles motion around spherical massive bodies using the Schwarzschild metric. In **Sections 3 and 4**, with the Schwarzschild metric, we apply the Lagrange formalism to obtain trajectories of particles with mass and null mass around spherical massive bodies. In **Section 5** is calculated the bending of the light rays around a massive spherical body, the Sun, for instance. In **Section 6** is estimated the retardation of light signals passing close to the Sun performing the path Earth → Venus → Earth.

(1) Lagrange´s Equations for General Particle motion.

"When Lagrange´s equations is applied, the adopted generalized coordinates can be usually used with no specific rules given *a priori*. This vagueness in the procedure of assigning coordinates has been criticized by certain mathematicians, who maintain that a satisfactory theory must begin by describing the observational methods - actual or conceptual - through which the coordinates are to be set up. Such a description on the freedom of the investigator does not seem to be legitimate and an analogy drawn from Newtonian mechanical may clarify this point. Systems can often be described by means of Lagrange´s equations and great advantages result by so doing. For the setting up of generalized coordinates for which no specific a priori rules are usually given seems to be essential. We believe that the whole value of Lagrange´s method is due, precisely in this lack of rules."[1,2]

(2)Particles Moving in the Vicinity of Spherical Massive Body.

Paths of particles with mass moving in the vicinity of a *spherical massive body* are given by ***timelike geodesics*** of spacetime, while the paths of *photons* (or null mass particles) are given by ***null geodesics***.[1-5]

First, let us remember that the fundamental space time line element ds^2 in General Relativity is given by

$$ds^2 = c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \textbf{(2.1),}$$

where τ = *proper time*, $\mu,\nu = 1,2,3,4$; $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = x^o = ct$. In the vicinity of a spherical body with radius R and mass M, putting $\mathbf{m \equiv GM/c^2}$, the Schwarzschild spacetime ds^2 is given by[1-5]

$$ds^2 = c^2 d\tau^2 = (1-2m/r)c^2 dt^2 - (1-2m/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad \textbf{(2.2).}$$

If we take a slice given by t = constant **Eq.(2)** becomes, putting dt =0,

$$ds^2 = c^2 d\tau^2 = (1-2m/r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad \textbf{(2.3).}$$

(3)Massive particle (timelike geodesic).

For a *timelike geodesic* we may use its proper time τ as an affine parameter to obtain the 4 geodesic equations given by:[3]

$$d(\partial L/\partial \dot{x}^\mu)d\tau - (\partial L/\partial x^\mu) = 0 \qquad \textbf{(3.1),}$$

where $\dot{x} = dx/d\tau$ and the Lagrangian $L(\dot{x}^\sigma, x^\sigma)$ is given by[3]

$$L(\dot{x}^\sigma, x^\sigma) = (1/2)\, g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu =$$

$$= (1/2)\{ c^2(1-2m/r)t^{*2} - (1-2m/r)^{-1} r^{*2} - r^2(\theta^{*2} + \sin^2\theta\, \phi^{*2})\} \; \textbf{(3.2),}$$

where the symbol *denotes derivatives, with respect to τ, of the coordinates $x^o = t$, $x^1 = x$, $x^2 = \theta$ and $x^3 = \phi$. That is, $t^* = dt/d\tau$, $x^* = \dot{x} = dx/d\tau$, $\theta^* = d\theta/d\tau$ and $\phi^* = d\phi/d\tau$.

Because of the spherical symmetry, there is no loss of generality in confining our attention to particles moving in the equatorial plane given by $\theta = \pi/2$. With this value for θ, the third ($\mu = 2$) of **Eqs.(3.1)** is satisfied, and the second of these ($\mu = 1$) reduces to

$$(1-2m/r)^{-1}[d^2r/d\tau^2] + (mc^2/r^2)[dt/d\tau]^2 - (1-2m/r)^{-2}(m/r^2)[dr/d\tau]^2 - r[d\phi/d\tau]^2 = 0 \quad \textbf{(3.3).}$$

As t and ϕ are cyclic coordinates,[2] that is, $\partial L/\partial t^* =$ const and $\partial L/\partial t^* =$ const and taking into account that $\theta = \pi/2$ we obtain the two integration constants k and h,[3]

$$(1-2m/r)[dt/d\tau] = k \quad \textbf{(3.3a)} \quad \text{and} \quad r^2[d\phi/d\tau] = h \quad \textbf{(3.3b).}$$

Taking into account these results, remembering that $\theta = \pi/2$, putting $u = 1/r$ and $m = Gm/c^2$, **Eq.(3.3)** becomes,

$$(du/d\phi)^2 + u^2 = E + (2GM/h^2)u + (2GM/c^2)u^3 \quad \textbf{(3.4),}$$

where $E \equiv c^2(k^2-1)/h^2$.

Note that, in similar conditions, the equation of motion of the particle in the classical Newtonian limit would be given by [6]

$$(du/d\phi)^2 + u^2 = E + (2GMm/h^2)u \quad \textbf{(3.5),}$$

where $u = 1/r$, E is a constant related to the energy of the orbit and $h = r^2(d\phi/dt)$ is the angular momentum per unit of mass. The solution of **Eq.(3.5)** is well known from classical mechanic as [6]

$$u = 1/r = (GM/h^2)[1 + \varepsilon \cos(\phi - \phi_o)] \quad \textbf{(3.6),}$$

where ϕ_o is a constant of integration and $\varepsilon^2 = 1 + Rh^2/G^2M^2$ (ε is the eccentricity of a conic section).

Comparing **Eq.(3.4)** with **Eq.(3.50)** the term $(2GM/c^2)u^3$ would be, in a given sense, a relativistic correction of the classical **Eq.(3.5).** It would be responsible by the perihelion advance (precession) in planetary orbits.[4]

(4) Null rest mass particles ("null geodesics").

For photons or any other particles with null rest mass it will be assumed, according to Foster and Nightingale[3] that the proper time τ *cannot be taken* as a faithful parameter. So, let ω be any affine parameter along the geodesic, and let dots now denote derivatives with respect to ω.

For photons moving in the equatorial plane, equations **(3.3a)** and **(3.3b)** remain the same, but **Eq.(3.4)** must be replaced by

$$c^2(1 - 2m/r)\,[dt/d\omega]^2 - (1 - 2m/r)^{-1}[dr/d\omega]^2 - r^2\,[d\phi/d\omega]^2 = 0 \quad \textbf{(4.1).}$$

Consequently, **Eq.(3.4)** will be substituted by

$$(du/d\phi)^2 + u^2 = F + (2GM/c^2)u^3 \qquad \textbf{(4.2),}$$

where $F \equiv c^2k^2/h^2$. This equation will be used in **Section 5** when analyzing the bending of light by the Sun. To complete this section let us discuss an important consequence of the null geodesic equations**.**

We verify that **Eq.(4.1)** shows that is possible of having photons in a **circular orbit.** When $dr/d\omega = d^2r/d\omega^2 = 0$ we see from Eq.(4.1) that

$$[d\phi/d\omega]^2/[dt/d\omega]^2 = [d\phi/dt]^2 = c^2(1 - 2m/r)/r^2 \qquad \textbf{(4.3).}$$

which shows that only when $\mathbf{r = 3m}$ the photon can go into an orbit. This shows that a massive objects (**Black Holes,** for instance) can have considerable effect on light.[4]

(5)Bending of Light.

In **Section 4** we have already noted that a massive object can be able to modify *light trajectories*. In this **Section** we intend to estimate modest deflections of light passing close to massive objects.[3]

In a first approach let us consider the case of a photon travelling very far from a star, in its equatorial plane. Thus, putting M = 0 at the right side of **Eq.(4.2)** we obtain[3]

$$(du/d\phi)^2 + u^2 = F \qquad \textbf{(5.1),}$$

which has as a particular solution,

$$u(r) = 1/r = (1/r_o)\sin(\phi) \quad \text{or} \quad r_o = r\sin(\phi) \qquad \textbf{(5.2),}$$

where $u_o^2 = 1/r_o^2 = F$. This solution represents the straight-line path of the photon from the infinity in the direction $\phi = 0$ and going to infinity in the direction $\phi = \pi$ as shown in **Figure (5.1).** The straight-line path passes by the nearest point r_o of the star.

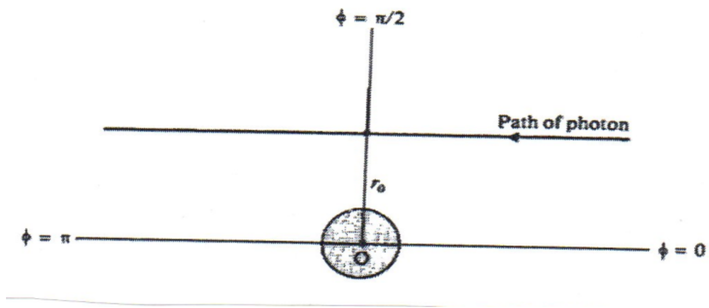

Figure (5.1). Photon path in the equatorial plane of the star with M = O.[3]

Taking into account **Eq.(4.2)** putting $M = 2GM/c^2 = \varepsilon$ we have

$$(du/d\phi)^2 + u^2 = F + \varepsilon u^3 \qquad \textbf{(5.3).}$$

Solving this equation they obtained,[3] for values $r > \varepsilon$ the contribution of term εu^3 as a relativistic correction to the flat spacetime, ignoring squares and higher terms of ε. They have shown that the photon does not go off to infinity in direction π, but in a direction π + α, (see **Figure (5.2)**) where

$$\alpha = 4MG/r_o c^2 \qquad \textbf{(5.4).}$$

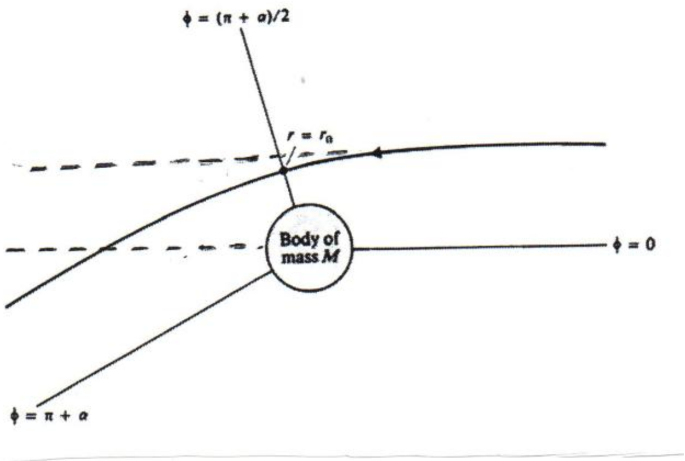

Figure (5.2). Photon path in the equatorial plane of the star with M > 0.[3]

Taking for M_Θ and r_o the accepted values for the **Sun** mass and radius[4] they found that $\alpha = 4M_\Theta G/r_o c^2 = 1.75''$ is the total deflection of light originating and terminating at infinity. That is in good agreement with experimental results of deflections of light and radio waves.[4]

(6)Retardation of the Light Signals.

Another observable effect is the *time delay* suffered by radar signals sent from Earth to a planet and reflected back to Earth passing close to the Sun.[4] In **Figure (6.1)** is seen Sun, Earth and Venus.

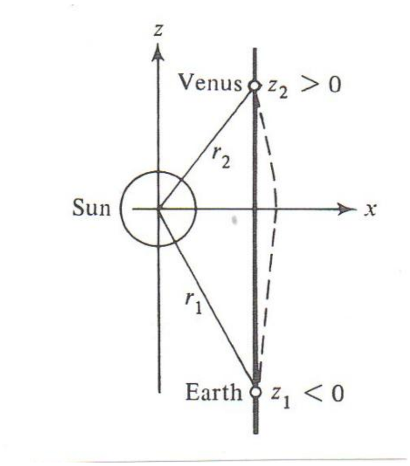

Figure (6.1). Path of light signal (**dotted line**) between Earth(z_1) and planet(z_2).

It is observed[4] that the time T expended by radar signals, going from Earth to a planet and reflected back to Earth is bigger than the time T* = 2Dc, where D is the distance between Earth and planet and c the light velocity. That is, there is a *time delay*, T > T*.

Of course, due to light deflection the true path is longer than exactly straight. However, this difference is very small and cannot explain the observed time delay.[4]

The *radial velocity* v_r of the light pulse can be estimated using the Schwarzschild line element $ds^2 = 0$, putting $d\theta = d\phi = 0$, obtaining,

$$(c^2-2m/r)dt^2 = dr^2/(1-2m/rc^2),$$

that is,
$$(dr/dt) = v_r(r) \approx c/(1 + 2GM_\odot/r) \qquad (6.1),$$

showing that the light *radial velocity* $v_r(r)$ of is *smaller* than the velocity c. Measurements[4] of the light velocity close to the Sun surface (using *ideal clocks* and *good meter sticks*) must give $c = 3\ 10^{10}$cm/s.

So, using **Eq.(6.1)** the travel time Δt for light or radio signal between z_1 and z_2 (see **Figure (6.1)**), where $r = (z^2 + b^2)^{1/2}$ and $b \approx R_\odot$ = Sun radius, would be,

$$\Delta t = \int dz/v(z) = \int dz\{1 + 2GM_\odot/(z^2 + b^2)^{1/2}\}/c \qquad (6.2),$$

that is,

$$\Delta t = (z_2 - z_1)/c + (2GM_\odot/c)\ \ln\{[z_2 + (z_2^2 + b^2)^{1/2}]/\ [z_1 + (z_1^2 + b^2)^{1/2}]\}, \qquad \textbf{(6.3),}$$

showing that the *delay time* τ would be obtained multiplying by 2 the *second term* of **Eq.(6.3)** that can be written as[4]

$$\tau = 2(GM_\odot/c)\ln(4|z_1||z_2|/b^2) \qquad \textbf{(6.4).}$$

Putting $b \approx R_\odot = 6.96\ 10^{10}$ cm, z_1 =(radius of Earth orbit) = $14.9\ 10^{12}$ cm and z_2 = (radius of Mercury orbit) = $5.8\ 10^{12}$ cm we obtain the "delay time" $\tau \approx 200$ μs, observed for the light path Earth-Sun-Venus-Sun-Earth, in good agreement with experimental results.[4]

Acknowledgements. *The author thanks the librarian Maria de Fatima A. Souza and the administrative technician Tiago B. Alonso for their invaluable assistances in the publication of this paper.*

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