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by

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ABSTRACT

We present here a general method to deduce topological macroscopic quantum waves (MQWs) solutions in non relativistic local field theories. It is shown that every theory $\lambda|\phi|^n$ (with $\lambda>0$ and $n>2$) exhibits subsonic MQWs. Explicit solutions are given for the $|\phi|^4$ and $|\phi|^6$ models. The fact that these topological waves are a common feature of a wide class of bosonic theories is an important support for our conjecture[1,2] that they exist in liquid ^{4}He.

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Recently[1,2] we found that Bogoliubov's theory of superfluidity[3] displays some topological macroscopic quantum waves (MQWs). This achievement led us to propose a new microscopic description of superfluidity, since the MQWs partition the condensate into domains of phase and provide a beaultiful explanation of what could be the λ-transition. Besides that, bound in the topological waves, there exist a new type of quasi-particles, which differ from Bogoliubov's phonons, and have a spectrum similar to liquid ^{4}He spectrum[1,2].

In references (1) and (2) our argumentation is entirely developed in the realm of the $|\phi|^4$ theory. The purposes of this paper are: (a) to deduce the MQWs solutions from the equations of motion; and (b) to show that these waves also exist in any one of the non relativistic $|\phi|^n$ models with $n > 2$. The fact that the MQWs are a common feature of a large class of bosonic theories comes to favor our proposal that they should exist in liquid Helium.

The Hamiltonian and the equation of motion of the $|\phi|^n$ theory are respectively (take $n>2$)

$$H = -\frac{1}{2m} \int d\vec{x}\ \phi^* \nabla^2 \phi + \frac{2\lambda}{n} \int d\vec{x} |\phi|^n, \tag{1}$$

and

$$-\frac{1}{2m} \nabla^2 \phi + \lambda |\phi|^{n-2} \phi = i\partial_t \phi \tag{2}$$

ϕ is a bosonic non relativistic field - the ^{4}He atom field, for example.

In the classical version of the $|\phi|^n$ theory, the ground states of systems of density ρ are solutions of (2), which do not depend on $\vec{x}$,

$$\Omega_{\theta_0} = \sqrt{\rho}\ \exp i\{\theta_0 - (\sqrt{\rho})^{n-2} \lambda t\} \tag{3}$$

They have a degeneracy of infinite degree because θ_0 can be any real constant.

Consider a particular coordinate x. The macroscopic quantum waves are solutions W of Eq.(2), such that[1,2]

$$\lim_{x\to-\infty} W = \Omega_{\theta_1} \qquad \text{and} \qquad \lim_{x\to+\infty} W = \Omega_{\theta_2} \tag{4}$$

These new solutions have a topological charge that is just the phase difference $\theta_2-\theta_1$. Therefore a MQW must be stable, since its topology is inequivalent to that of the ground state.

Phonons are small excitations of the ground state and in order to study their motion we define the field

$$\phi = (\sqrt{\rho} + \eta)\ \exp\{-i(\sqrt{\rho})^{n-2}\ \lambda t\} \tag{5}$$

where η represents a small fluctuation. After plugging this representation of ϕ in Eq.(2), and retaining only terms which are linear in η , we get the phonons equation of motion:

$$i\partial_t\eta + \frac{1}{2m}\nabla^2\eta = mc^2\ (\eta^*+\eta) \tag{6}$$

In the last equation we have

$$c = \{(n-2)(\sqrt{\rho})^{n-2}\ \lambda/2m\}^{1/2}\ . \tag{7}$$

c is the velocity of large wave length phonons (see Eq.(9)).

The eigensolutions of (6) are

$$\eta_{\vec{k}} = \frac{A}{\sqrt{V}}\{(\frac{k^2}{2m} + mc^2 + \omega)^{1/2} \exp i(\vec{k}\vec{x}-\omega t) - (\frac{k^2}{2m} + mc^2 - \omega)^{1/2} \exp - i(\vec{k}\vec{x}-\omega t)\} \tag{8}$$

Here A is a real constant, V is the volume where we define the model, and ω is the frequence which depends on the wave number $\vec{k}$:

$$\omega(\vec{k}) = ck\ \{1 + (k/2mc)^2\}^{1/2} \tag{9}$$

To quantize the phonons one can employ, for instance, the field theoretic version of the Bohr-Sommerfeld quantization rule, as is done in ref.(2).

Now, let V be a real number such that $|V| \leq 1$. To look for MQWs we define the auxiliary variable

$$\xi = mc(x - cVt) \tag{10}$$

(cV is the MQW's velocity and x is a particular coordinate, along which the MQW will move), and seek solutions of Eq.(2) that can be written in the form

$$W_V = \sqrt{\rho\ s(\xi)}\ \exp i\theta(\xi)\ \exp\{-i(\sqrt{\rho})^{n-2}\lambda t\} \tag{11}$$

where $s(\xi)$ and $\theta(\xi)$ are real functions, $s(\xi)$ being also non negative. Plugging W_V in Eq.(2) we conclude that $s(\xi)$ and $\theta(\xi)$ obey the following system of coupled equations:

$$\frac{1}{2}\partial_\xi^2\sqrt{s} + \frac{2}{n-2}\ \{\sqrt{s} - (\sqrt{s})^{n-1}\} + V\sqrt{s}\ \partial_\xi\theta - \frac{1}{2}\sqrt{s}(\partial_\xi\theta)^2 = 0 \tag{12a}$$

$$\frac{1}{2}\sqrt{s}\ \partial_\xi^2\theta + \partial_\xi\sqrt{s}\ \partial_\xi\theta = V\partial_\xi\sqrt{s} \tag{12b}$$

Note that conditions (4) implies that

$$\lim_{|\xi|\to\infty} s(\xi) = 1 \tag{13a}$$

$$\lim_{|\xi|\to\infty} \frac{\partial\theta}{\partial\xi} = 0 \tag{13b}$$

Eq.(12b) can be integrated to give (after using (13))

$$\frac{d\theta}{d\xi} = V(1-\frac{1}{s}) \tag{14}$$

Once $s(\xi)$ is known, $\theta(\xi)$ can be obtained by integrating the above equation.

Substituting (14) in (12a), it follows that

$$\frac{1}{2}\partial_\xi^2\sqrt{s} + \frac{V^2}{2}(1-\frac{1}{s^2})\sqrt{s} + \frac{2}{n-2}\{\sqrt{s}-(\sqrt{s})^{n-1}\} = 0 \tag{15}$$

If we multiply Eq.(15) by $(\partial_\xi s)/\sqrt{s}$ and integrate the resulting expression (imposing (13a) we obtain

$$\frac{1}{2}(\partial_\xi\sqrt{s})^2 + \frac{V^2}{2}(s + \frac{1}{s}) + \frac{2s}{n-2} - \frac{4(\sqrt{s})^n}{n(n-2)} = V^2 + \frac{2}{n-2} - \frac{4}{n(n-2)} \tag{16}$$

Finally, multiplying this last equation by $8s$, we get

$$(\partial_\xi s)^2 + {}^nU_V(s) = 0 \tag{17}$$

where

$${}^nU_V(s) = -\frac{32}{n(n-2)}(\sqrt{s})^{n+2} + \frac{16}{(n-2)}s^2 + 4V^2(s^2+1) - 8(V^2+\frac{2}{n-2} - \frac{4}{n(n-2)})s \tag{18}$$

Therefore, in order to get MQWs solutions in the $|\phi|^n$ theory, it is necessary to: (a) solve Eq.(17), imposing the boundary conditions (13a); (b) integrate Eq.(14); and (c) plug $s(\xi)$ and $\theta(\xi)$ so obtained in Eq.(11).

To implement the first step is equivalent to solve an ordinary problem of classical mechanics: the frictionless motion of a particle, whose mass is 1/2 and whose energy is zero, under the action of the potential ${}^{n}U_V(s)$.

Let us list some properties of ${}^{n}U_V(s)$. First of all we observe that ${}^{n}U_V(1)=0={}^{n}U'_V(1)$, and ${}^{n}U_V(0)=4V^2>0$.

A simple analysis shows that (recall that n>2) if $0<|V|<1$ [4], ${}^{n}U_V$ has the following properties:

(a.i) - The point $s=1$ is a point of <u>maximum</u>.

(a.ii) - In the interval $0\leq s\leq 1$, ${}^{n}U_V$ has one and only point of minimum.

Then, when $0<|V|<1$, the shape of ${}^{n}U_V(s)$ is like that of fig. 1; and, for any real number ξ_o, Eq.(17) shall have a solution $s(\xi)$, which obeys the boundary condition (13a) and the inequality

$$1 \geq s(\xi) \geq s(\xi_o) = a \tag{19}$$

where $a \neq 1$ is given by ${}^{n}U_V(a)=0$ (see fig.1).

By using Eq.(14) we get

$$\theta(\xi) = V \int_b^{\xi} \{1 - \frac{1}{s(\xi')}\} \, d\xi' \tag{20}$$

Inequality (19) implies that the integrand of (20) is regular. Note that b is an arbitrary real constant. Such an arbitrariness comes from the gauge invariance of the model, since, if W is a solution of Eq.(2), W exp $i\theta_o$ (where θ_o is any real constant) shall be another one.

Plugging $s(\xi)$ and $\theta(\xi)$ in Eq.(11), we obtain a macroscopic quantum wave, moving with velocity cV. We conclude, therefore, that every theory of the type $|\phi|^n$ (with n>2) describes subsonic MQWs.

It is very easy to see that, if $|V|>1$, the properties of

${}^{n}U_V(s)$ are:

(b.i) - The point $s=1$ is a point of minimum.

(b.ii) - In the interval $0 \leq s \leq 1$, ${}^{n}U_V(s)$ is positive and monotonic.

Fig.2 shows the shape of ${}^{n}U_V(s)$ when $|V|>1$. In this case the only solution compatible with the boundary conditions (13a) shall be the constant $s(\xi) = 1$, and there are no supersonic MQWs.

Nevertheless, in the supersonic case, Eq.(17) describes excitations bouncing around the minimum at $s=1$ (see fig.2). To treat the small excitations we define $r=1-s$, and take into account, in Eq.(17), only terms of second order in r, so obtaining

$$(\partial_\xi r)^2 + 4(V^2-1)r^2 = 0 \tag{21}$$

This last equation describes phonons which move in Bogoliubov's condensate with velocity cV. By solving (21) and using (14), we are lead to solutions of type (8).

Now we will present explicit examples of MQWs.

In the $|\phi|^4$ theory, ${}^{4}U_V$ is

$${}^{4}U_V = 4\,\{-s^3 + (2+V^2)s^2 - (2V^2+1)s + V^2\} \tag{22}$$

So that Eq.(17) can be integrated to give (observe that conditions (13a) are satisfied)

$$s^4(\xi) = 1 - \gamma^2 \operatorname{sech}^2\gamma\xi \tag{23}$$

where

$$\gamma = \sqrt{1 - V^2} \tag{24}$$

Taking the arbitrary constant b of Eq.(20) to be zero, we get

$$\theta^4(\xi) = - \text{arc tg } (\frac{\gamma}{V} \text{ tgh } \gamma\xi) \tag{25}$$

If we plug (23) and (25) in (11), we find the MQWs of the $|\phi|^4$ theory:

$$W_V^4 = (V - i\gamma \text{ tgh } \gamma\xi) \sqrt{\rho} \exp(-imc^2t) \tag{26}$$

The momentum per unit of area carried by W_V^4 is[2]

$$P^4(V) = - 2 \rho V \sqrt{1-V^2} \tag{27}$$

To deduce the energy it carries per unit of area, we must use Eq.(1) and subtract the ground state energy[2]. This procedure leads to

$$\sigma^4(V) = \frac{4}{3} c \rho \gamma^3 \tag{28}$$

The topological charge associated to W_V^4 is given by

$$\theta^4(\infty) - \theta^4(-\infty) = -2 \text{ arc cos } V \tag{29}$$

On the other hand, in the $|\phi|^6$ theory, the potential ${}^6U_V$ shall be

$${}^6U_V = \frac{4}{3} \{-s^4 + 3(1+V^2)s^2 - 2(3V^2+1)s + 3V^2\} \tag{30}$$

In this case, after imposing (13a), the integration of (17) leads to[5]

$$s^6(\xi) = 1 - \frac{3\gamma^2}{2 + \sqrt{4 - 3\gamma^2} \cosh 2\gamma\xi} , \tag{31}$$

whereas integration (20) gives

$$\theta^6(\xi) = -\ \text{arctg}\ \left\{\frac{3\ \gamma V\ \text{tgh}\ \gamma\xi}{\sqrt{4-3\gamma^2} + 2 - 3\gamma^2}\right\} \tag{32}$$

Hence, the macroscopic quantum waves of the $|\phi|^6$ theory are

$$W_V^6 = \sqrt{2}\ \frac{(\cos\ \delta\ \cosh\ \gamma\xi - i\ \text{sen}\ \delta\ \text{senh}\ \gamma\xi)}{\{(2/\sqrt{4-3\ \gamma^2}) + \cosh\ 2\ \gamma\xi\}^{1/2}}\ \sqrt{\rho}\ \exp\left(\frac{-imc^2t}{2}\right) \tag{33}$$

where

$$\delta = \text{arc tg}\ \left\{\frac{\gamma}{\sqrt{4-3\gamma^2} + 2 - 3\ \gamma^2}\right\} \tag{34}$$

The topological charge of W_V^6 is -2δ, and the momentum and energy it carries per unit of area are respectively

$$P^6(V) = -\frac{\sqrt{3}}{2}\ \rho V\ \ \ln\left(\frac{2+\sqrt{3}\gamma}{2-\sqrt{3}\gamma}\right) \tag{35}$$

and

$$\sigma^6(V) = c\ \rho\ \left\{\gamma + \frac{\sqrt{3}}{4}\gamma^2\ \ln\ \left(\frac{2+\sqrt{3}\gamma}{2-\sqrt{3}\gamma}\right)\right\} \tag{36}$$

The deductive method presented here can be used to look for topological waves in any theory of the type $P(|\phi|)$, where $P(|\phi|)$ is a polinomial.

In a forthcoming paper, we shall show that a wide class of <u>non local</u> models (which includes the Yukawa potential theory) also displays MQWs solutions.

These facts are, of course, a support for our conjecture that MQWs exist in Helium II.

ACKNOWLEDGMENTS:

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FOOTNOTES AND REFERENCES:

1) I.Ventura; Theory of Superfluidity: Macroscopic Quantum Waves, to appear.

2) I.Ventura; Theory of Superfluidity, to be published in Rev.Bras. Fis.9 (1979) see also São Paulo preprint, IFUSP/P-162 (1978).

3) N.N.Bogoliubov, J.Phys. USSR 11 (1947) 23.

4) We are avoiding the value V=0 solely for technical reasons, because to perform integration (20) it is necessary that $s(\xi)>0$, i.e., ${}^{n}U_V(0)>0$. However, to obtain the MQW W_o it is enough to take the limit $\lim_{V\to 0} W_V$.

5) To integrate Eq.(17) in the cases of ${}^{4}U_V$ and ${}^{6}U_V$ one can employ respectively expressions (14.186) and (14.283) of M.R.Spiegel, Mathematical Handbook of Formulas and Tables, McGraw-Hill (1968).

FIGURE CAPTIONS

FIG. 1 - The potential ${}^{n}U_V(s)$ in the subsonic case ($|V|<1$) that displays MQWs.

FIG. 2 - The potential ${}^{n}U_V(s)$ of the supersonic case ($|V|>1$). In this situation, Eq.(17) has only phonon like solutions.

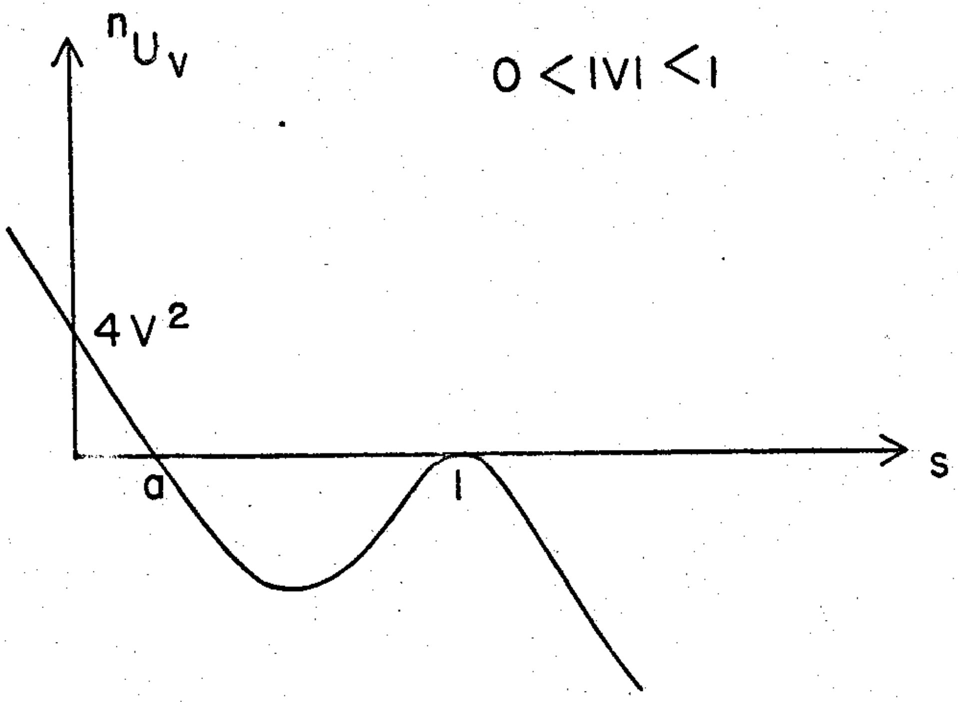

fig. 1

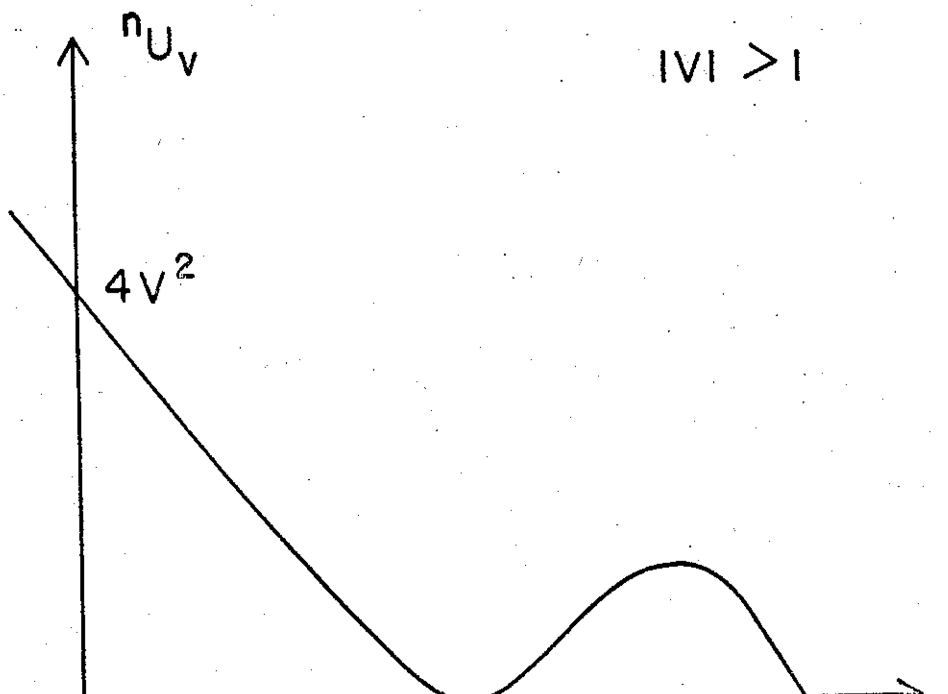

fig. 2