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Abstract : We study the relevance of the longitudinal gluons in the derivation of the Feynman rules in the temporal gauge, via the canonical approach. It is then shown that these gluons are also fundamental for the asymptotic freedom of the Yang-Mills theory.

(1979)

The temporal gauge is very appropriate for the investigation of various aspects of gauge theories, like the analysis of instantons[1] and the study of Drell-Yang processes[2]. With the help of the canonical approach, several methods have recently been developed which allow for the elimination, in this gauge, of the unphysical degrees of freedom corresponding to the gauge symmetry of the theory[3].

In this note, using the canonical quantization procedure, we will give a simple derivation of the Feynman rules, which were previously obtained by the path-integral method[4]. In this derivation, the longitudinal gluons play a relevant role. Moreover, we will show that they are also very important for the asymptotic freedom of the Yang-Mills theory.

We start with the Lagrangean density[5]

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \tag{1a}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{1b}$$

In this expression, a denotes an internal symmetry index, g is the coupling constant of the gluon fields A^a_μ and f^{abc} are the antisymmetric structure constants of the Yang-Mills theory. In the temporal gauge, characterized by the condition $A^a_0 = 0$, the canonical momenta E^a_i are simply given by $\partial_0 A^a_i$. In terms of the canonical variables A^a_i and E^a_i, the Hamiltonian density in the Heisenberg picture is given by

$$\mathcal{H} = E^a_i \partial_0 A^a_i - \mathcal{L} = \frac{1}{2} E^a_i E^a_i + \frac{1}{4} F^a_{ij} F^a_{ij} \tag{2}$$

From here, the equations of motion can be obtained as usual, with the help of the following equal-time commutation relations

$$[A^a_i(\vec{x},t), A^b_j(\vec{x}',t)] = [E^a_i(\vec{x},t), E^b_j(\vec{x}',t)] = 0 \tag{3a}$$

$$[A^a_i(\vec{x},t), E^b_j(\vec{x}',t)] = i\,\delta_{ab}\,\delta^3(\vec{x}-\vec{x}')\,\delta_{ij} \tag{3b}$$

In terms of the Hamiltonian $H = \int d^3x\, \mathcal{H}$ one finds

$$\partial_0 A_j^a(\vec{x},t) = i[H, A_j^a(\vec{x},t)] = E_j^a(\vec{x},t) \tag{4a}$$

$$\partial_0 E_j^a(\vec{x},t) = i[H, E_j^a(\vec{x},t)] = \partial_i F_{ij}^a + g f^{abc} A_i^b F_{ij}^c \tag{4b}$$

Note that Gauss law

$$(\partial_j \delta_{ab} - g f^{abc} A_j^c) E_j^b \equiv D_j^{ab} E_j^b = 0 \tag{5a}$$

which follows from the Euler-Lagrange field equations does not involve time derivatives, and therefore cannot be obtained as one of Hamilton's equations of motion. Instead, this law must be imposed as a constraint on the physical states(3)

$$D_j^{ab} E_j^b \,|\,\Psi\rangle = 0 \tag{5b}$$

Now, let us pass to the interaction picture, where, for simplicity of notation, we denote the field variables by the same symbols. In this picture, the canonical commutation relations remain unchanged and the fields satisfy free-field equations given by (see (4))

$$\partial^2 A_j^a - \partial_j \partial_i A_i^a = 0 \tag{6}$$

In order to solve this equation, we remark that, in view of the commutation relation (3b), we need $\partial_i A_i^a$ to be non-zero. Correspondingly, we decompose A_i^a into a sum of transversal and longitudinal fields as follows

$$A_i^a(\vec{x},t) \equiv A_i^{a,T}(\vec{x},t) + A_i^{a,L}(\vec{x},t) \tag{7a}$$

where the transverse field $A_i^{a,T}(\vec{x},t)$ satisfies

$$\partial_i A_i^{a,T}(\vec{x},t) = 0 \qquad , \qquad \partial^2 A_i^{a,T}(\vec{x},t) = 0 \tag{7b}$$

That is, the field A^T represents just the usual transverse free-field, which can be expanded in terms of one-particle creation and annihilation operators.

The longitudinal field $A_i^{a,L}(\vec{x},t)$ can be written as $\partial_i \phi^a(\vec{x},t)$ so that from (6) we see that ϕ^a satisfies

$$\partial_0^2 \phi^a(\vec{x},t) = 0 \tag{8a}$$

Solving this equation, we can express the field A_i^L in the following convenient form

$$A_i^{a,L}(\vec{x},t) = \partial_i [\nabla^{-2} A^{a,-}(\vec{x})t + A^{a,+}(\vec{x})] \tag{8b}$$

where $A^{a,\pm}$ denote hermitian operators which will be specified by the commutation relations (3) and the subsidiary condition. In order to satisfy (3) we need that

$$[A^{a,-}(\vec{x}), A^{b,-}(\vec{x}')] = [A^{a,+}(\vec{x}), A^{b,+}(\vec{x}')] = 0 \tag{9a}$$

$$[A^{a,-}(\vec{x}), A^{b,+}(\vec{x}')] = i\delta_{ab}\,\delta^3(\vec{x}-\vec{x}') \tag{9b}$$

Furthermore, it is necessary that the longitudinal and transversal fields commute

$$[A_i^{a,L}(\vec{x},t), A_j^{b,T}(\vec{x}',t)] = 0 \tag{10a}$$

while the transversal fields satisfy the usual commutation relation

$$[A_i^{a,T}(\vec{x},t), \partial_0 A_j^{b,T}(\vec{x}',t)] = i\delta_{ab}(\delta_{ij} - \nabla^{-2}\partial_i\partial_j)\,\delta^3(\vec{x}-\vec{x}') \tag{10b}$$

To get a physical insight into the meaning of the operators $A^{a,\pm}$ and complete their specification, we will now consider the subsidiary condition in the interaction picture. To obtain this, we perform the usual unitary transformations on the canonical variables and on the physical states which connect the Heisenberg and interaction pictures. We then obtain from (5b)

$$D_j^{ab} E_j^b |\Psi(t)\rangle = 0 \tag{11a}$$

where the physical state $|\Psi(t)\rangle$ has its time evolution given by

$$i\partial_0 |\Psi(t)\rangle = H_{int} |\Psi(t)\rangle \tag{11b}$$

In the interaction picture, $H_{int} = \int d^3x\, \mathcal{H}_{int}$ is determined by (see (2))

$$\mathcal{H}_{int} = \frac{1}{2} g f^{abc} (\partial_i A_j^a - \partial_j A_i^a) A_i^b A_j^c + \\ + \frac{1}{4} g^2 f^{abc} f^{ab'c'} A_i^b A_j^c A_i^{b'} A_j^{c'} \tag{11c}$$

In order that the constraint equation (11a) be satisfied at all times, it is necessary that its time derivative vanishes at any time. It is straightforward to check, using (6), (11b), (11c) as well as the commutation relations (9) and (10), that indeed

$$\partial_0 [D_j^{ab} E_j^b | \psi(t) >]^{\cdot} = 0 \tag{12}$$

so that, provided (11a) is satisfied initially, it will remain valid at any latter time.

For the treatment of scattering processes, we will assume that there is no interaction at $t = \pm\infty$, so that g will vanish at these times. Then, from (11a), we see that

$$\partial_j E_j^a | \psi(t = \pm\infty) > = 0 \tag{13a}$$

With the help of (7b) and (8b), we can write this as

$$A^{a,-}(\vec{x}) | \psi(t = \pm\infty) > = 0 \tag{13b}$$

Thus, since there are no longitudinal gluons in the initial and final states, it follows that A^- must represent the annihilation operator for the longitudinal gluon. Furthermore, in view of the commutation relations (9), we interpret A^+ as being the creation operator for a longitudinal gluon.

Now we are in a position to calculate the vacuum expectation value of the time ordered product of two gluon fields

$$T[A_i^a(\vec{x},t) A_j^b(\vec{x}',t')] = \theta(t-t') A_i^a(\vec{x},t) A_j^b(\vec{x}',t') + \\ + \theta(t'-t) A_j^b(\vec{x}',t') A_i^a(\vec{x},t) \tag{14a}$$

The T-product can be written as a sum of a commutator and an anti-commutator as follows

$$T[A_i^a(\vec{x},t) A_j^b(\vec{x}',t')] = \frac{1}{2} \varepsilon(t-t') [A_i^a(\vec{x},t), A_j^b(\vec{x}',t')] + \\ + \frac{1}{2} \{ A_i^a(\vec{x},t), A_j^b(\vec{x}',t') \} \tag{14b}$$

where $\varepsilon(t-t') = \theta(t-t') - \theta(t'-t)$

This way of writing the T-product is motivated by the following consideration. There seems to be an ambiguity when one considers the expectation values of relation (9b) between physical states at $t = \pm\infty$ on the one hand, and condition (13b) on the other. This situation is not peculiar to the temporal gauge, but arises, as discussed in reference (6), in all cases where we have to quantize systems subject to subsidiary conditions(*). The way out, as shown by Dirac, is that the constraints should not be used under the commutation relations. The correct way of proceeding consists in first calculating the commutators that occur in the expressions of the quantities of interest and then imposing the constraint. Proceeding in this manner and then taking the vacuum expectation value of (14b), we will obtain the propagator in a way which does not hamper the canonical fields from acting as independent variables under the comutation relations.

Since the longitudinal and transversal fields commute (see (10a)), the commutator in (14b) splits into a longitudinal and a transversal part. Using (8) and (9), we obtain for the commutator of the longitudinal fields

$$[A_i^{a,L}(\vec{x},t), A_j^{b,L}(\vec{x}',t')] = i(t-t')\delta_{ab}\nabla^{-2}\partial_i\partial_j\delta^3(\vec{x}-\vec{x}') \tag{15}$$

We can now take the vacuum expectation value of (14b) and apply condition (13b). Then, since the longitudinal and transversal fields are independent, the resulting propagator will be a sum of two parts

$$D_{ij}^{ab}(x,x') = D_{ij}^{ab,L}(x,x') + D_{ij}^{ab,T}(x,x') \tag{16a}$$

The first part contains the longitudinal fields and arises only from the commutator (15)

$$D_{ij}^{ab,L}(x,x') = \frac{i}{2}\delta_{ab}\,\varepsilon(t-t')(t-t')\,\nabla^{-2}\partial_i\partial_j\,\delta^3(\vec{x}-\vec{x}') \tag{16b}$$

The other part contains the transversal fields and arises from both the commutator and the anticommutator. It is given as usual by

$$D_{ij}^{ab,T}(x,x') = \frac{\delta_{ab}}{i(2\pi)^4}\int\frac{d^4k}{k^2}\left(\delta_{ij}-\frac{k_i k_j}{\vec{k}^2}\right)e^{ik\cdot(x-x')} \tag{17}$$

In order to write the propagator of the longitudinal fields in the momentum space, we use the following integral representation for $\varepsilon(t-t')$

(*) We encounter a similar case even in ordinary Quantum Mechanics by considering, for example, the canonical commutator $[q,p]=i$ Taking its vacuum expectation value, we again have an ambiguity since $p|0\rangle = 0$.

$$\mathcal{E}(t-t') = \frac{1}{i\pi}\int_{-\infty}^{+\infty} dk_0 \, P\left(\frac{1}{k_0}\right) e^{ik_0(t-t')} \tag{18a}$$

where P denotes the principal value prescription

$$P\left(\frac{1}{k_0}\right) = \frac{1}{2}\left(\frac{1}{k_0+i\varepsilon} + \frac{1}{k_0-i\varepsilon}\right) \tag{18b}$$

With the help of this relation, we obtain from (16b) the result

$$D_{ij}^{ab,L}(x,x') = \frac{\delta_{ab}}{i(2\pi)^4}\int d^4k \, \frac{k_i k_j}{k^2}\left(\frac{1}{\vec{k}^2} - \frac{1}{k_0^2}\right) e^{ik.(x-x')} \tag{19}$$

In this expression, $-1/k_0^2$ is to be understood as the derivative, with respect to k_0 of the principal value prescription (18b). From (16a), (17) and (19), we then obtain the propagator in the temporal gauge

$$D_{ij}^{ab}(x,x') = \frac{\delta_{ab}}{i(2\pi)^4}\int d^4k \, \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{k_0^2}\right) e^{ik.(x-x')} \tag{20}$$

which is precisely the same as the one derived by the path-integral method. Furthermore, in view of the gauge condition $A_0^a = 0$, the interaction Hamiltonian (11c) is equivalent to the corresponding one given in the path-integral approach[4].

Although there are no longitudinal gluons in the initial and final states, these are, of course, present during the interaction. In fact, as we will now show, they are fundamental for the asymptotic freedom of the Yang-Mills theory in the temporal gauge. (For a discussion of asymptotic freedom in other gauges, see reference (7)). To see this, let us recall that in this gauge the ultraviolet divergent part of the self-energy function Π_{ij}^{ab} has the following form [4]

$$\left(\Pi_{ij}^{ab}\right)^u = \delta_{ab}\,(p_i p_j - \delta_{ij}\, p^2)\,\Pi^u \tag{21a}$$

where the ultraviolet divergent part Π^u is related to the wave-function renormalization constant Z_3 by the relation

$$\Pi^u = 1 - Z_3 \tag{21b}$$

Furthermore, because of the Ward identities, which are similar to those in QED, Z_3 is also equal to the renormalization function of the coupling constant. We have

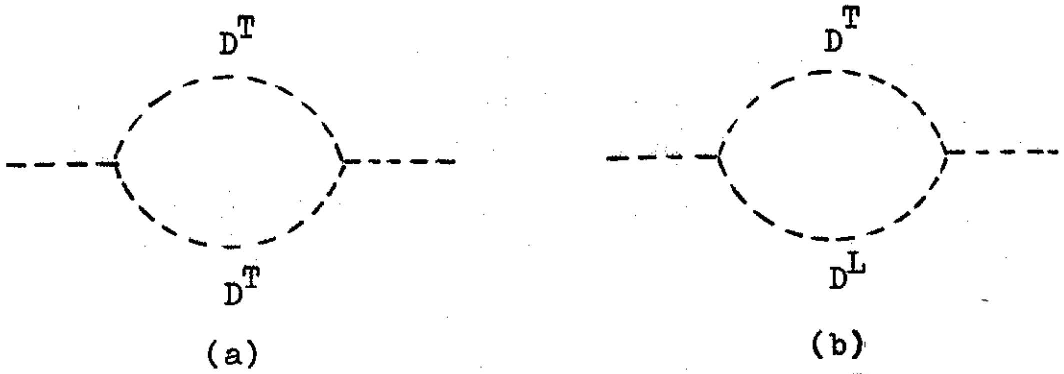
$$1 - \left(\frac{g_r}{g}\right)^2 = 1 - Z_3 \tag{21c}$$

where g and g_r denote respectively the bare and the renormalized coupling constants.

In QED, because the polarization of the vacuum shields the bare charge, we have $g^2 > g_r^2$, which implies that the theory is not asymptotically free. In order to understand the reason for the negative sign of $(1 - g_r^2/g^2)$ in the Yang-Mills theory, and hence of its asymptotic freedom, we note that, using (21a), (21b) and (21c), we obtain

$$1 - \left(\frac{g_r}{g}\right)^2 = \frac{1}{p_0^2}\left(\Pi^{ab}_{ii}\right)^u_{a=b;\ \vec{p}^2 = p_0^2} \qquad (22)$$

Now we will calculate the right-hand side of this equation, using the Feynman rules expressed in (11c), (17) and (19). The relevant diagrams are shown in the following figure.



Self-energy graphs with transverse (D^T) and longitudinal (D^L) gluon propagators.

The contributions to $1 - g_r^2/g^2$ resulting from the diagrams above are given respectively by

$$I_a = \frac{13C}{48\pi^2} g^2 \ln \Lambda^2 \qquad (23a)$$

$$I_b = \frac{-24C}{48\pi^2} g^2 \ln \Lambda^2 \qquad (23b)$$

Here the constant C is defined by $C\delta_{aa'} = f^{abc} f^{a'bc}$ and Λ denotes the ultraviolet cut-off. (A graph containing two longitudinal gluons gives zero as a consequence of the principal value prescription (18b)). Note that diagram (a), which corresponds to vacuum polarization by a pair of transverse gluons, contributes with a positive sign. (This is similar to the contribution of a particle-antiparticle pair in QED). The cause of the overall negative sign of $(1 - g_r^2/g^2)$

$$1-\left(\frac{g_r}{g}\right)^2=-\frac{11\,C}{48\pi^2}\,g^2\ln\Lambda^2 \qquad (24)$$

is therefore isolated in graph (b). This diagram, which is crucial for achieving the asymptotic freedom, corresponds to the polarization of the vacuum by a transverse and a longitudinal gluon.

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