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LECTURES ON PLASMA PHYSICS

by

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LECTURES ON PLASMA PHYSICS†

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These seminars are specialized on aspects of Plasma Physics which can be of importance for Fusion Theory. In the applications, the emphasis is put on Tokamaks.

The first approach will be Ideal Magnetohydrodynamics and its applications. This will be done in the first three seminars.

The fourth and fifth lectures will be on Dissipative MHD.

CONTENTS

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IDEAL MAGNETOHYDRODYNAMICS I

GENERALITIES

A gas is an assembly of atoms. If the atoms are fully ionized, it is called a plasma.

The "fluid" or "continuum" approximation is very useful for gases. We treat the plasma as a "fluid", able to interact with the electromagnetic field.

The "particle" aspect of the plasma will not be treated here and the mathematical derivation of the fluid equations from "first principles" (Liouville equation) is one of the subjects of statistical mechanics (see books of Spitzer[1] and Balescu[2]).

A plasma can carry charges and currents. A charge separation over a distance of the order of the Debye[1] length $\left(\sqrt{\frac{KT_e}{4\pi ne^2}}\text{, cgs esu}\right)$ is quickly restored in a time $\omega_{p_e}^{-1} = \left(\frac{4\pi ne^2}{m}\right)^{-1}$.

If one ignores high frequency and short range phenomena, it is plausible to assume quasineutrality which is obviously compatible with electrical currents.

We are going to assume for these currents a perfect conductivity which leads to zero electric field in the system of the fluid. We assume also non-relativistic motion.

While motion of matter is governed by Newton's law, we assume "pre Maxwell's" equations for the electromagnetic field, which is compatible with the assumption of non-relativistic motion.

These assumptions lead to the following system of partial differential equations written in cgs esu units:

MHD EQUATIONS

(1) $$c^2 \nabla \times \underline{B} = 4\pi \underline{j} + \cancel{\frac{1}{c}\dot{\underline{E}}}$$

(2) $$\nabla \times \underline{E} = -\dot{\underline{B}}$$

(3) $$0 = \underline{E}'_{Fl.} = \underline{E} + \underline{v} \times \underline{B}$$

(4) $$\nabla \cdot \underline{B} = 0$$

where $\underline{j}$ is the electrical current in the fluid and $\underline{v}$ the local velocity of the fluid. Quasi-neutrality implies that
$\nabla \cdot \underline{E} = -\nabla \cdot \underline{v} \times \underline{B} \approx 0$

The change in $\underline{v}$ in the case of an ideal fluid with scalar pressure is given by Navier Stokes[3] equations:

(5) $$\rho\left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right] = \underline{j} \times \underline{B} - \nabla p$$

where the pressure p is governed by an adiabatic law (Heat conductivity neglected):

(6) $$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$ along the motion

and the mass density ρ behaves in such a way that the continuity equation is satisfied:

(7) $$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \underline{v}$$

The system of equations (1) to (7) is Galilei invariant which is due to the non-relativistic approximation in the motion and in Maxwell's equations.

In the case of astrophysical plasmas, a gravitational force should be added to equation (5) which is the expression of Newton's law for a fluid (see a Book[3] on Hydrodynamics or Gasdynamics). Equation (4) is satisfied at any time if it is satisfied initially.

EQUILIBRIUM IN CYLINDRICAL GEOMETRY[4]

We assume a static plasma as well as a time-independent field. The equations (1) to (7) reduce to:

$$\underline{j} \times \underline{B} = \nabla P \tag{8}$$

$$c^2 \nabla \times \underline{B} = 4\pi \underline{j} \tag{9}$$

$$\nabla \cdot \underline{B} = 0 \tag{10}$$

If we consider a system of cylindrical coordinates r, θ, z and if the components of $\underline{B}$ are B_r, B_θ, B_z, equation (10) becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

If there is a symmetry with respect to θ and z it follows that $r B_r = ct.$ and this constant must be zero because of the finiteness of B_r at $r = 0$ which leads to $B_r = 0$.

Using $B_r = 0$ and the symmetry in θ and z, equation (9) leads to

$$j_r = 0 \;,\quad \frac{4\pi}{c^2}\, j_\theta = -\frac{\partial B_z}{\partial r} \;,\quad \frac{4\pi}{c^2}\, j_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r B_\theta\right)$$

Inserting this in equation (8) we obtain:

$$-\frac{c^2}{4\pi}\left(B_z \frac{\partial B_z}{\partial r} + \frac{B_\theta}{r}\frac{\partial}{\partial r}\left(r B_\theta\right)\right) = \frac{\partial p}{\partial r} \tag{11}$$

REMARK

Equation (11) contains 2 free functions, for example: $p(r)$ and $B_\theta(r)$ or $j_z(r)$ and $j_\theta(r)$.

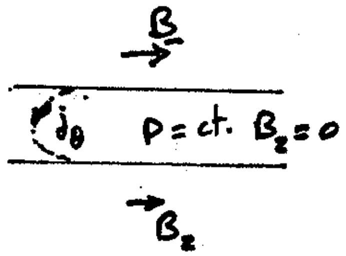
This freedom allows us to distinguish between types of equilibria:

θ pinch

Let $j_z = 0$ which leads to $B_\theta = 0$ then

$$\frac{\partial p}{\partial r} + \frac{c^2}{4\pi}\frac{\partial}{\partial r}\left(\frac{B_z^2}{2}\right) = 0$$

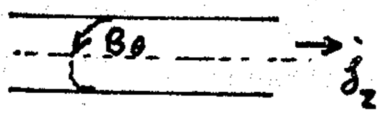
example: see Fig.



$B_z(r)$ or $p(r)$ is free

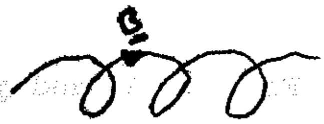
z pinch

$j_\theta = 0$ and $B_z = 0$



Screw Pinch and Tokamak

None of $B_\theta, B_z, j_\theta, j_z$ is zero. This leads to helical magnetic and current lines



The helicity of the magnetic lines is expressed in the "rotational transform" or in its inverse, the so-called "safety factor".

Let r be the radius of the plasma cylinder, L its length, then let us define the "safety factor" q as:

$$q = \frac{r B_z}{L B_\theta} \tag{12}$$

$q = \infty$	for the θ pinch
$q = 0$	for the z pinch
$q \doteq O(1)$	for the Tokamak

Another quantity which usefully characterizes the equilibrium is the

$$\beta = \frac{\text{kinetic pressure}}{\text{Magnetic pressure}}$$

It can be defined in several ways

$$\beta = \frac{P(r=0)}{c^2 \frac{B_{out}^2}{8\pi}} \text{ in ues}, \qquad \frac{P(r=0)}{\frac{B_{out}^2}{8\pi}} \quad B \text{ in Gauss} \tag{13}$$

or $$\bar{\beta} = \frac{\int p\, r dr}{\int B_\theta^2/8\pi\; r dr}.$$

For the θ and z pinch $\quad \beta \approx O(1)$

For the Tokamak

$$\beta = \frac{p}{B_\theta^2/8\pi}\, q^{-2} \left(\frac{r}{L}\right)^2 \qquad \text{rather small.}$$

<u>Force-Free Fields</u>

These are fields having the property

(14) $$\underline{j} = \lambda \underline{B}$$

<u>Exercise</u>

(a) What is the shape of Force-Free Fields having $\lambda = ct.$, in cylindrical symmetry?

(b) Compute the q(r).

<u>Exercise</u>

(a) What is the shape of B_θ and B_z and p if $j_z = ct.$ and $j_\theta = 0$?

(b) Compute the q(r).

TOROIDAL EQUILIBRIUM

From equation (8) $\underline{j} \times \underline{B} = \nabla p$ it is easy to obtain $\underline{B} \cdot \nabla p = 0$ and $\underline{j} \cdot \nabla p = 0$. This means that the pressure is constant along the magnetic lines and the current lines.

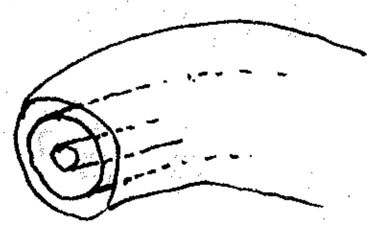
The magnetic lines are given by

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}$$ in cartesian

coordinates.

A magnetic line can be either closed or not closed. If it is not closed it may fill up a toroidal "magnetic surface" or it may fill up a volume depending upon the non-linear dependence of B_x, B_y, B_z upon x,y,z. (This problem is similar to the problem of the motion of planets if more than 2 bodies are involved).

Good confinement can be reached if most of the magnetic lines fill up a nested set of toroidal magnetic surfaces. To be sure of that is, in general, a very difficult problem of non-linear mappings (see KAM Theory for example in HCP/T2865-01 by Treve(5)).



Fortunately this problem is much easier for axi-symmetric equilibria. Let us try to make it clear.

Let us consider the meridional and the toroidal components of the magnetic field and denote them by $\underline{B}_{mer}$ and $\underline{B}_{tor}$. Again let r, φ, z be the cylindrical coordinates.

The divergence of $\underline{B}_{tor}$ is zero because of the symmetry in the φ coordinate, so that

$$\underline{B}_{tor} = \underline{e}_\varphi \, \frac{T}{r}(r,z)$$

with T an arbitrary function of r and z. If $T = ct.$, $\underline{B}_{tor}$ is a vacuum field.

If we want $\underline{B}_{mer}$ to lie on the surfaces $\Psi = ct.$ and be divergence free, it will be of the form

$$\underline{B}_{mer} = \frac{\underline{e}_\varphi}{r} \times \nabla \Psi$$

and

$$\underline{j}_{mer} = \frac{c^2}{4\pi} \nabla \times \underline{B}_{tor} = \frac{c^2}{4\pi} \frac{\nabla T}{r} \times \underline{e}_\varphi$$

$$\underline{j}_{tor} = \frac{c^2}{4\pi} \nabla \times \underline{B}_{mer} = \frac{c^2}{4\pi} \left[-\frac{1}{r^2} \frac{\partial \Psi}{\partial r} \underline{e}_\varphi + \nabla . \left(\frac{\nabla \Psi}{r} \right) \underline{e}_\varphi \right]$$

Inserting $\underline{j}$ and $\underline{B}$ into equation (8) we obtain:

$$-\frac{c^2}{4\pi} \frac{1}{r} \left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r} \frac{\partial^2 \Psi}{\partial z^2} \right) \nabla \Psi - \frac{c^2}{4\pi} \left(\frac{\underline{e}_\varphi}{r} . \nabla T \times \nabla \Psi \right) \frac{\underline{e}_\varphi}{r} =$$

$$\frac{c^2}{4\pi} \frac{1}{r^2} T \nabla T + \nabla p$$

From the toroidal component we obtain $T = T(\Psi)$ and from $\underline{B}_{mer} \cdot \nabla p = 0$ $p = p(\Psi)$. These two functions are arbitrary in Ψ. From the meridional component we obtain

$$\frac{c^2}{4\pi} \frac{1}{r} \left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} \right] = -r \frac{dp}{d\Psi} - \frac{T}{r} \frac{dT}{d\Psi} \frac{c^2}{4\pi}$$

This is a non-linear elliptic equation already known in hydrodynamics (Incompressible flows). It can be shown[6] (see Courant, Hilbert) that if $p(\psi)$ and $T(\psi)$ are specified there is a solution which under certain conditions is unique.

Exercise

(a) In the case of $T^2 = \alpha + \beta\psi$ and $\frac{dp}{d\psi} = ct.$

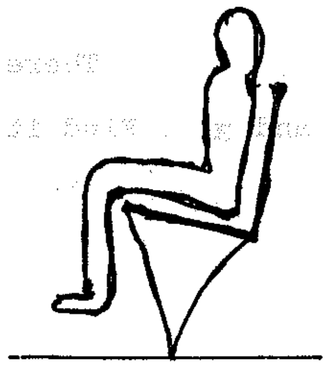
There is a polynomial solution of 4^{th} degree in r and z. Find it!

(b) Plot the $\psi(r,z) = ct.$ surfaces.

IDEAL MAGNETOHYDRODYNAMICS II

WHY STABILITY?

It is unfortunately not enough to have an equilibrium. There are many examples in current life to demonstrate this. You would not build a chair reposing on one point despite the fact that there is an equilibrium. The reason is that it is unstable.



If you let the chair rotate like a top, it may be dynamically stabilized but you would probably not like to sit down on such a chair.

A plasma, even in the MHD model, has a continuum of degrees of freedom, so its stability will be, of course, much more difficult to study than points or solids in mechanics. Perhaps we can learn from previous stability studies in hydrodynamics and gas dynamics.

The simplest example in that field is the Rayleigh-Taylor instability[7]. If a heavy liquid is on top of a lighter one, it would tend to go down to the state "light on top". The potential energy in the first state is higher that in the end state. If we consider two thin layers of different mass densities we can compute the potential energy for the 2 states.

STATE 1 $\quad E_p^1 = \rho_1 g h_1 + \rho_2 g h_2$

STATE 2 $\quad E_p^2 = \rho_2 g h_1 + \rho_1 g h_2$

If $\quad h_1 > h_2 \quad$ and $\quad \rho_1 > \rho_2 \quad$ it follows $\quad E_p^1 > E_p^2$

The second state "light on top" has a lower potential energy.

State 1 could be maintained, however, if auxiliary constraints (like a separating membrane) are introduced.

The fact that we assumed in Lecture I that the plasma is perfectly conducting is certainly a constraint which is not exactly verified. As we shall see, precisely this constraint makes stabilization in the ideal case possible.

<u>MHD STABILITY (MATH.THEORY)</u>

A study of the non-linear or full problem of stability is practically not accessible to analysis. Only linear stability can be investigated with rather great generality. A necessary and sufficient condition in the form of an "energy principle" can only be deduced for <u>static equilibria</u>.

Let us perturb the MHD equations around a static equilibrium ($\underline{v}_0 = 0$) so that

$$\underline{B} = \underline{B}_0 + \underline{B}_1 + \cdots$$
$$\underline{v} = \underline{v}_0 + \underline{v}_1 + \cdots$$
$$\rho = \rho_0 + \rho_1 + \cdots$$
$$\vdots$$

we obtain the following set of equations:

(1) $$\rho_0 \dot{\underline{v}}_1 = \underline{j}_0 \times \dot{\underline{B}}_1 + \underline{j}_1 \times \underline{B}_0 - \nabla p_1$$

(2) $$\dot{p}_1 = -\underline{v}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{v}_1$$

(3) $$\dot{\underline{B}}_1 = -\nabla \times \underline{E}_1 = \nabla \times (\underline{v}_1 \times \underline{B}_0)$$

(4) $$\dot{\underline{j}}_1 = \frac{c^2}{4\pi} \nabla \times \underline{B}_1$$

<u>Exercise</u>

Derive these equations from the MHD equations of lecture I.

From equations (1) to (4) it is possible to obtain one vector equation in $\underline{v}_1$

Take the time derivative of eq.(1) and then eliminate $\dot{\underline{B}}_1$, $\dot{\underline{j}}_1$, $\dot{p}_1$ from equations (2), (3) and (4). This leads to:

(5) $$\rho_0 \ddot{\underline{v}}_1 = \frac{c^2}{4\pi} \left[\nabla \times \nabla \times (\underline{v}_1 \times \underline{B}_0)\right] \times \underline{B}_0 + \underline{j}_0 \times \nabla \times (\underline{v}_1 \times \underline{B}_0) - \nabla \left(\gamma p_0 \nabla \cdot \underline{v}_1 - \underline{v}_1 \cdot \nabla p_0\right)$$

or

(6) $$\rho_0 \ddot{\underline{v}}_1 = - F(\underline{v}_1)$$

It can be proved that $F(\underline{v}_1)$ is a symmetric[8] operator. This proof can be done in an elegant way using the Lagrangean (see Newcomb[9] IAEA Salzburg Conf. 1961) properties. There is also a general theorem by Vainberg[10] (Variational Methods for the study of Non-linear Operators, Holden Day, San Francisco, Cal. 1964) saying "essentially" that if $\rho_0 \ddot{\underline{v}}_1 + F(\underline{v}_1) = 0$ can be derived from a Lagrangian $F(\underline{v}_1)$ is symmetric and vice-versa.

Exercise (very difficult)

Prove directly that $F(\underline{v}_1)$ is symmetric, under the boundary conditions $n_0 \cdot \underline{v}_1 = 0$ at a perfectly conducting wall.

Symmetry means: $(\eta, F\xi) = (\xi, F\eta)$ for all $\underline{\eta}, \underline{\xi}, \in L^2$ with

$$(\eta, F\xi) = \int_V \underline{\eta} \cdot F(\underline{\xi})\, d\tau$$

NECESSARY AND SUFFICIENT CONDITION FOR STABILITY

(a) Statement: If for any $\underline{\xi}$ of L^2

$$\delta W \equiv (\xi, F(\xi)) < 0$$

the system is exponentially unstable.

(b) Statement: If for all $\underline{\xi}$ in L^2

$$\delta W \equiv (\xi, F(\xi)) > 0$$

the system is stable.

PROOF OF (b) OR "SUFFICIENCY"

Multiply equation (6) by $\dot{\underline{\xi}}_1$ scalarly and integrate:

(7) $$\frac{1}{2}\left(\dot{\underline{\xi}}_1, \rho_0 \dot{\underline{\xi}}_1\right)^{\cdot} + \frac{1}{2}\left(\underline{\xi}_1, F(\underline{\xi}_1)\right)^{\cdot} = 0$$

(to obtain this equation use has been made of the symmetry of $F(\underline{\xi}_1)$

(8) or $$\left(\dot{\underline{\xi}}_1, \rho_0 \dot{\underline{\xi}}_1\right) + \left(\underline{\xi}_1, F(\underline{\xi}_1)\right) = 2E$$

If $\left(\underline{\xi}_1, F(\underline{\xi}_1)\right) > 0$ for all $\underline{\xi}_1 \in L^2$

the two terms on the left-hand side of eq.(8) are positive and their sum has to be a positive constant. Neither of them can increase indefinitely. The system is stable.

PROOF OF (a) OR "NECESSITY"(11)

Multiply equation (6) by $\underline{\xi}_1$ scalarly and integrate over the volume:

(9) $$\left(\underline{\xi}_1, \rho_0 \ddot{\underline{\xi}}_1\right) + \left(\underline{\xi}_1, F(\underline{\xi}_1)\right) = 0$$

and assume that $\left(\underline{\xi}_1, F(\underline{\xi}_1)\right) < 0$ for some $\underline{\xi}_1 = \underline{\xi}_{10}$

Eq. (9) can also be written as

$$\frac{1}{2}\left(\underline{\psi}_1, \rho_0 \underline{\psi}_1\right)^{\cdot\cdot} = 2\left(\dot{\underline{\psi}}_1, \rho_0 \dot{\underline{\psi}}_1\right) - 2E$$

where use has been made of eq.(8).

Let $I = \left(\underline{\psi}_1, \rho_0 \underline{\psi}_1\right)$ and use the Schwarz inequality:

$$\frac{\dot{I}^2}{4} = \left(\underline{\psi}_1, \rho_0 \dot{\underline{\psi}}_1\right)^2 \leq I\left(\dot{\underline{\psi}}_1, \rho_0 \dot{\underline{\psi}}_1\right)$$

from which it follows that

(10) $$\frac{1}{2}\ddot{I} \geqslant \frac{1}{2}\frac{\dot{I}^2}{I} - 2E$$

Eq. (6) being of second order in time we can choose $\dot{\underline{\psi}}_{10} = \alpha \underline{\psi}_{10}$ So that eq. (8) at $t = 0$ $\left(\underline{\psi}_1 = \underline{\psi}_{10}, \dot{\underline{\psi}}_1 = \dot{\underline{\psi}}_{10}\right)$ gives $\alpha^2 I_0 + \left(\underline{\psi}_{10}, F(\underline{\psi}_{10})\right) = 0 = E$ with α real.

Now equation (10) leads to:

$$\frac{\ddot{I}}{\dot{I}} \geqslant \frac{\dot{I}}{I} \Rightarrow \frac{\dot{I}}{\dot{I}_0} > \frac{I}{I_0} \Rightarrow \frac{\dot{I}}{I} > \frac{\dot{I}_0}{I_0}$$

so $$I > I_0 \exp\left[\frac{\dot{I}_0}{I_0} t\right] = I_0 e^{2\alpha t}$$

which proves exponential instability with the rate α

PHYSICAL INTERPRETATION

The quantity $\delta W \equiv \left(\underline{\xi}, F(\underline{\xi})\right)$ can be interpreted as the variation of potential energy induced by the virtual displacement $\underline{\xi}$. In fact $\left(\underline{\xi}, F(\underline{\xi})\right)$ is the variation of the potential part of the Lagrangean to 2^{nd} order. If this variation is positive for all $\underline{\xi}$ in L^2 it means that there is no displacement $\underline{\xi}(\underline{r})$ for all functions $\underline{\xi}(\underline{r}) \in L^2$ which is able to make the potential energy smaller. This is similar to the situation "light on top" for the Rayleigh-Taylor stability problem.

If there is any trial function $\underline{\xi}(\underline{r})$ able to make $(\underline{\xi}, F(\underline{\xi})) < 0$ the system may decrease its potential energy following this path. It is similar to the situation "heavy on top".

QUALITATIVE STABILIZING AND DESTABILIZING TERMS

Using eq.(5) we find

$$\delta W \equiv \frac{c^2}{4\pi} \int \left\{ \left[\nabla \times (\underline{\xi} \times \underline{B}_0) \right]^2 - \frac{4\pi}{c^2} \underline{\xi} \times \underline{j}_0 \cdot \nabla(\underline{\xi} \times \underline{B}_0) \right\} d\tau \tag{11}$$

$$+ \int \left\{ \gamma p_0 (\nabla \cdot \underline{\xi})^2 - (\underline{\xi} \cdot \nabla p_0) \nabla \cdot \underline{\xi} \right\} d\tau$$

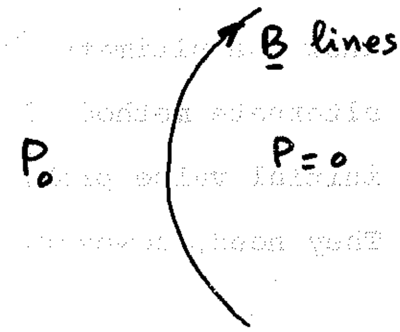
If we consider force-free fields(12), then

$p_0 \approx \nabla p_0 \approx \underline{j}_{\perp 0} \approx 0$ and $\underline{j}_0 \approx \lambda \underline{B}_0$

Only the first integral survives, the first integrand being statilizing and the second one destabilizing. If it is possible to make $\nabla \times (\underline{\xi} \times \underline{B}_0)$ small, then the second term may overcome the first one. So one should try to prevent $\nabla \times (\underline{\xi} \times \underline{B}_0)$ from becoming too small. It turns out that the "shear" (defined as $\frac{dq}{dr}/q$) tends to increase $\nabla \times (\underline{\xi} \times \underline{B}_0)$ in magnitude (over the radius of the plasma).

If $\underline{j}_{0\parallel}$ is small $\underline{j}_{0\perp}$ depends upon the sign of ∇p_0, and the sign of δW will be a combination of the sign of ∇p_0 and some derivatives of $\underline{B}_0$ (which are related

to the curvature by thė Serret-frenet formula $\frac{\underline{B}}{|\underline{B}|}\cdot\nabla\left(\frac{\underline{B}}{|\underline{B}|}\right) = -\frac{\underline{n}}{R}$)
It turns out that if P_0 decreases to the outside the curvature has to be concave toward the plasma.



This could also be visualized roughly by saying that the centrifugal force plays the role of gravity and the stable situation is "light on top of heavy".

These are two main types of instabilities, the first one is called "kink" and the second "flute" (it is more local). Of course the full problem of evaluating the energy principle is much more difficult and real instabilities are always a combination (13) of the two main types .

EXTERNAL MODES

Until now we have assumed $\underline{n}_0 \cdot \underline{\xi} = 0$ at the boundary of the plasma. If there is an interface between plasma and vacuum, the plasma boundary can move (14) and new modes (some very dangerous like "external kinks") may occur. They can be reduced by "shear" and by reducing $j_{\parallel 0}$

MATHEMATICAL ADVANTAGES OF THE "ENERGY PRINCIPLE"

The remarkable thing is that it is possible to decide about stability without having to solve the equations with time dependence. If for any test function you find a negative δW you know it is unstable. If on the contrary the curvature is good everywhere like in the "cusp" it is possible to prove that $\delta W > 0$ for all $\underline{\xi}$

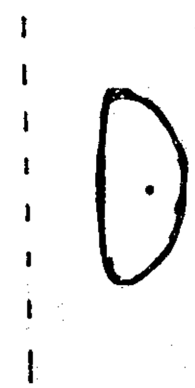
In most real cases a test function study is not enough (for example if one wants to know the ultimate β value in the Tokamak) and alternate methods like eigenmode analysis or initial value problems may become competitive. They need, however, an important numerical effort.

IDEAL MAGNETOHYDRODYNAMICS III

NECESSITY OF EVALUATING TOROIDAL EFFECTS

There are many reasons to consider MHD in toroidal geometry. The principal reason is topological (qualitative). The torus is finite and has an internal and external part. This makes confinement by the own field impossible as we shall see (in contrast to the cylindrical case). The stability may also be qualitatively different: (a) because the magnetic lines have a favorable curvature on the inner side (at large q values) and an unfavorable curvature on the outerside; (b) because the wave length along the torus is now limited by the dimensions of the system. Precisely this leads under specific conditions to the Kruskal-Shafranov limit.

The second reason not to neglect toroidal geometry, is quantitative. The trend is for compact (more economical) Tokamaks. A typical example is the JET experiment which has an aspect ratio of ≈ 2.5 and by no means could be approximated by a cylinder.



Unfortunately, even in the axisymmetric case (2-dimensional), the mathematical problems are going to be very tough. The numerical schemes will have to be sophisticated and the programming and computing effort will be rather large.

Impossibility of Axisymmetric Confinement by the own field + a toroidal external field or the need for a vertical field

To prove this, we are going to use a procedure similar to the virial theorem in mechanics. Consider first the r component of j x B = ∇p which is:

$$(1) \qquad j_\varphi B_z - j_z B_\varphi = \frac{\partial p}{\partial r}$$

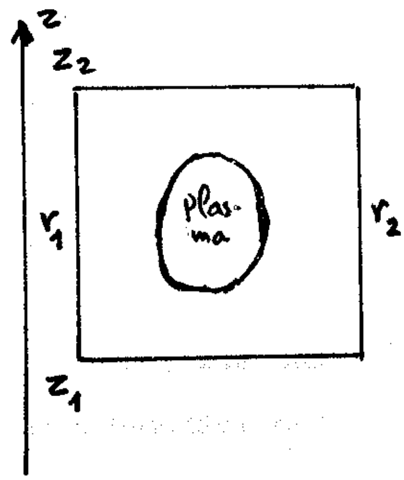
From $\quad \underline{j} = \frac{c^2}{4\pi} \nabla \times \underline{B} \quad$, we have

$$\frac{4\pi}{c^2} j_\varphi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \quad , \quad \frac{4\pi}{c^2} j_z = \frac{1}{r} \frac{\partial}{\partial r}(r B_\varphi)$$

so eq. (1) becomes

$$(2) \qquad \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) B_z - \frac{B_\varphi}{r} \frac{\partial}{\partial r}(r B_\varphi) = \left(\frac{c^2}{4\pi}\right)^{-1} \frac{\partial p}{\partial r}$$

Multiply eq.(2) by r^2 and integrate over r and z in the rectangle $(z_2 - z_1 , r_2 - r_1)$



$$\frac{c^2}{4\pi} \int_{r_1}^{r_2} dr \int_{z_1}^{z_2} dz\, r^2 \left[\frac{\partial B_r}{\partial z} B_z - \frac{1}{2} \frac{\partial B_z^2}{\partial r} \right] - \frac{c^2}{4\pi} \int_{z_1}^{z_2} \left[r^2 B_\varphi^2 \right]_{r_1}^{r_2} dz =$$

$$- 2 \int_{r_1}^{r_2} r dr\, dz\; p$$

From $\quad \nabla \cdot \underline{B} = 0 \quad$ we have $\quad \frac{\partial B_z}{\partial z} = - \frac{1}{r} \frac{\partial}{\partial r}(r B_r)$

Integrating the first integrand by parts with respect to z and the 2nd with respect to r, we obtain

$$\int_{r_1}^{r_2} dr\, r^2 \left[B_r B_z\right]_{z_1}^{z_2} + \int_{r_1}^{r_2} dr \int_{z_1}^{z_2} dz\; r B_r \frac{\partial}{\partial r}(r B_r) - \frac{1}{2} \int_{z_1}^{z_2} dz \left[r^2 B_z^2\right]_{r_1}^{r_2} +$$

$$+ \int_{r_1}^{r_2} dr \int_{z_1}^{z_2} dz\; r\, B_z^2 = - \left(\frac{c^2}{8\pi}\right)^{-1} \int_{r_1}^{r_2} dr \int_{z_1}^{z_2} dz\; r p$$

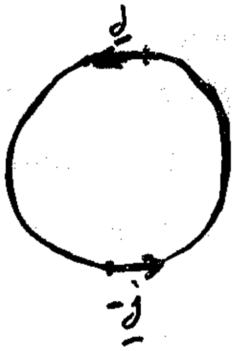
finally we have

$$\int_{r_1}^{r_2} dr\, r^2 \left[B_r B_z\right]_{z_1}^{z_2} + \frac{1}{2} \int_{z_1}^{z_2} \left[r^2 B_r^2\right]_{r_1}^{r_2} dz - \frac{1}{2} \int_{z_1}^{z_2} dz \left[r^2\, B_z^2\right]_{r_1}^{r_2}$$

$$+ \int_{r_1}^{r_2} \int_{z_1}^{z_2} dr\, dz\; r\, B_z^2 = -\left(\frac{c^2}{8\pi}\right)^{-1} \int_{r_1}^{r_2} dr \int_{z_1}^{z_2} dz\; r p$$

If B_r and B_z are fields due to the plasma only the first 3 integrals will vanish for $z_2 \to +\infty$, $z_1 \to -\infty$ $r_2 \to \infty$, $r_1 \to 0$ and we remain with a contradiction + = −.

A simple physical picture of that result is that currents diametrically opposite (inacircular wire) repel each other, so a vertical external field is needed which increases the own field on the outer side, unless the wire can be maintained by elastic forces.



Shift of Magnetic Axis

Let us first prove that the magnetic surfaces cannot be concentric around the axis in a finite region.

Putting a solution of the type $\psi \approx (r-r_0)^2 + z^2$ in the equilibrium equation we have

$$\frac{c^2}{4\pi}\left(4 - \frac{2(r-r_0)}{r}\right) = - r^2 \frac{dp}{d\psi} - \frac{T\,dT}{d\psi}\frac{c^2}{4\pi}$$

which cannot be fulfilled unless $\psi = f(r)$. Contradiction!!

The next step is to assume circles for magnetic surfaces but shifted to the outside with respect to each other. In that case a formula can be given for the shift (approximate) which was first derived by Shafranov (see Galvão(4) lectures, or Mercier's book(15)).

In general the equilibrium will have to be calculated.

Safety Factor and β

In the first lecture q has been defined as a function of r . The natural extension is to introduce a $q(\psi)$ by taking

$$q = \frac{d\,(\text{Tor. Flux})}{d\,(\text{Pol. Flux})}$$

which in the case of a rational surface can be shown (Mercier's book) to be equal to $\frac{n}{m}$, n being the number of turns of the magn. line around the symmetry axis and m the number of turns of the magn. line around the magnetic axis. $q(\psi)$ cannot be calculated analytically in general. The same is true for β whose extension is more straightforward

$$\bar{\beta} = \frac{\int p \, dV}{\frac{c^2}{8\pi} \int B^2 \, dV} \quad \text{in ues}$$

Equilibrium with plasma-vacuum interface

The nonlinear equation in Ψ obtained in the first lecture (sometimes called Grad-Shafranov) is meaningful if the boundary $\Psi_b = ct.$ is given. In real situations the boundary is not given and attention should be payed to that problem[16].

MHD Stability in the Torus

Mercier's criterion[17] which generalizes Suydam's criterion to toroidal geometry is a very succesful application of the "Energy Principle". The fact that no eigenmodes are necessary allows one to concentrate on classes of test functions which may be easily handled mathematically and may seem dangerous for intuitive reasons.

Mercier considered test displacements $\underline{\xi}$ localized in the neighbouring of a rational magnetic surface with $q(\Psi_0) = \frac{n}{m}$ and having a weak dependence along the magnetic field. He succeeded in finding the minimum of δW within this class explicitly so that if the equilibrium is known it can be decided whether δW is > 0 ,or < 0 within this defined class of perturbations. The detailed calculations are in the literature[17] (see for example his book or/and the reference therein). The result is rather complicated:

$$\left[\frac{1}{2}\left(\frac{\iota}{2\pi}\right)'_{\Psi} + \int \frac{\underline{B} \cdot \underline{\Xi}}{|\nabla\Psi|^3} \, ds\right]^2 + \int \frac{B^2 \, ds}{|\nabla\Psi|^3} \left[\frac{dp}{d\Psi}\left(\frac{\iota}{2\pi}\right)^2 V''_{\Psi} - \frac{|\underline{\Xi}|^2}{B^2}\right] \geq 0$$

(Mercier[15])

on every surface for stability

(in Gauss rationalized)

If the cylindrical limit is taken in a proper way one obtains the Suydam criterion

$$\left(\text{Suydam}^{(18)}\right) \quad \frac{1}{4}\left(\frac{q'}{q}\right)^2 + \frac{2}{r}\frac{p'}{B_z^2} \geqslant 0 \qquad \text{(in Gauss rationalized)}$$

The notations are also explained in the book.

Let us make some remarks about the criterion. In the toroidal case we recognize the positive shear term $(\iota')^2 \approx \left(\frac{q'}{q}\right)^2$ and the curvature term V''_{Ψ}. They do not show up in the same simple form as in the cylinder (Suydam) but they still play an important role for the stabilization.

It can be shown that Suydam cannot be satisfied near the axis if $\dot{j}_z \neq 0,\ p' < 0$. Mercier's criterion can be satisfied in the torus if q is large enough on axis[15]. For circular surfaces near axis the answer for stability is $q > 1$. For more sophisticated applications (elongated cross sections...) see his book and Küppers, Tasso[19] (Z. für Naturforschung, 27a, 23, 1972).

"Ballooning" Displacements

The displacements considered in Mercier's criterion (sometimes called "flute" perturbations) are weably dependent on the coordinate along the field line. But in the torus the curvature of the lines has both signs and the "flutes" do not need to be the worst case.

Perturbations emphasizing the bad curvature along the lines (called "balloonings") are very important in the torus. Their mathematical treatment being more complicated, their study has been rather late (B.J.Taylor[20], C.Mercier[21] at Innsbruck IAEA Conf. and Correa-Restrepo[22] Z. für Naturforschung, all in 1978).

Even the "balloonings" may not be the worst among the internal modes but a complete study of the modes necessitates (the "balloonings" is localized in the neighbouring of rational surfaces) a minimization of δW in all L^2 space of functions. To do this analytically seems out of the question at present. So we are naturally led to use numerical techniques.

Numerical Codes

To evaluate the energy principle it is necessary to know the equilibrium. The first study of δW for extended modes and a plasma vacuum interface has been done for a polynomial equilibrium (see exercise 1st lecture). This study (Küppers, Pfirsch, Tasso[23], Madison IAEA Conf. 1971) has been carried out using a symbolic computation system named "Reduce". It was quite suitable for the polynomial equilibrium. This code has been refined subsequently by Kerner, Tasso[24] (Tokyo IAEA Conf. 1974) and Kerner[25] (Nucl.Fusion 16, 643, 1976). The fact that the code was symbolic allowed very high precision to be obtained (m=14,n=13 has even be detected).

The fact that it could handle only the polynomial equilibrium was a handicap but it furnished a basic test for purely numerical codes which were able to consider more general equilibria.

These codes have been developed in Lausanne and Princeton (see Berger et al.[26] and Johnson et al.[27] Berchtesgaden IAEA Conf. 1976 and a review article by Grimm et al.[28] in Methods in Comput. Physics, Fusion (1976).

These codes are still in a development and refinement phase. Recently Kerner was able to reproduce the analytical marginal curve found by Bussac et al. for the internal "kink" modes.

Both equilibrium and stability codes are very sophisticated and it may not be the right place to describe them. Those who are interested could read the literature.

The problem of 2-dimensional ideal MHD stability has not been touched here but a nice working code for surface currents has been written by Rebhan and Salat[52] (Nucl.Fusion, 1976) and an extended review of the MHD stability of Tokamaks has been published by J.Wesson[53] (Nucl.Fusion. 1978).

Finally let me mention that 3-d codes are under study by Garabedian[29] et al. at Courant Inst. (New York) and by Schlüter[30] et al. (Garching). They have already delivered some results for stellarators but results concerning the Tokamak have not yet been reported.

DISSIPATIVE MAGNETOHYDRODYNAMICS PART I

The assumption of perfect conductivity is not exactly verified in a plasma but we know that the plasma resistivity (see Spitzer[1], Interscience Publishers, 1962) behaves like $1/T^{3/2}$ and that, for a temperature of 1 keV, the resistivity of a hydrogen plasma has approximately the value of the resistivity of copper. At thermonuclear temperatures the conductivity will be 1 or 2 orders of magnitudes larger than in copper. It is then reasonable to believe that, for fast motion, the reaction of the plasma will be approximated by the one of a perfect conductor.

For slow motions, things will be different not only quantitatively but also qualitatively because of the lack of a constraint imposing flux conservation during the motion of the plasma. This has been mentioned already in previous lectures but let us make it more quantitative following Spitzer's book[1]. Consider a surface element following the motion of the fluid and compute the change of flux through it.

$$\frac{d\phi}{dt} = \underbrace{\iint_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{s}}_{\text{due to change in } \underline{B}} + \underbrace{\int_C \underline{B} \cdot \underline{v} \times d\underline{s}}_{\text{due to motion of contour}}$$

Using Maxwell's equation and Stokes theorem it can also be written as:

$$\frac{d\phi}{dt} = -\iint_S \underline{ds}\cdot\nabla\times\underline{E} - \iint_S \underline{ds}\cdot\nabla\times(\underline{v}\times\underline{B})$$

In the ideal case $\underline{E}+\underline{v}\times\underline{B}=0$ so that the flux through a surface during the motion is invariant. If the plasma is resistive, Ohm's law in the fluid system is $\underline{E}'=\eta\underline{j}$ and in the laboratory system is

(1) $$\underline{E}+\underline{v}\times\underline{B}=\eta\underline{j}$$

so that

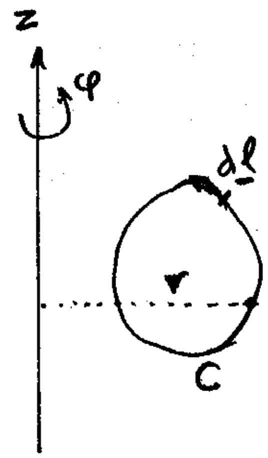
$$\frac{d\phi}{dt} = -\frac{c^2}{4\pi}\iint_S \nabla\times\eta\nabla\times\underline{B}\cdot\underline{ds}$$

The flux through a surface element during the motion will not be an invariant any more.

<u>Equilibrium in the Resistive Case</u>

Integrate equation (1) along a poloidal contour

$$\int_C \underline{E}\cdot\underline{dl} + \int_C \underline{v}\times\underline{B}\cdot\underline{dl} = \int_C \eta\underline{j}\cdot\underline{dl}$$



If $\dot{\underline{B}}=0$ the first integral vanishes by Stokes theorem because no flux change takes place in equilibrium through a meridional cross section so that the equilibrium cannot be static unless $\underline{j}_{pol}\equiv 0$

Integrate now equation (1) along a toroidal contour, we have, because the equilibrium is compatible with a linear change in time of the transformer flux

$$\int_{C'} \underline{E} \cdot d\underline{\ell} = 2\pi r \, E_{tor} = ct.$$

It follows that

$$(2) \qquad j_{tor} = \frac{\underline{v} \times \underline{B} \cdot \underline{e}_\varphi}{\eta(\Psi)} + \frac{ct.}{2\pi r \, \eta(\Psi)}$$

We see from equation (2) that a static plasma is not compatible with the equilibrium in the torus for ideal MHD (see first lecture) because $j_{tor.} = -r \frac{dP}{d\Psi}$ when $j_{pol} \equiv 0$ This means that in a torus there will always be a flow and in general a net outward flow for diamagnetic plasmas (see Shafranov[31], Review of Plasma Physics, Translation from Russian). This flow should not be too strong so that an ideal MHD equilibrium should still be reasonable in the presence of resistivity. This may be critical for FCT equilibria[32] (Flux Conserving Torus, Oak Ridge)

Remarks

(a) In the straight case the toroidal r dependence is absent so that static cases are possible if $j_{pol} \equiv 0$ and $\eta j_z = E_z = ct.$

(b) η has been taken as a function of the magnetic surface because the resistivity is a function of temperature and the magnetic surfaces are roughly isothermal because of the high thermal conductivity along the magnetic lines.

RESISTIVE INSTABILITIES

The non-conservation of flux leads to a freedom whose influence on stability has been pointed out already by Dungey[33] (Cosmic Electrodynamics, Cambridge Univ. Press. 1958).

In 1963, Furth, Killeen and Rosenbluth [34] published a well known paper (see also B.Coppi[35]) in Physics of Fluids, where they investigated the stability of a resistive pinch having plane symmetry. In 1969 Barston[36] (Phys. of Fluids) published an "Energy Principle" - like approach to the same problem. In 1975 an 1977 two papers (see Tasso[37,38], Plasma Phys.) on 2-dim. and helical resistive instabilities respectively extended Barston's approach[36] of the planar pinch to realistic geometries.

Paper[37] is going to be the main line of approach of the next section.

TWO-DIMENSIONAL RESISTIVE INSTABILITIES

Let us limit ourself to the straight case because it can be taken static with the hope that under favorable circunstances it will be a good approximation to the torus.

The equilibrium is given by

$$\underline{j}_0 = \underline{e}_z J_0(\psi)$$

$$\underline{E}_0 = \underline{e}_z \eta_0(\psi) J_0(\psi) = ct.$$

$$\underline{B}_0 = \underline{e}_z \times \nabla\psi + \underline{e}_z B_{z0} \tag{3}$$

$$J_0(\psi) = \frac{c^2}{4\pi} \nabla^2\psi = -\frac{dP_0}{d\psi}$$

which is the Grad-Shafranov equation in the straight case.

Let us study now the stability with respect to small perturbations around the equilibrium by linearizing the time dependent equations and assuming <u>incompressible motion</u>

(4) $$\rho_0 \ddot{\underline{\xi}} + \nabla p_1 - \underline{j}_1 \times \underline{B}_0 - \underline{j}_0 \times \underline{B}_1 = 0$$

(5) $$\nabla \cdot \underline{\xi} = 0$$

(6) $$\dot{\underline{A}}_1 + \frac{c^2}{4\pi} \eta_0 \nabla \times \nabla \times \underline{A}_1 + \eta_1 \underline{j}_0 - \dot{\underline{\xi}} \times \underline{B}_0 = 0$$

(7) $$\underline{B}_1 = \nabla \times \underline{A}_1$$

(8) $$\eta_1 = - \underline{\xi} \cdot \nabla \eta_0$$

with $\underline{v}_1 = \dot{\underline{\xi}}$

The last equation (8) results from the assumption that the resistivity is a property of the fluid and will be transported by the fluid according to $\frac{d\eta}{dt} = 0$

This is only partially true because the very high heat conductivity will smear out the temperature and resistivity along the lines and surfaces unless the lines do not form surfaces (closed magnetic lines).

Restriction to 2-dim. perturbations leads to

$$\underline{\xi} = \underline{e}_z \times \nabla u \quad , \quad \underline{A}_1 = \underline{e}_z A$$

This allows, by elimination in (4) to (8), to obtain 2 partial differential equations in the scalars A and U.

(See Tasso[37], Plasma Physics, 1975 for the details of the calculations).

$$\begin{pmatrix} -\nabla\cdot\rho_0\nabla & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \ddot{u} \\ \ddot{A} \end{pmatrix} + \begin{pmatrix} -(\underline{B}_0\cdot\nabla)^2 & -\dfrac{\underline{B}_0\cdot\nabla}{\eta_0} \\ \dfrac{\underline{B}_0\cdot\nabla}{\eta_0} & \dfrac{1}{\eta_0} \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{A} \end{pmatrix} +$$

(9)

$$\begin{pmatrix} -J_0(\psi)\,\underline{B}_0\cdot\nabla\,\underline{e}_z\times\dfrac{\nabla\eta_0}{\eta_0}\cdot\nabla & -\nabla\times\underline{j}_0\cdot\nabla \\ \nabla\times\underline{j}_0\cdot\nabla & -\dfrac{c^2}{4\pi}\nabla^2 \end{pmatrix}\begin{pmatrix} u \\ A \end{pmatrix} = 0$$

Use has been made of the equilibrium equations (3) also.

It can be proved (see previous reference) that, if equation (9) is written as

(10) $$N\ddot{\underline{\xi}} + M\dot{\underline{\xi}} + Q\underline{\xi} = 0$$

N, M and Q are symmetric and that $M > 0$ which means $(\underline{\xi}, M\underline{\xi}) > 0$ for $\underline{\xi} \in L^2$.

Nec. and Suf. Condition for Stability

For equation (10) it can be proved that the necessary and sufficient condition for stability[36] is (see Barston, Phys.Fluids, 1969)

(11) $$\delta W \equiv (\underline{\xi}, Q\underline{\xi}) > 0$$

Proof

(a) Multiply eq. (10) by $\dot{\underline{\xi}}$ scalarly and integrate over the volume of the plasma

(12) $$\frac{1}{2}(\dot{\underline{\xi}}, N\dot{\underline{\xi}})^{\cdot} + \frac{1}{2}(\underline{\xi}, Q\underline{\xi})^{\cdot} = -(\dot{\underline{\xi}}, M\dot{\underline{\xi}})$$

If $(\underline{\xi}, Q\underline{\xi}) > 0$ for $\underline{\xi} \in L^2$

$(\dot{\underline{\xi}}, N\dot{\underline{\xi}})$ nor $(\underline{\xi}, Q\underline{\xi})$ can grow so that the system is stable.

(b) If for some $\underline{\xi} = \underline{\xi}_0$, $(\underline{\xi}_0, Q\underline{\xi}_0) < 0$ let us integrate equation (12) with respect to time assuming $\dot{\underline{\xi}}_0 = 0$ and $\underline{\xi}(t=0) = \underline{\xi}_0$

(13) $$(\underline{\xi}, Q\underline{\xi}) = (\underline{\xi}_0, Q\underline{\xi}_0) - (\dot{\underline{\xi}}, N\dot{\underline{\xi}}) - \int_0^t (\dot{\underline{\xi}}, M\dot{\underline{\xi}})\, dt'$$

From eq.(13) it follows that $(\underline{\xi}, Q\underline{\xi})$ remains negative and the last integral will diverge because $(\dot{\underline{\xi}}, M\dot{\underline{\xi}}) > 0$ and is at least finite in average.

With a little more effort (see Barston[36],1969) it is possible to prove exponential instability.

APPLICATIONS

From eqs.(11) and (9) we obtain

$$\delta W = \int d\tau \left(-\frac{dJ_0}{d\psi}\right)\left(\underline{e}_z \times \nabla\psi \cdot \nabla u\right)^2 + 2\int d\tau \left(-\frac{dJ_0}{d\psi}\right) A \left(\underline{e}_z \times \nabla\psi \cdot \nabla u\right) + \frac{c^2}{4\pi}\int d\tau\, |\nabla A|^2 \tag{14}$$

The so-called "rippling mode"(34,39) can be demonstrated easily by making $A \approx 0$ and emphasizing u on the places where $\frac{dJ_0}{d\psi} > 0$ (current density increasing outward).

If the current density decreases to the outside $\frac{dJ_0}{d\psi} < 0$ the "rippling" is stable but "magnetic" perturbations can increase if δW can be made negative by them:

δW can be minimized with respect to u to give

$$\delta W = \frac{c^2}{4\pi}\int d\tau\, |\nabla A|^2 - \int d\tau\, J_0 \frac{\eta_0'}{\eta_0}\left(A^2 - \bar{A}^2\right)$$

$$= \frac{c^2}{4\pi}\int d\tau\, |\nabla A|^2 + \int d\tau\, \frac{dJ_0}{d\psi}\left(A^2 - \bar{A}^2\right)$$

with $\bar{A} = \dfrac{\oint \frac{dl}{B} A}{\oint \frac{dl}{B}}$

The minimization with respect to A leads to an Euler-Lagrange equation of the Schroedinger type

$$\frac{c^2}{4\pi}\nabla^2 A + (A-\bar{A})\left(-\frac{dJ_0}{d\psi}\right) = \lambda A \qquad (15)$$

where λ is the Lagrange factor due to the normalization of A. If $\lambda > 0$ it is unstable.

The solution of (15) is the solution of an eigenvalue problem in 2 dimensions which cannot be done, in general, analytically. A numerical application which has been made by Jensen and McClain(40) (J.Plasma Physics, 1978) delivered results for the stability of "doublet". It turns out that putting a boundary shaped like in Fig. 1 is more favorable than in Fig. 2 in which case the plasma may divide in two droplets (Fig.3).

FIG.1

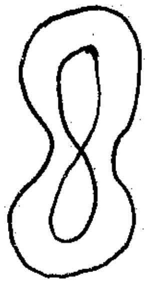
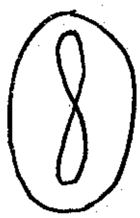
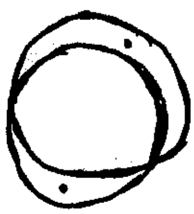


FIG.2



A very interesting application which has not been made yet is to apply the code to an "$m = 2$ island" in a Tokamak whose helicity has a long pitch so that the 2-d approximation would hold.

FIG.3



"m=2 island"

This may be a way to answer with precision how large the island will be. The island would grow at the beginning and will cease to grow if the criterion (11) or (15) were barely verified.

Why an island exists at all is due to helical "tearing"(34),(39) modes which will be investigated in the linear phase in the next lecture.

Exercise

Minimize δW from equation (14) for a plasma having cylindrical symmetry and circular cross section and prove that if the current density decays outward it is stable.

DISSIPATIVE MAGNETOHYDRODYNAMICS PART II

In the previous lecture we considered 2-dimensional perturbations which makes sense for shaped cross-sections of the plasma. For a circular cross-section and a current density decaying outwards, the 2-dim. perturbations are stable (see exercise at end of 4^{th} lecture).

On the other hand we know already from ideal MHD that kink-like perturbations, having more or less the same helicity as the magnetic lines, are rather dangerous. Suppose now that the kinks are stabilized by shear for a "bell" shaped current density distribution, so we would like to know what will happen to such perturbations if resistivity is taken into account.

Let us consider helical resistive perturbations in a plasma cylinder with a circular cross-section. We are going to derive an "energy principle" (see Tasso[38], Plasma Physics, 1977) for these helical perturbations and then apply it to a physically important class of test functions the "tearing modes" and the "rippling" modes.

Derivation of the "Energy Principle"

To study these perturbations it is convenient to represent the equilibrium and the perturbations in helical symmetry, although the equilibrium has a cylindrical geometry. For that purpose let us introduce a coordinate (see Mercier[41], 1960).

$$u = m\theta - hz$$

$(r, \theta, z$ being the usual cylindrical coordinates) and the vector

$$\underline{u} = \frac{m\underline{e}_z + rh\underline{e}_\theta}{m^2 + r^2h^2}$$

It can easily be shown that

(2) $$\nabla \cdot \underline{u} = 0 \quad , \quad \nabla \times \underline{u} = \frac{2hm}{m^2 + r^2h^2}\,\underline{u}$$

Now, in a similar way to the axisymmetric case, we can write down the solution of $\nabla \cdot \underline{B} = 0$ using $\nabla \cdot \underline{u} = 0$

(3) $$\underline{B} = f(r,u)\,\underline{u} + \underline{u} \times \nabla F(r,u)$$

(4) $$\underline{j} = \frac{c^2}{4\pi}\left\{\left[\frac{2hm}{m^2 + h^2r^2} f + (m^2 + h^2r^2)\, LF\right]\underline{u} + \nabla f \times \underline{u}\right\}$$

with $$LF = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{m^2 + r^2h^2}\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial u^2}$$

If we, now, write down the equilibrium equation $\underline{j} \times \underline{B} = \nabla p$, we would obtain the Grad-Shafranov equation for helical equilibria (see Mercier[41] 1960). But we wish to restrict to cylindrical symmetry, otherwise the stability problem could become very tough. In that case the magnetic field at equilibrium is

(5) $$\underline{B}_0 = f_0(r)\,\underline{u} + \underline{u} \times \nabla F_0(r)$$

For the stability analysis we linearize as usual the time dependent system of equations to obtain

(6) $$\rho_0 \ddot{\underline{\xi}} + \nabla p_1 - \underline{j}_1 \times \underline{B}_0 - \underline{j}_0 \times \underline{B}_1 = 0$$

(7) $$\nabla \cdot \underline{\xi} = 0$$

(8) $$-\dot{\underline{B}}_1 + \nabla \times (\underline{\xi} \times \underline{B}_0) + \nabla \times (\eta_0 \underline{j}_1 + \eta_1 \underline{j}_0) = 0$$

(9) $$\nabla \cdot \underline{B}_1 = 0$$

(10) $$\eta_1 = -\,\underline{\xi} \cdot \nabla \eta_0$$

Equations (7) and (9) can be solved respectively using Eq.(3)

(11) $$\underline{B}_1 = f(r,u,t)\,\underline{u} + \underline{u} \times \nabla F(r,u,t)$$

(12) $$\underline{\xi}_1 = g(r,u,t)\,\underline{u} + \underline{u} \times \nabla G(r,u,t)$$

As one can already see, helical perturbations are more difficult to analyse because we are going to obtain 4 equations for the 4 quantities f,F,g,G instead of 2 equations in the 2-dim. case. Another difficulty is that the system of 4 equations do not have the symmetry properties required to derive an "energy principle".

At this level, we have to make the approximation of the thin cylinder $\frac{r}{L} \approx \frac{B_\theta}{B_z} \approx \varepsilon \ll 1$ which is usually assumed in the literature. In that case it is possible to obtain a system of 2 partial differential equations in F and G which can be reduced to the form

$$N\ddot{\underline{\xi}} + M\dot{\underline{\xi}} + Q\underline{\xi} = 0$$

with N, M, Q symmetric and $M > 0$. $\delta W \equiv (\underline{\xi}, Q\underline{\xi})$ is given in this case[38] by

$$(13) \quad \delta W = \int_0^{r_0} \left(-\frac{dj_0}{dr}\right)\left(\underline{u}\times\underline{e}_r\cdot\nabla G\right)\left(\underline{u}\times\underline{\nabla}F_0\cdot\nabla G\right)\, d\tau$$

$$+ 2\int_0^{r_0}\left(-\frac{dj_0}{dr}\right)\left(\underline{u}\times\underline{e}_r\cdot\nabla G\right)F\, d\tau - \frac{c^2}{4\pi}\int_0^{r_0} FLF\, d\tau$$

where only the lowest order in $hr \approx \varepsilon$ should be kept. This leads to

$$\frac{\delta W}{2\pi} = -\int j_0' F_0' \frac{1}{m^2}\left(\frac{\partial G}{\partial\theta}\right)^2 r\,dr - 2\int j_0' \frac{F}{m}\frac{\partial G}{\partial\theta}\, r\,dr$$

$$+ \frac{c^2}{4\pi}\int\frac{r\,dr}{m^2}F'^2 + \frac{c^2}{4\pi}\int\frac{dr}{m^2 r}\left(\frac{\partial F}{\partial\theta}\right)^2$$

and making the ansatz

$$F = \tilde{F}(r)\, e^{im\theta} \quad , \quad G = \tilde{G}(r)\, e^{im\theta}$$

we obtain using the complex conjugates

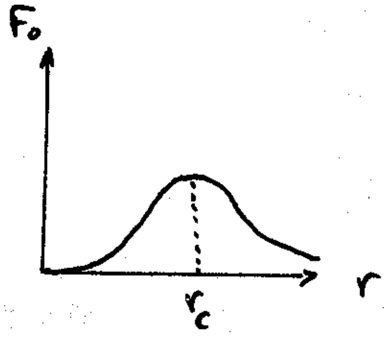
$$(14) \quad \frac{\delta W}{2\pi} = -\int j_0' F_0' \tilde{G}^2\, r\,dr - 2\int j_0' \tilde{F}\tilde{G}\, r\,dr$$

$$+ \frac{c^2}{4\pi}\int\frac{r\,dr}{m^2}\tilde{F}'^2 + \frac{c^2}{4\pi}\int\frac{dr}{r}\tilde{F}^2$$

For a certain value of $r = r_c$, F_0' may vanish which means that $\underline{B}_0$ is parallel to $\underline{u}$ and the pitch of the perturbation is the same as that of the magnetic line at the point $r = r_c$.

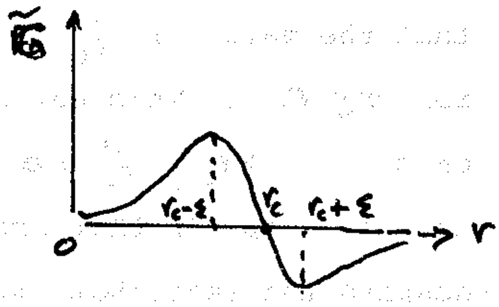
"Tearing" Mode (34) (39)

If we look for the extrema of the first 2 integrals with respect to $\tilde{G}$ we find



(15) $$\tilde{G} = -\frac{\tilde{F}}{F_0'}\,, \quad r \neq r_c$$

Inserting $\tilde{G}$ from equation (15) into equation (14) we obtain



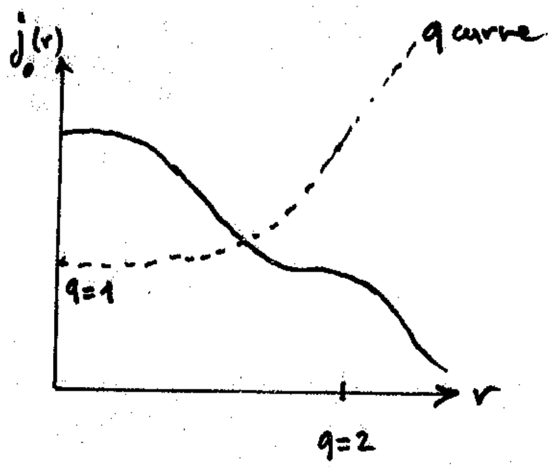
$$\frac{\delta W}{2\pi} = \frac{c^2}{4\pi}\int_0^{r_c-\varepsilon}\frac{r\,dr}{m^2}\,\tilde{F}'^2 + \frac{c^2}{4\pi}\int_{r_c+\varepsilon}^{r_0}\frac{r\,dr}{m^2}\,\tilde{F}'^2 \quad +$$

$$(16) \qquad + p.p. \int_0^{r_0} dr \left(\frac{j_0'}{F_0'}\, r + \frac{c^2}{4\pi}\frac{1}{r} \right) \tilde{F}^2 + \frac{c^2}{4\pi}\,\frac{r_c}{m^2}\,\tilde{F}(r_c-\varepsilon) \left[\frac{\tilde{F}'}{\tilde{F}} \right]_{r_c-\varepsilon}^{r_c+\varepsilon}$$

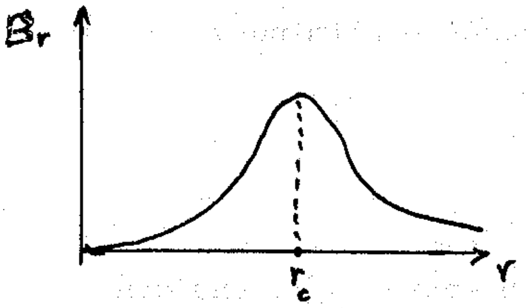
The jump in the logarithmic derivative of $\tilde{F}$ is usually called Δ' in the literature on the "tearing" mode.

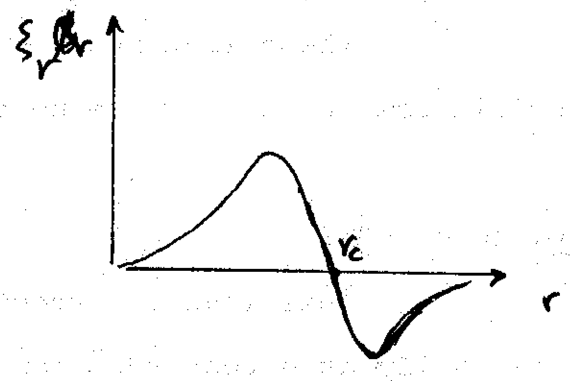
It remains to minimize expression (16) with respect to $\tilde{F}(r)$; that is to find the shape of $\tilde{F}(r)$ which makes δW as small as possible. This has been done numerically recently by Glasser, Furth and Rutherford[42] in Phys.Rev.Lett. 38, 234 (1977), for several current distributions $j_0(r)$. It turns out that the value of j_0' near $r = r_c$ is very important, especially for $r \lesssim r_c$. This can be understood from expression (16) because at $r \approx r_c$, $F_0' \to 0$. So a small change in the current density shape in that area may be stabilizing or destabilizing despite the fact that $q(r)$ does not change appreciably.

This fact seems to explain the results found on Pulsator by Karger et al.[43] (IAEA Conf. Tokyo, 1974). A shape of $j_0(r)$ similar to the one plotted here was stable to all "tearing" modes according to Glasser[42] et al.



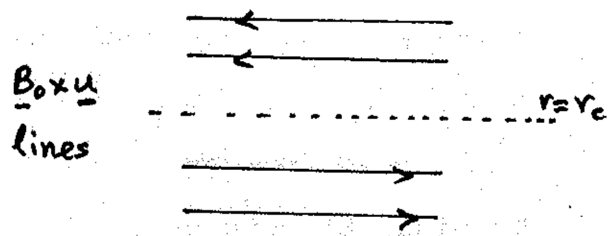
In the case of an unstable shape the functions ξ_r and B_r can be obtained from $\tilde{F}$ and $\tilde{G}$ and are plotted below:



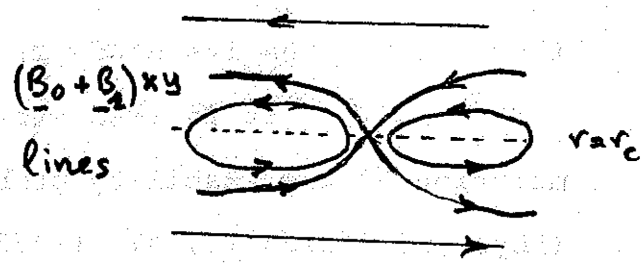


The fact that B_r does not change~~s~~ sign at $r = r_c$ means that a topological change in the magnetic surfaces occurs, which is qualitatively drawn in the next figures.

In fact a helical tube having the cross section of an island centered on a rational value of q (for ex. $q = 2$) could be formed in the unstable case.



If this island remains small in size compared to the radius of the plasma it may be harmless for the gross properties of the plasma. But the size may increase in some circunstances, as computed for example by Waddell[44] et al. (See Theor. and Comp. Plasma Phys., Trieste 1977). Recently Carrera et al[45] and Biskamp and Welter[46] (Workshop on Disruptions, Garching 1979) considered numerically the interaction of 2 unstable modes



of different helicities which may lead to a randomization of magnetic lines especially in the torus.

These last results are rather preliminary because the models used up to now are possibly too simple.

"Rippling mode"

This mode is essentially electrostatic and can be seen easily as a test function in expression (14) by choosing $\tilde{F} \approx 0$ and by emphasizing $\tilde{G}$ on the negative values of $-j_0'' F_0'$. The magnetic "tearing" component of this mode could also lead to island formation but probably of smaller size, especially because the smearing out of resistivity due to the high heat conductivity along the magnetic lines.

Nevertheless its effect may be important on the anomalous heat conductivity across the plasma (but this could also be produced by drift waves).

Toroidal Effects on Resistive Modes

In the previous lecture we already mentioned that the equilibrium on the resistive time scale is going to have a flow which makes its study mathematically more difficult.

It is also very difficult to make a nice formulation of the stability problem. Glasser, Green, Johnson[47] (Phys. of Fluids 18, 875 (1975)) have made a very comple analysis involving several scalings which seems to show a favorable influence of toroidal geometry.

Toroidal force-free field stability

(See Tasso[48] in Theor. and Comp. Phys., Trieste 1977)

For the case of $\underline{j} = \lambda \underline{B}$ with $\lambda = ct.$ and for a good conducting surrounding wall it may be proved[48][49] that if

$$\delta W_R \equiv \int d\tau \; \nabla\times\underline{A}_1 \cdot \nabla\times\underline{A}_1 \; - \int d\tau \; \lambda \; \underline{A}_1 \cdot \nabla\times\underline{A}_1$$

is positive for all $\underline{A}_1 \in L^2$ and $\underline{n}\times\underline{A}_1 = 0$ at the boundary then the plasma is stable against MHD + Resistive Modes and this is an exact result.

This means that λ has to be small enough or that the current along the magnetic field has to be small enough.

Bussac Furth and Rosenbluth[50] reported a similar result at the Innsbruck Conf. (IAEA, 1978) using Taylor's[51] invariant (Phys. Rev. Lett. 33, 1139, 1974). On this basis they proposed the configuration "Spheromak".

Final Remarks

It is difficult to believe that resistive MHD alone is able to describe quantitatively (and even qualitatively) the experiment. My feeling is that much more work should be done with more refined models.

Some lectures using a two-fluid plasma model will be given later by I.L.Caldas which cover essentially the subject of his thesis[54][55] which is a detailed investigation of the two-fluid "Energy Principle" found in ref. (56).

Exercise

Derive equations (11) and (12) out of eq. (7) and (9).

Exercise

Derive expression (13) and (14) using a publication in Plasma Physics (by Tasso[38] 19, 177, 1977).

* * *

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REFERENCES

(1) Spitzer L. - "Physics of Fully Ionized Gases", Interscience Publishers - John Wiley and Sons, 1962.

(2) Balescu R. - Statistical Mechanics of Charged Particles- New York, Interscience, 1963.

(3) Lamb H. - "Hydrodynamics", Cambridge Univ. Press, 1953.

(4) Galvão R.M. - "Lectures at IFUSP", 1978 and 1979.

(5) Treve - "Assessment of Contemporary Methematical Methods for Magnetic Fusion Research" - HCP/T2865-01 DOE Report.

(6) Courant Hilbert - "Methods of Math. Phys.", Interscience Publishers, 1953.

(7) Chandrasekhar - Hydrodynamic and Hydromagnetic Stability Clarendon Press, Oxford, 1961.

(8) Bernstein et al., Hain et al. - Proc. Roy. Soc. London, A244, 17, 1958. Z. für Naturforschung, A12, 833, 1957.

(9) Newcomb - IAEA Salzburg Conf. 1961 on "Plasma Physics and Contr. Nucl. Fusion".

(10) Vainberg M.M. - Variational Methods for the Study of Nonlinear Operators - Holden Day, San Francisco 1964.

(11) Laval, Mercier, Pellat - Nuclear Fusion 5, 156, 1965.

(12) Voslamber, Callebaut - Phys. Rev. 128, 2016, 1962.

(13) Tayler R.J. - Proc. Phys. Soc. B70, 1049, 1957.

(14) Kruskal, Tuck - Proc. Roy. Soc. A245, 222, 1958.

(15) Mercier C. - Lectures in Plasma Physics (Commission of European Communities, EUR 5127e)

(16) Lackner K. - Proc. of "Symposium on Computational Physics" Garching, 1977.

(17) Mercier C. - Nucl. Fusion 1, 47, 1960.

(18) Suydam B. - Proc. of 2nd United Nations Conf. on Peaceful Uses of Atomic Energy", Geneva, 1958, vol. 31, 157.

(19) Küppers G., Tasso H. - Z. fur Naturforschung 27a, 1,23, 1972.

(20) Taylor J.B. - IAEA Conf. on Plasma Physics and Controlled Nucl. Fusion, Innsbruck, 1978.

(21) Mercier C. - Ibidem, Innsbruck, 1978.

(22) Correa-Restrepo - Z. für Naturforschung 33a, 789, 1978.

(23) Küppers, Pfirsch, Tasso - "MHD Stability of Axisymmetric Plasmas". Proc. of IAEA Conf on Plasma Physics and Controlled Nucl. Fusion Research, Madison 1971, Paper CN-28/F-14 vol. 2 p.529.

(24) Kerner, Tasso - Kink Instabilities for Shaped Tokamaks Proc. IAEA Conf. Tokyo (1974). CN 33/A-13-1 vol. 1, Vienna 1975 p.475.

(25) Kerner - Nucl. Fusion 16, 643, 1976.

(26) Berger et al. - IAEA Conf. on Plasma Physics and Contr. Nucl. Fusion - CN 35/B-11-4, Berchtesgaden 1976.

(27) Johnson J.L. et al. - Ibidem IAEA - CN 35/B-11-2.

(28) Grimm et al. - Methods in Comput. Phys. Fusion, 1976.

(29) Garabedian et al. - Innsbruck IAEA Conf. 1978.

(30) Schlüter et al. - Ibidem.

(31) Shafranov - Review of Plasma Physics.

(32) Callen et al. - IAEA Conf. on Plasma Physics and Thermonuclear Fusion, Berchtesgaden 1976

(33) Dungey - Cosmic Electrodynamics, Cambridge Univ. Press. (1958).

(34) Furth, Killeen, Rosenbluth - Phys. of Fluids 6, 459, 1963.

(35) Coppi B. - Propagation and Instabilities, Stanford Univ. Press, 70, (1963).

(36) Barston E.M. - Phys. of Fluids 12, 2162, 1969.

(37) Tasso H. - Plasma Physics, 17, 1131, 1975.

(38) Tasso H. - Plasma Physics, 19, 177, 1977.

(39) Furth H. - Propagation and Instabilities, Stanford Univ. Press, 70, (1963).

(40) Jensen T.H., McClain F.W. - J. of Plasma Phys. 20, 61, 1978.

(41) Mercier C. - Euratom - CEA Rapp. Int. 74, 1960.

(42) Glasser, Furth, Rutherford - Phys. Rev. Lett. 38, 234, 1977.

(43) Karger F. et al. - IAEA Conf. on Plasma Phys. and Contr. Thermonuclear Fusion 1974, page 207 vol. 1.

(44) Waddell et al. - Theor. and Comput. Plasma Phys., Trieste 1977.

(45) Carrera et al. - Symposium on Disruptions (IAEA, Garching) 1979.

(46) Biskamp, Welter - Ibidem, 1979.

(47) Glasser, Green, Johnson - Phys. of Fluids 18, 875, 1975.

(48) Tasso H. - Theor. and Comput. Plasma Phys., Trieste 1977.

(49) Tasso H. - MPI-IPP Report 6/151, 1976.

(50) Bussac, Furth, Rosenbluth - IAEA Conf. on Plasma Phys. and Contr. Thermonuclear Fusion", at Innsbruck, 1978.

(51) Taylor J.B. - Phys. Rev. Lett. 33, 1139, 1974.

(52) Rebhan, Salat - Nucl. Fusion, 16, 805, 1976.

(53) Wesson J. - Nucl. Fusion, 18, 1,87, 1978.

(54) Caldas I.L. - Instabilidades Dissipativas em Magnetohydro-dinâmica - Tese de Doutoramento IFUSP 1979.

(55) Caldas I. L., Tasso H. - Plasma Phys. 20, 1299, 1978.

(56) Tasso H. - Z. Naturforschung 33a, 257 , 1978.