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QUANTIZATION OF A CLASSICAL REAL CONFIGURATION IN CP^2 MODEL

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A B S T R A C T

Starting from the O(3) instanton and meron solution of the NLS-model, which are also (topologically trivial) solutions of the CP^2 model, we calculate the effect of fluctuations in the CP^2 case, which has some interesting features.

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I- INTRODUCTION

The CP^{n-1} models discovered by Eichenherr[1] and studied by Luscher et al[2] has many interesting properties. Besides assymptotic freedom, the model presents instanton solutions for all n[2], and is 1/N expandable[2], which makes of it a good laboratory to mimicry the formidable Yang Mills problem. Recently, the instanton gas was calculated in this model[3] providing some insights in its structure. It remains to study the role of other solutions the equations of motion, which are no longer of instanton type. In particular, all the solutions of the O(n) non-linear σ model are also solutions of the CP^{n-1} model. In this paper we treat the most simple case of the first instanton solution of the NLS-model, which is a solution of zero topological charge of the CP^2 model, and show that the meron solution of the NLS model is a highly unstable one in the CP^2 case, and so should not be taken into account.

We find that the perturbation around such classical configurations produces negative eigenvalues, which are reminiscent of the instability, and has some interesting consequences in one of the cases, about which we speculate.

The paper is divided as follows: in section II we develop the perturbation around the classical solution, proving that one part of the partition function is the same as the one from NLS-model and there is a remaining part caracteristic of this model; in section III we calculate the new amplitude, showing that there is a problem associated with a negative eigenvalue of the quadratic form making aparent the instability of the solution (which has zero topological charge). In section III we give arguments to show that the effect of the negative eigenvalue is to produce a renormalization of the coupling constant in the n-point functions. In section V we calculate the quantum correction. In section VI we treat the (NLS-meron) solution, and show that it is a highly unstable one. Section VII is a brief discussion of the results.

II- PERTURBATION AROUND THE O(3) INSTANTON CLASSICAL CONFIGURATION

The CP^2 model has a lagrangian of the form

$$\mathcal{L} = \frac{1}{2f}\left(\overline{\partial_\mu z}\,\partial_\mu z + (\bar{z}\,\partial_\mu z)^2\right) \qquad \text{II.1}$$

with $\sum_{\alpha=1}^{3} \bar{z}_\alpha z_\alpha = \bar{z} z = 1$

If we put

$$z_\alpha = \left(1 - f|\eta|^2\right)^{1/2} q_\alpha + \eta_\alpha \sqrt{f} \qquad \text{II.2}$$

where q_α $(\alpha = 1, 2, 3)$ is the O(3) instanton solution[4] and η_α $(\alpha = 1, 2, 3)$ some small complex fluctuation, we can write II.1 as follows

$$\mathcal{L} = (1 - f|\eta|^2)\mathcal{L}_0 + \overline{\partial_\mu \eta}\,\partial_\mu \eta - 2\,|\bar{\eta}\,\partial_\mu q|^2$$

$$+ (\bar{\eta}\,\partial_\mu q)^2 + (\eta\,\partial_\mu q)^2 \qquad \text{II.3}$$

with $\mathcal{L}_0 = \dfrac{(\partial_\mu q)^2}{f} = 2\,\mathcal{L}_{0_{NL\sigma}}$

We used the equations of motion of q and discarded the terms in η of order higher than two.

Now, writing
$\eta = \eta_1 + i\,\eta_2$
the lagrangian splits into

$$\mathcal{L} = 2\,\mathcal{L}_{0_{NL\sigma}} + \mathcal{L}_1 + \mathcal{L}_2 \qquad \text{II.4a}$$

$$\mathcal{L}_1 = (\partial_\mu \eta_1)^2 - (\partial_\mu q)^2\,\eta_1^2 \qquad \text{II.4b}$$

$$\mathcal{L}_2 = (\partial_\mu \eta_2)^2 - (\partial_\mu q)^2\eta_2^2 - 4\,\partial_\mu q_\alpha\,\partial_\mu q_\beta\,\eta_{2\alpha}\,\eta_{2\beta} \qquad \text{II.4c}$$

Our aim is to calculate the ratio of the transition amplitude in the presence of the real configuration

$$T = \int \mathcal{D}[\eta]\,\delta(\eta\cdot q)\,e^{-S[\eta]} \qquad \text{II.5}$$

to the vacuum transition amplitude. With II.4, we have

$$T = T_1\,T_2\;e^{-8\pi/f}$$

$$T_1 = \int \mathcal{D}[\eta_1]\,\delta(\eta_1\cdot q)\,e^{-S_1[\eta_1]} \qquad \text{II.6a}$$

$$T_2 = \int \mathcal{D}[\eta_2]\,\delta(\eta_2\cdot q)\,e^{-S_2[\eta_2]} \qquad \text{II.6b}$$

$$S_{1(2)} = \int \mathcal{L}_{1(2)}\,[\eta_{1(2)}]\,d^2x \qquad \text{II.6c}$$

T_1 is the same amplitude already calculated by Jevicki[5] some time ago for the NLS-model. So we only need to treat the amplitude T_2.

III- ABOUT THE AMPLITUDE T_2

$S_2[\eta]$ is given by

$$S_2[\eta] = \frac{1}{2}\int d^2x \, (\partial_\mu q)^2 \sum \eta_\alpha \, m^2_{\alpha\beta} \, \eta_\beta \qquad \text{III.1}$$

where

$$m^2_{\alpha\beta} = \left[-4 \left(\partial_\mu q_\alpha\right)^{-2} \partial^2 - 2 \right] \delta_{\alpha\beta}$$

$$-4 \frac{\partial_\mu q_\alpha \, \partial_\mu q_\beta}{\left(\frac{1}{2}\partial_\mu q_\alpha\right)^2} \qquad \text{III.2}$$

We must now solve the eigenvalue problem associated with the above operator. Aiming at this we rewrite the parameters that appear in m^2 in a more convenient way. We use the following coordinates

$$\xi = \frac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2 + 1} \qquad \text{III.3a}$$

$$\alpha = \operatorname{arctg} \frac{x_2}{x_1} \qquad \text{III.3b}$$

This set corresponds to those used in ref.(4) with R=1 and provides a compactification of $\mathbb{R}^2$. Note also that in this system the eigenvalues turn out to be discrete.

The classical solution q now reads

$$q = \left(\xi \, ; \, \sqrt{1-\xi^2} \cos \alpha \, ; \, \sqrt{1-\xi^2} \operatorname{sen} \alpha \right) \qquad \text{III.4}$$

It turns out to be more convenient to rewrite our equations in terms of complex fields and write q as

$$q = \left(\xi \, ; \, \frac{1}{\sqrt{2}} \sqrt{1-\xi^2} \, e^{i\alpha} \, ; \, \frac{1}{\sqrt{2}} \sqrt{1-\xi^2} \, e^{-i\alpha} \right) \qquad \text{III.5}$$

In this coordinates

$$|\partial_\mu q|^2 = 2 \, (1-\xi)^2 \qquad \text{III.6a}$$

$$\partial^2 = (1-\xi)^2 \, L^2 \qquad \text{III.6b}$$

$$L^2 = \left(1-\xi^2\right) \frac{\partial^2}{\partial \xi^2} - 2\xi \frac{\partial}{\partial \xi} + \frac{1}{1-\xi^2} \frac{\partial^2}{\partial \alpha^2} \qquad \text{III.6c}$$

we define also

$$e_{\pm} = e^{\mp i\alpha}\left(\partial_1 q \pm i\,\partial_2 q\right) =$$

$$= \left(\xi - 1\right)\left(-\sqrt{1-\xi^2}\,;\ \frac{1}{\sqrt{2}}\left(\xi \pm 1\right)e^{i\alpha}\,;\ \frac{1}{\sqrt{2}}\left(\xi \mp 1\right)e^{-i\alpha}\right) \qquad \text{III.7}$$

Our ansatz to solve the eigenvalue equation is

$$\eta_{\alpha\pm} = e_{\pm\alpha}\,\Sigma_{\pm} \qquad \text{III.8}$$

This ansatz generates terms proportional to q_α in this eigenvalue equation and we have no solution. However in virtue of the constraint n.q=0 in II.6 the eigenvalue equation for the operator m^2 can be generalized to

$$m^2_{\alpha\beta}\,\eta_\beta = \epsilon\,\eta_\alpha + c_n\,q_\alpha \qquad \text{III.9}$$

We do not need to case about the q_α term, so that we can go straithforwardly to the equation obeyed by $\Sigma_\pm$ so that using III.9 and III.8 we get

$$L^2\,\Sigma_\pm + 6\,\Sigma_\pm + 4\left(\xi - 1\right)\left(1-\xi^2\right)\cdot\frac{\partial\Sigma_\pm}{\partial\xi} \pm \partial_i\,\xi\,\frac{1-\xi}{1+\xi}\,\frac{\partial\Sigma_\pm}{\partial\alpha}$$

$$+\frac{\xi-1}{\xi+1}\left[-4\xi^2 - 4\xi + 2\right]\Sigma_\pm = \mathcal{E}\,\Sigma_\pm \qquad \text{III.10}$$

This can be shown to be solved by

$$\Sigma_{\pm} = \sqrt{\frac{1+\xi}{1-\xi}}\; h_n^{\pm} \qquad \text{III.11}$$

$$h_n^{\pm} = \left(1-\xi^2\right)^{-1/2} e^{im\alpha}\, h_{nm}^{\pm}(\xi) \qquad \text{III.12}$$

where

$$\left(L^2 - \frac{1 \pm 2m\xi}{1-\xi^2} + \varepsilon_n + 6\right) h_{nm}^{\pm} = 0 \qquad \text{III.13}$$

so that $h_{jm}^{\pm}(\xi) = P_{m,\mp 1}^{j}(\xi)$ III.14

and $\varepsilon_j = j(j+1) - 6 \quad , \quad j = 1, 2, \dots$ III.15

We see immediatly a negative eigenvalue, for j=1.

IV- THE NEGATIVE EIGENVALUE

If we continue the expansion of the lagrangian as a function of η to fourth order, we find:

$$\mathcal{L}^{3rd,4th} = -8\left(\eta_2 \partial_\mu q\right)\left(\eta_1 \partial_\mu \eta_2\right) - 3\left(\eta_1^2+\eta_2^2\right)\left(\partial_\mu q \cdot \partial_\mu \eta_1\right)$$

$$+\left(\eta_1 \partial_\mu \eta_1\right)^2 + \left(\eta_2 \partial_\mu \eta_2\right)^2 + 2\left(\eta_1 \partial_\mu \eta_1\right)\left(\eta_2 \partial_\mu \eta_2\right)$$

$$+\left(\eta_2 \partial_\mu \eta_1\right)^2 + \left(\eta_1 \partial_\mu \eta_2\right)^2 - 2\left(\eta_2 \partial_\mu \eta_1\right)\left(\eta_1 \partial_\mu \eta_2\right) \qquad \text{IV.1}$$

As the η_1 part of the partition function is convergent, we do not need to worry about terms in which η_1 appears. We see that the only term in which η_2 enters alone is $\left(\eta_2 \partial_\mu \eta_2\right)^2$ which is obviously a favourable term in the η_2 functional integration. The contribution of the negative mode will be some thing like

$$\int_{-\infty}^{\infty} dy\; e^{\lambda y^2 - g y^4} = \frac{\lambda}{g}\int_{-\infty}^{\infty} dy\; e^{\frac{\lambda^2}{g}\left(y^2-y^4\right)} \qquad \text{IV.2}$$

where $g = \mathcal{O}(f)$.

We must now face with this kind of integral, and see the contributions in the limit $g\to 0$, namely vanishing quartic term, to the physical objects of the theory.

Now, the integral in IV.2 can be calculated; defining

$$Y = \int dy\; e^{\nu\left(y^2-y^4\right)} \qquad \text{IV.3a}$$

we have, by direct calculation that:

$$Y = \frac{2e^{\nu/4}}{\sqrt{2\nu}} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{\sqrt{1 + 2y/\sqrt{\nu}}} - \frac{\sqrt{2}}{\nu} + \theta\left(\frac{1}{\sqrt{\nu^3}}\right) +$$

$$+ \theta\left(e^{-\nu/4} \sqrt{\nu}^{-1}\right) \qquad \text{IV.3b}$$

Using the saddle point method, we see that the first term in IV.3 is reproduced by the two non-trivial saddle points in $y = \pm \frac{1}{\sqrt{2}}$ The saddle point at y=0, when integrated by the imaginary axis gives a contribution $\sqrt{\frac{\pi}{\nu}}$ which is not of the desired form, in such a way that we suspect that the physics is not give by the unstable point, so far as we do not consider effects related to the instabilities of the model. In an n-point function of the field, the typical contribution will be of the form:

$$\frac{\left(\sqrt{\frac{\lambda}{g}}\right)^{2n+1} \int y^{2n} e^{\frac{\lambda^2}{g}(y^2-y^4)} dy}{\sqrt{\frac{\lambda}{g}} \int e^{\frac{\lambda^2}{g}(y^2-y^4)} dy} = \left(\sqrt{\frac{\lambda}{g}}\right)^{2n} \frac{1}{2^n} \left(1 + \theta\left(\sqrt{g/\lambda^2}\right)\right) \qquad \text{IV.4}$$

in such a way that it implements a renormalization of the coupling constant f equal to $\sqrt{\frac{\lambda}{2g}}$.

This mode is responsible for a renormalization, which is non-analitical in the coupling constant, because g~f; such feature is caracteristic of non-renormalizable interactions(6), causing some surprise, because this theory is renormalizable, so that we simply interpret it as a consequence of our inability to treat this unstable solution.

V- THE QUANTUM CORRECTION

Taking into account the small oscillation problem of section

III, and the discussion of section IV we now calculate the ratio of the transition amplitude

$$\int \mathcal{D}[\eta]\, \delta(\eta \cdot q)\, \exp - \frac{1}{2}(\eta, m^2 \eta) \qquad \text{V.1}$$

with

$$(\eta, m^2\eta) = \int d\mu(x)\, \eta \cdot m^2 \eta$$

and $d\mu(x) \equiv \frac{1}{2}\left(\partial_\mu q\right)^2 d^2x = \left(\frac{2}{1+r^2}\right)^2 d^2x$

to the ordinary vacuum transition amplitude

$$\int \mathcal{D}[\eta]\, \delta(\nu \cdot \eta)\, \exp - (\eta, m_0^2 \eta) \qquad \text{V.2}$$

for $\nu=(0,0,1)$ which represents the classical vacuum and

$$m_0^2 = -\left(\frac{1}{2}\partial_\mu q\right)^{-1} \partial^2 \qquad \text{V.3}$$

We obviously obtain a product of the ratio for η_1 and η_2, as shown in section II (see eq. II.6). The amplitude T_1 is given by[5]

$$T_1 = \pi f^{-2}\left(\frac{16}{3}\right)^2 \int \frac{d^2z_0\, d\lambda}{\lambda^3} \exp\left\{ \frac{-4\pi}{f(\mu_0^2)} + \ln(\mu_0^2 \lambda) - \right.$$

$$- 6\left(1 - \frac{1}{2}\sum_{s=2}^{3} \ln s\right)\Bigg\} \qquad V.4$$

For the zero modes problem we refer the reader to reference 5 section III. We go straigthforwardly to the evaluation of the contribution of the non-zero modes, which from Gaussian integration are:

$$\pi = \left(\prod_{\substack{n \\ \varepsilon_n > 0}} \varepsilon_n\right)^{-1/2} \qquad V.5$$

so we must calculate

$$\frac{\pi}{\pi_0} = -\frac{1}{2}\left[\sum_{j=3}^{\infty} \delta_j \ln \varepsilon_j - \sum_{j=1}^{\infty} \delta_j \ln \varepsilon_j^{(0)}\right] \qquad V.6$$

where
$$\varepsilon_j = j(j+1) - 6$$
$$\varepsilon_j^{(0)} = j(j+1)$$
$$\delta_j = 2(2j+1)$$

The expression V.6 has ultraviolet divergences which will be cancelled by a renormalization procedure. Therefore we regularize employing Pauli-Villars regulators with alternating metric[6]

$$e_0 = 1\ ;\quad M_0 = 0\ ,\quad \sum_{i=0}^{R} e_i = 0 \quad \sum_{i=0}^{R} e_i M_i = 0 \qquad \text{V.7}$$

$$\sum_{i=1}^{R} e_i \ln M_i = - \ln M \qquad \text{V.8}$$

so that we get

$$\ln \left(\frac{\pi}{\pi_0} \right)_{reg} = -2 \sum_{i=0}^{R} e_i \left[A^{M_i}\left(\tfrac{5}{2}\right) - A^{M_i}\left(\tfrac{1}{2}\right) \right] \qquad \text{V.9}$$

with

$$A^{M_i}(a) = \sum_{s=1+a}^{\Lambda} s \ln \left(s^2 - a^2 + M_i^2 \right) \qquad \text{V.10}$$

It is easily seen that

$$A^{M_i}(a) = -\frac{1}{2} \left(a^2 \ln \Lambda + a \ln M^2 \right) + \theta \left(\frac{1}{\Lambda}, \frac{1}{M^2} \right) \qquad \text{V.11}$$

For $A^0(a)$ we have

$$A^0(a) = \sum_{s=1+2a}^{\Lambda+a} (s-a) \ln a + \sum_{s=1}^{\Lambda-a} (s+a) \ln s = \sum_{s=1}^{2a} (a-s) \ln s +$$

$$+ \sum_{s=1}^{\Lambda-a/2} 2 s \ln s + \sum_{s=\Lambda+\frac{1}{2}}^{\Lambda+a} (s-a) \ln \Lambda - \sum_{s=\Lambda+1-a}^{\Lambda-1/2} (s+a) \ln s$$

$$\text{V.12}$$

Using the above formulas into V.9 we have

$$\ln\left(\frac{\pi}{\pi_0}\right)_{reg} = -2\left\{\left(A^{M}\left(\tfrac{5}{2}\right)-A^{M}\left(\tfrac{1}{2}\right)\right)-\left(A^{(0)}\left(\tfrac{5}{2}\right)-A^{(0)}\left(\tfrac{1}{2}\right)\right)\right\} =$$

$$= -2\left\{6 + 2\ln M + \sum_{s=2}^{5}\left(\frac{5}{2}-s\right)\ln s\right\} \qquad \text{V.13}$$

We made the above calculation using a regulator with space dependent mass given by

$$\mu^2(x) = \frac{4M^2}{(1+r^2)^2} \qquad \text{V.14}$$

and there is an additional contribution coming from a finite renormalization effect in relation to a standard fixed mass regulator μ_0, given by:

$$-\Delta A = \int d^2x\,(\partial_\mu \varphi)^2\,\frac{1}{2\pi}\ln\frac{\mu(x)}{\mu_0} = 4\ln\frac{2M}{\mu_0} - 4 \qquad \text{V.15}$$

and also, instead of the zero modes contribution, we have the correponding modes for the regulator fields, which contributes with a factor μ_0^4.

We have then

$$\ln\left(\frac{\pi}{\pi_0}\right)_{reg} = -2\left\{8 - \ln 2 + 2\ln\mu_0 + \sum_{s=2}^{5}\left(\frac{5}{2}-s\right)\ln s\right\} \qquad \text{V.16}$$

Now for the total contribution, including zero modes factors, we have the expression

$$\pi f^{-2}\left(\frac{16}{3}\right)^2 \int d^2 z_0 \, \frac{d\lambda}{\lambda^3} \, exp \left\{ 2 \ln \mu_0^2 \lambda - 2 (8 - \ln 2 \right.$$

$$\left. + \sum_{s=2}^{5} \left(\frac{s}{2} - s \right) \ln s) \right\} \qquad \text{V.17}$$

VI- PERTURBATION AROUND THE O(3) MERON CLASSICAL CONFIGURATION

Now we turn to the meron solution

$$q = \left(\frac{x^1}{\sqrt{x^2}} \; ; \; \frac{x^2}{\sqrt{x^2}} \; ; \; 0 \right) \qquad \text{VI.1}$$

The eigenvalue problem can be put again into the form:

$$\partial^2 \eta_\alpha + (\partial_\mu q)^2 \eta_\alpha + 2 \, \partial_\mu q_\alpha \, \partial_\mu q_\beta \, \eta_\beta = \mp \varepsilon \, \eta_\alpha \qquad \text{VI.2}$$

The elements in VI.2 can be immediatly written down as :

$$q = (\cos \alpha \; ; \; sen \, \alpha \; ; \; 0)$$

$$(\partial_\mu q)^2 = \frac{1}{r^2}$$

$$\partial_1 q = \frac{\sin\alpha}{r}\left(\sin\alpha\,,\,-\cos\alpha\,;\,0\right)$$

$$\partial_2 q = \frac{\cos\alpha}{r}\left(-\sin\alpha\,,\,\cos\alpha\,,\,0\right)$$

$$\partial^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2} \qquad \text{VI.3}$$

and obviously: $x_1 = r\cos\alpha$

$x_2 = r\,\text{sen}\,\alpha$

With the ansatz

$$\eta_\alpha = \left(\sin\alpha\,,\,-\cos\alpha\,,\,0\right)\Sigma \qquad \text{VI.4}$$

we can trivially derive the equation:

$$\partial^2\Sigma + \frac{2}{r^2}\Sigma = -\mathcal{E}\,\Sigma \qquad \text{VI.5}$$

or, for $\chi = \sqrt{r}\; e^{im\alpha}\,\Sigma$

$$\frac{\partial^2\chi}{\partial r^2} + \left(\frac{13}{4} - m^2\right)\frac{\chi}{r^2} = -\mathcal{E}\,\chi \qquad \text{VI.6}$$

This system has already been discussed by Landau[7], which prooved that it has no bounded value for the energy ε (i.e. $\varepsilon\to-\infty$) in such a way that the solution (VI.1) is highly unstable, and there is no meaning in perturbing around it.

VII- DISCUSSION

The appearance of unstable modes (negative eigenvalues) must be a general fact of solutions with zero topological charge. However, some of this solutions, having a small number of unstable modes can contribute to the physics. A more detailed discussion of this problem is unfortunately beyond the scope of this article. We find it worth to mention that unstable states (such as the positronium) must contribute to the thermodynamics of a gas of photons.

In view of the relations between CP^{n-1} models in 2 dimensions and Yang-Mills theory in 4 dimensions we suspect that similar events occur when dealing with classical solutions of the non-abelian gauge theories having trivial topology.

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