KAPITZA PENDULUM: SIMPLE ANALYSIS.

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Abstract. Kapitza pendulum is a rigid pendulum in a gravitational field whose pivot point vibrates, up and down, in a vertical direction. Many papers have been published on this subject. We only present a rough description of this pendulum motion to undergraduate students of Physics. Our main intention is to suggest a simple explanation for the phenomenon of dynamic stabilization of the inverted pendulum whose pivot is forced to oscillate along *the vertical.*
key words: *inverted pendulum; dynamic stabilization.*

(I)Introduction.

In **Figure (I.1)** is shown a *Kapitza pendulum*[1] a rigid planar pendulum of length ℓ with a point mass M, where all mass of the pendulum is assumed to be concentrated, which the pivot point vibrates in a vertical direction. A motor rotates a crank with a high speed moving the lever arm, up and down, with the rigid pendulum attached to a pivot.[1,2]

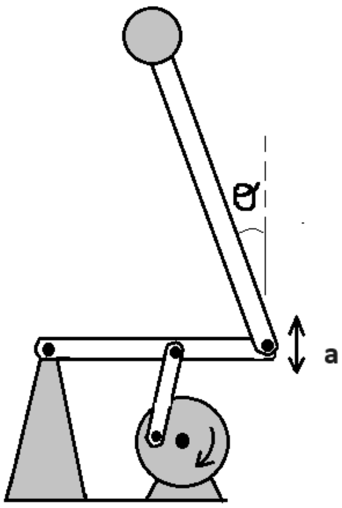

Figure (I.1). Drawing of a Kapitza pendulum; a motor rotates a crank at high speed, the crank vibrates a lever arm up and down, which the pendulum is attached to with a pivot.[1]

This kind of pendulum was first described by A. Stephenson in 1908[3] and a vast list of papers have been published on the subject.[2,3] Up to 1950 there was no faithful explanation for the counterintuitive and unusual phenomenon that was observed for this pendulum. Pyotr Kapitza in 1951[1] was the first to present a satisfactory explanation. As in our "Laboratorio de Demonstrações" (*Lab. Demo.*) the experiments are performed to undergraduate students of Physics and we present a simple mathematical approach to estimate the Kapitza oscillations.(see also Tiago dissertation[4]) In **Section 1** is shown the equation of motion for simple pendulum oscillations.[5,6] In **Section 2** is deduced the equation of motion for the Kapitza pendulum. A simple physical explanation is suggested in this paper for the phenomenon of dynamic stabilization of the inverted pendulum whose pivot is forced to oscillate along *the vertical. The* commonly known criterion of stability is obtained on the basis of the developed approach, and this criterion is verified by numerical simulations .

(1)Simple Pendulum.

As well known from basic Physics course[5,6]**(Fig.(1.1))** the

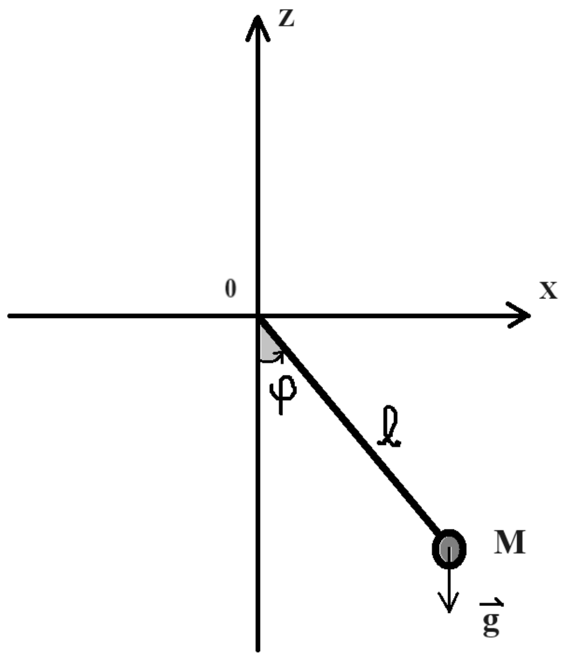

Figure (1.1). Simple pendulum in a gravitational field g.

the angular equation of motion for $\varphi(t)$, with lentgh ℓ and mass M, in a uniform gravitational field **g,** is given by

$$d^2\varphi/dt^2 = -\,(Mg\ell/I)\sin\varphi = -\,(g/\ell)\sin(\varphi) \qquad \textbf{(1.1)},$$

where $I = M\ell^2$ is the moment of inertia of the pendulum. The weight torque $\boldsymbol{\tau} = -\,M\mathbf{g} \times \mathbf{f}$ forces the pendulum to swing harmonically around $\varphi = 0$.[5,6] Note that when there is no mechanical energy dissipation, that would given by $-2\gamma(d\gamma/dt)$, the pendulum would swing eternally.

(2)Kapiza Pendulum.

In this case the origin **O** where is fixed rigid pendulum (**Figure (2.1)**) moves with an acceleration $\mathbf{a} = a(t)\,\mathbf{k}$ along the z-axis.

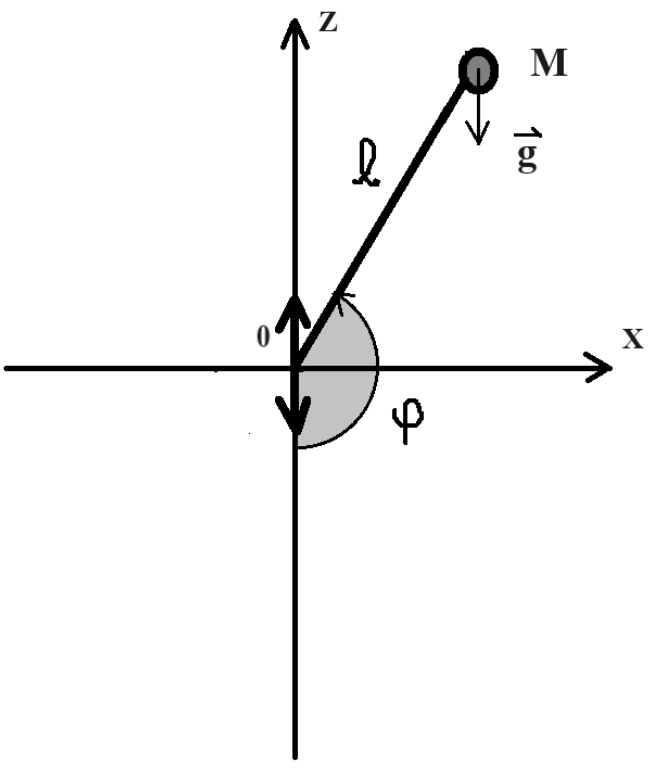

Figure (2.1)

In this way, to describe the pendulum motion it is necessary to take into account a *non-inertial frame*[7] of reference with origin at the point O. In this new frame, pendulum equations of motion will also depend of the "pseudo" force of inertia $\mathbf{F}_{inertia} = -\,M\mathbf{a}$. If the pivot is forced to execute an harmonic oscillation with frequency ω (*"pivot frequency"*) along the vertical axis z, that is, $z(t) = A\cos(\omega t)$ we have

$$\mathbf{F}_{inertia} = - M(d^2z/dt^2)\,\mathbf{k} = MA\omega^2\cos(\omega t)\,\mathbf{k} \qquad \mathbf{(2.1).}$$

$\mathbf{F}_{inertia}$ is responsible by a torque which contrary to the torque created by weight force - Mg**k**. So, it will be become submitted to **two torques**, one due to the gravitational force - Mg**k** and another due to the non-inertial force $\mathbf{F}_{inertia}$**.** So, taking into account these torques we have, instead of **Eq.(1.1)**, the following equation for φ(t),[2]

$$d^2\varphi/dt^2 = - 2\gamma(d\varphi/dt) - (g/\ell)\sin\varphi + (A\omega^2/\ell)\cos(\omega t)\sin\varphi \qquad \mathbf{(2.2).}$$

Where now we are also introducing the effect of the dissipative force - 2γ(dφ/dt) due the interaction of the pendulum with the air.[2] This damping effect can be estimated by the factor $Q = \omega_o/2\gamma$.

When the "inertial" amplitude $\Gamma = A\omega^2/\ell$ is much smaller than the frequency $\omega_o = (g/\ell)^{1/2}$, that is, when $\omega_o >> \Gamma$ we would have the non-dissipative pendular motion.[5,6]

It is **important to note** that, in the general case, the equation of motion φ(t) can be obtained *only* by numerical integration of **Eq.(2.2).**

(2.a) Vertical Dynamic Stabilization.

When $\Gamma >> \omega_o$, integrating numerically **Eq.(2.2)** and with convenient initial conditions we get a "*vertical dynamic stabilization"* of the pendulum shown in **Fig.(2a).**[2] In this particular case it oscillates **vertically** between 152° and 208°.

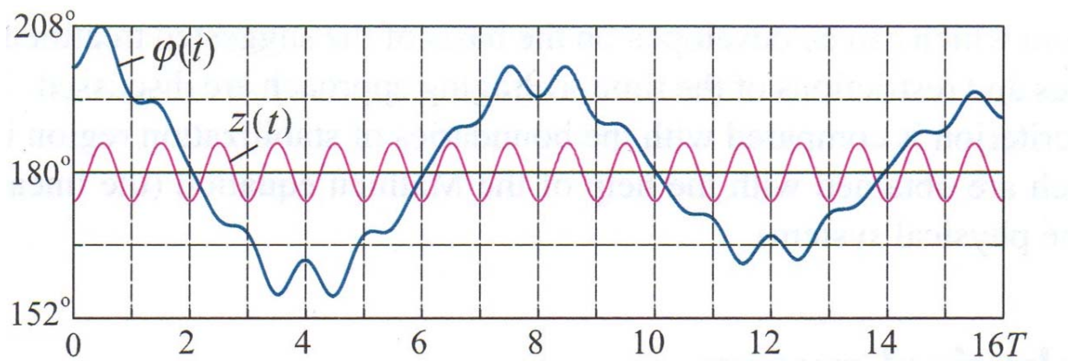

Figure (2a) Vertical dynamic stabilization. φ(t) obtained by numerical integration of **Eq.(2.2)** taking the initial deflection φ(o) = 200°, (dφ/dt)(o) = 0, $\omega = 16\,\omega_o$, $A = 0.20\ell$ and *quality factor Q* = 5.0 (dissipative effect); z(t) = A cos(ωt); T = 2π/ω.

From **Fig.(2a)** we verify that the pendulum oscillates is in a inverted position, that is, when 180° + Δ > φ > 180° - Δ , where Δ ~ 30° . The *inertial* torque, given by the second term of **Eq.(2.1),** tends to decrease the *amplitude of angular oscillation* Δ .The *weight* torque tends to increase Δ. The oscillation occur in the interval 208º - 152º. The mean value of the *inertial* torque with respect to the z-axis is not zero. The value of the *gravitational* torque is zero.

When the rod is slightly deviated from the vertical, it executes relatively slow oscillations about the vertical line on the background of rapid oscillations of the pivot point **O**. According to Kapitza: "*our eyes cannot follow the fast small movements caused by vibrations of the pivot, so the behavior of the pendulum in the inverted position seems perplexing and astonishing...*"

In next **Figure (2b),** when the pendulum is oscillating between 260º and 100º, are shown the oscillations of the pendulum around the circle of radius ℓ, perpendicularly to the plane (z,x).[2]

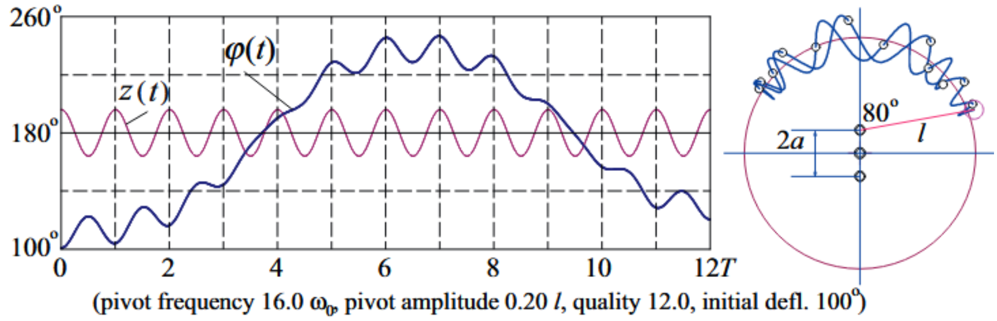

Figure (2b). Pendulum oscillations in the circle with radius ℓ in plane (z,x).

(2.b) Bottom Oscillating Position.

Depending on adequate initial conditions we can have also oscillations at the bottom z-axis, that is, with the pendulum oscillating hanging down below the pivot O. For instance, with a numerical integration of the exact **Eq.(2.2)** are shown in **Figure (2c)** bottom oscillations between 23º and -23º around the z-axis.[2]

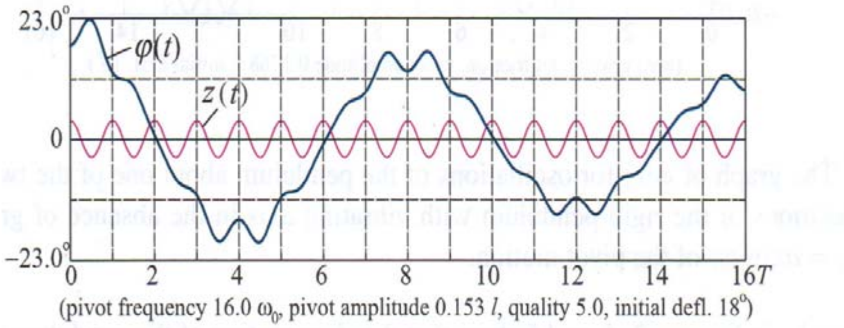

Figure (2c). Vertical bottom oscillations between 23º and -23º

Any tiny deviation from the vertical increases in amplitude with time and **chaotic effects** can be realized.[1]

(3) Experimental set up.

In our laboratory we set up a very simple kapitza pendulum. It consists of two pieces of plastic straws connected by an axis made of thin cooper wire. The pendulum has a length of about 8 cm and can oscillate freely. The other straw is attached to the center of a speaker, using adhesive tapes. Furthermore, this straw passes through a hole in an acrylic plate, so that the straw oscillates preferentially in the vertical direction.

Figure (3a) shows the assembly with the speaker.

Figure (3a). The speaker and the pendulum.

The set is connected to an audio generator with amplifier. When a sinusoidal signal above 50 Hz and speaker oscillation amplitude of approximately 5 mm, the pendulum balances with the center of mass above the support axis, as shown in the **Figure (3b).**

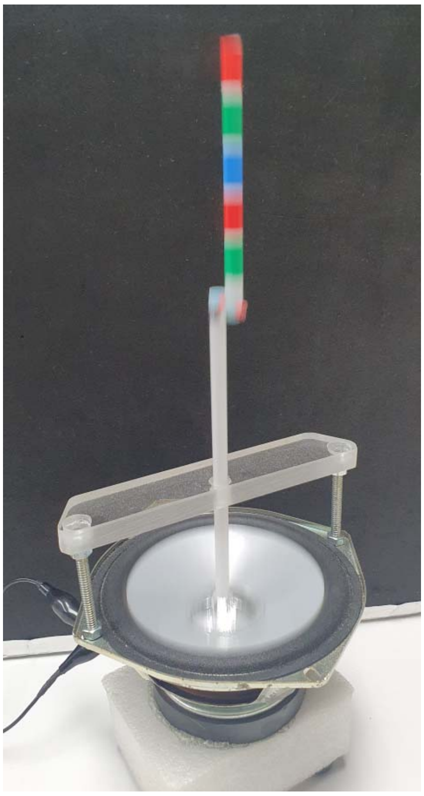

Figure (3b). Pendulum balancing vertically with the center of mass above the support axis.

***Acknowledgements**. The author thanks the librarian Maria de Fatima A. Souza and the administrative technician Tiago B. Alonso for their invaluable assistances in the publication of this paper.*

REFERENCES.

[1]https://en.wikipedia.org/wiki/Kapitza%27s_pendulum

[2]http://butikov.faculty.ifmo.ru/InvPendulumCNS.pdf

[3]A.Stephenson. Philosophical Magazine 6.15(86):233-236 (1908).

[4]Tiago P. dos Santos. https://www.if.ufrj.br/~pef/producao_academica/dissertacoes/2023_Tiago_Santos/dissertacao_Tiago_Santos.pdf (IFUFRJ(2023)).

[5]Física 1 - F. W. Sears and M. W. Zemansky. LTD.Vol.1(1975).

[6]P. A. Tipler. Física 1. Guanabara Dois (1978).

[7] https://www.google.com/search?client=firefox-b-e&q=non-inertial+frame+of+reference