Planetary Motion in Classical Mechanics and in General Relativity; comments on Restricted 3-body Problem in General Relativity and on Chaos and Stability in Solar System.

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Abstract. This paper was written to graduate and postgraduate students of Physics. In **Sections 1** and **2** is seen the Planetary Motion, that is, Earth and Moon, for instance, in **Classical Mechanics** and in **Section 3** in **General Relativity**. In **Section 4** is briefly analyzed the **Restricted 3-body Problem** in the relativistic and non Relativistic approaches**.** Finally, are done brief comments on chaos and stability of the Solar System.

***Key words**: planetary motion: classical mechanics; general relativity; chaos in solar system.*

(1)Motion in a Central Field in Classical Mechanics.

First are presented main aspects of a planet motion submitted to an attractive central field following Landau & Lifchitz.[1]

When a body with mass m moves in a central attractive potential field U(r) it will be submitted to a force **F**(r) = - [dU(r)/dr]$\mathbf{r_o}$. Its angular momentum **L = r x p** is conserved, where **p** = m**v**. As the vectors **L** and **r** are perpendiculars, during all time, its trajectory **r(t)** remains in the plane (**r**,**p**), perpendicular to **L**.

In polar coordinates r and θ the Lagrange function Ł(r,φ) is written[1]

$$Ł = (m/2)[r^{*2} + r^2\theta^{*2}] - U(r) \qquad \textbf{(1.1),}$$

where r*= dr/dt and θ* = dθ/dt. As this function does not depend explicitly of θ, it is a *cyclic* coordinate, that is,

$$d\{\partial Ł/\partial\theta^*\}/dt = \partial Ł \partial\theta = 0 \qquad \textbf{(1.2),}$$

showing that the generalized impulse $p_\theta = \partial Ł/\partial\theta$ = constant. That is, the *angular momentum* L_θ :

$$L_\theta = p_\theta = mr^2\theta^* = mr^2 (d\theta/dt) = \text{constant.} \qquad \textbf{(1.3).}$$

Since the motion is always in a plane perpendicular to **L** we can write

$$L_z = mr^2(d\theta/dt) = L = \text{constant} \qquad \textbf{(1.4).}$$

and its energy E = constant would be given by

$$E = (m/2)\{r^{*2} + r^2\theta^{*2}) + U(r) = mr^{*2}/2 + \mathbf{L}^2\theta^{*2}/(2mr^2) + U(r) \qquad \textbf{(1.5).}$$

From **Eq.(1.5)** we get, since $r^* = dr/dt$ and $md\theta = (L/r^2)dt$,

$$t = \int dr/\{(2/m)[E-U(r)] - (L^2/m^2r^2)\}^{1/2} + \text{constant} \qquad \textbf{(1.6),}$$

and

$$\theta = \int (Ldr/r^2)/\{(2/m)[E-U(r)] - (L^2/m^2r^2)\}^{1/2} + \text{constant} \qquad \textbf{(1.7).}$$

Equations **(1.6)** and **(1.7)** give the general solution of the problem. The first one **(1.6)** gives **r = r(t)** and the second **(1.7)**, gives $\theta = \theta(r)$.

It is important to note that from **Eq.(1.4)** that $d\theta/dt$ never change of signal: or it is *clockwise* or *counter clockwise* In this way[1] the body motion can be finite or infinite. If it comes from infinite it goes to infinite (as will be analyzed latter). It can also be limited, moving between r_{min} and r_{max}. Its trajectory will be inside a ring limited by circles with r_{min} and r_{max} (**Fig. 1**).

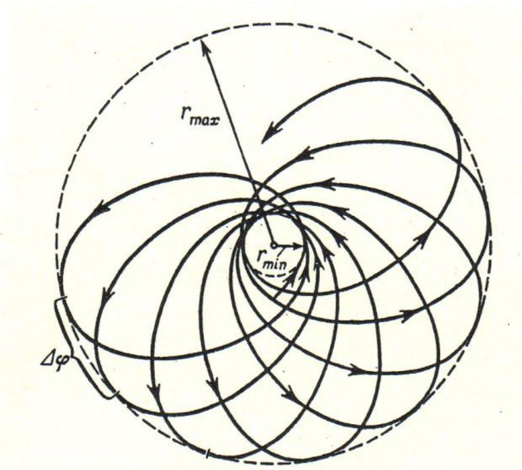


Figure 1.Body trajectory between points r_{min} and r_{max} and angle $\Delta\varphi$.[1]

At r_{min} and r_{max} the radial velocity $dr/dt = r^* = 0$ but the angular velocity is always $\theta^* = d\theta/dt \neq 0$, because $L = mr^2(d\theta/dt)$ is constant.

When r goes from r_{max} and $_{rmim}$ and, after again to r_{max}, the vector radius is displaced by an angle $\Delta\varphi \equiv \Delta\theta$ = given by (**Figure 1**),

$$\Delta\theta = 2\int (Mdr/r^2) / \{(2/m)[E-U(r)] - M^2/r^2\}^{1/2} \qquad \textbf{(1.8)}$$

To have a **closed trajectory** it is **necessary and sufficient** that $\Delta\theta = 2\pi(m/n)$ where *m* and *n* are **integer numbers.** This means that after **n** repetitions of this period of time the vector radius, been performed **m** complete turns, will be back to its initial value. In this way the trajectory becomes **closed**. There are only two kinds of central fields that obey this condition: when **U(r) ~1/r** and **U(r) ~ r^2.** In other cases the trajectories of finite movements are **not closed.** They pass infinite times by r_{min} and r_{max} (**Figure 1**) and after an infinite time they fill completely the ring area between these distances.

As a last comment, we must note[1] that a particle can "fall" to the center when $U(r) \sim - 1/r^{\beta}$ only if $\beta > 2$.

(2)Planetary Motion: Kepler Problem.

In what follows will be analyzed a planetary system (for instance, **Sun and planets**). In this case the central field is $U(r) = - \alpha/r$, where α = constant = GMm, where M = solar mass and m = planet mass This analysis will be done following Landau & Lifchitz[1], Goldstein[2] and Symon.[3]

According to Goldstein,[2] taking into account that energy E = constant (**Eq.1.5)**, that the angular momentum $L = m r^2\theta^* = m r^2(d\theta/dt)$ = constant and defining u = 1/r we verify that

$$d^2u/d\theta^2 + u = m\alpha/L \qquad \textbf{(2.0)}$$

Putting $y(t) = 1/r - m\alpha/L^2$, from **Eq.(1.9)** we get

$$d^2y/dt^2 + y = 0 \qquad \textbf{(2.1),}$$

which has the immediate solution

$$y(t) = b \cos(\theta - \theta_o) \qquad \textbf{(2.2)}$$

where b and θ_o are constants of integration, that is,

$$1/r = (m\alpha/L^2)\{1 + \varepsilon \cos(\theta - \theta_o) \}\textbf{...................} \qquad \textbf{(2.3),}$$

where $\varepsilon = b[L^2/m\alpha]$, where b is a constant to be determined as a function of E = T - U = constant and $L = m r^2(d\theta/dt)$ = angular momentum = constant.

In this way, one we can show that[2]

$$\theta = \theta_o - \int du/\{2mE/L^2 - 2\alpha mu/L^2 - u^2\}, \qquad \textbf{(2.4).}$$

Integrating **Eq.(2.4),**[2]

$$\theta - \theta_o = - \text{arc cos}\{(uL^2/m\alpha -1)/(1+ 2EL^2/m\alpha^2)^{1/2}\} \qquad \textbf{(2.5).}$$

Thus, as u =1/r we get

$$1/r = (m\alpha/L^2)\{1 + (1+ 2EL^2/m\alpha^2)^{1/2} \cos(\theta - \theta_o) \} \qquad \textbf{(2.6),}$$

where the constant θ_o can now be identified as one of the turning angles of the **orbit (**see **Fig.1).**

(1.b)General Equation of a Conic.

The general equation of a conic with one focus at the origin is [2]

$$1/r = C \{1 + \varepsilon \cos(\theta - \theta_o)\} \qquad \textbf{(2.7),}$$

where C is a constant and ε is the eccentricity of the conic section. By comparison with **Eq.(2.6)** we verify that

$$C = m\alpha/L^2 \quad \text{and} \quad \varepsilon = (1+ 2EL^2/m\alpha^2)^{1/2} \qquad \textbf{(2.8).}$$

The orbit nature depends on E and ε, according to the following scheme,

$\varepsilon > 1$, $E > 0$ → hyperbola

$\varepsilon = 1$, $E = 0$ → parabola

$\varepsilon < 1$, $E < 0$ → ellipse

$\varepsilon = 0$, $E = -m\alpha^2/2L^2$ → circle,

Figures and details of these orbits are shown, for instance, by Landau[1], Goldstein[2] and Symon.[3]

Hyperbola.

In this case **E > 0** and the motion is infinity. The body comes from infinity and goes to infinity. It contours the center of the potential which is the *focus* of the trajectory.

Parabola.

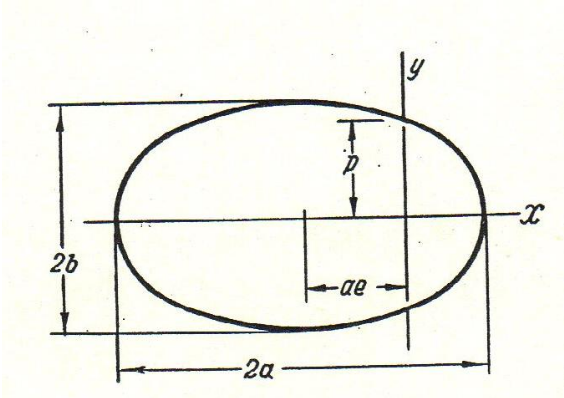
In this case **E = 0** and the motion is also infinity. The body comes from infinity and goes to infinite. It contours the center of the potential which is in the *focus* of the parabola.

Ellipse.

In the case of a *planet moving around the Sun* the energy E is given by $E = T\text{-}U(r) = mr^2(d\theta/dt)^2 - \alpha/r < 0$.[1] That is, it is in a *bound state* and its energy passes by a minimum value $E_{min} = -\alpha^2 m/2L^2$.[1] In these conditions its trajectory is an **ellipse** and **Eq.(2.7)** can be written as[1]

$$p/r = 1 + \varepsilon \cos(\theta) \qquad \textbf{(2.9),}$$

where $p = L^2/m\alpha$ and $\varepsilon = \{1+2EL^2/m\alpha^2\}^{1/2}$. It corresponds to an ellipse with focus at the origin of coordinates (where is the Sun) and $\theta_o = 0$ **(Fig. (2.2)).**



Figure(2). Elliptic orbit with the Sun at the origin of coordinates(x,y).

The large **a** and small **b** axis of the ellipse are given, respectively, by $a = p/(1-\varepsilon^2) = \alpha/(2|E|)$ and $b = p/(1-\varepsilon^2)^{1/2} = L/(2m|E|)^{1/2}$.The ellipse. area is given by $S = \pi ab$ and the period by $T = 2\pi\alpha^{3/2}(m/\alpha)^{1/2}$.

Elliptic orbits give good descriptions of the planetary motions in Solar System. The elliptical orbits have the Sun at one *focus*. However, it is observed that as the planets describe their orbits, their major axes slowly rotate about the Sun. There is a shifting of the line from the Sun to the perihelion through an angle $\Delta\theta$ during each orbit. This shifting is referred to as the *precession of the perihelion*. To estimate these *orbit precessions* it is necessary to take into account the **General Theory Relativity**. It will be done in next **Section 3.**

(3)Planetary Motion in General Relativity.

Here will be briefly studied the **planetary motion** like, for, instance**, Earth and Moon,** within the General Relativity context. As this analysis involves **extensive calculations** they are shown explicitly elsewhere.[4-6]

We present two different mathematical approaches to study the planetary motion: **Geodesic Equations** and **Lagrange Equations.**

(3.1)Geodesics Equations.

The planetary *geodesics equations* of motion[4-6] in *General Relativity* are given by

$$d^2x^\alpha/ds^2 + \{_{\mu}{}^{\alpha}{}_{\nu}\}(dx^\mu/ds)(dx^\nu/ds) = 0 \qquad \textbf{(3.1),}$$

where $\{_{\mu}{}^{\alpha}{}_{\nu}\}$ are the Christoffel symbols[4-6] and

$$ds^2 = c^2 d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu \qquad \textbf{(3.2),}$$

where τ = *proper time*, μ,ν =1,2,3,4; $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = x^o = ct$. With **Eqs.(3.1)** and **(3.2)** and solving *Einstein field equations*

$$R_{\mu\nu} - (1/2)g_{\mu\nu} = (8\pi G/c^2)\, T_{\mu\nu} \qquad \textbf{(3.2a)}$$

where $T_{\mu\nu}$ is the matter tensor[4-6] for a spherical body (Sun), with radius R and mass M, putting $И = GM/c^2$, ds^2 is given by[4-6]

$$ds^2 = c^2 d\tau^2 = (1 - 2И/r)c^2dt^2 - (1 - 2И/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \qquad \textbf{(3.3).}$$

When $И/r = GMc^2/r << 1$, solving the geodesic equations **(3.1)**[see detailed calculations elsewhere[4-6]] we get, instead of **Eq.(2.1),**

$$d^2u/d\theta^2 + u - GM/A^2 - 3GMu^2 = 0 \qquad \textbf{(3.4)}$$

where A = L/m, that is, $GM/A^2 = m\alpha/L$. In the Newtonian limit[1] we have

$$d^2u/d\theta^2 + u = (GM/A^2) \qquad \textbf{(3.5).}$$

As $GM/rc^2 \sim 10^{-8}$ is very small,[4-6] even for the planet Mercury, compared with the other ones it would be sufficient to solve **Eq.(3.4)** using the method of successive approximations. So, beginning the calculation with u given by the Newtonian equation **(3.5)** one can show putting, for simplicity $\theta_o = 0$, that[4-6]

$$1/r = (GMm^2/L^2)\{1 + \varepsilon \cos[\theta - 3(GMm/L)^2\theta]\} \qquad \textbf{(3.6),}$$

which is a precessing ellipse orbit, putting A = L/m,[4-6]

$$1/r = (GM/A^2)\{1 + \varepsilon \cos[\theta - 3(GM/A)^2\theta]\} \qquad \textbf{(3.7),}$$

showing a *perihelion precession*[1-3] or *perihelion shift* = Δ

$$\Delta = 6\pi\,(GM/A)^2 \qquad \textbf{(3.8).}$$

Note that the precession is in the direction of the motion. (**Fig.(3)**).

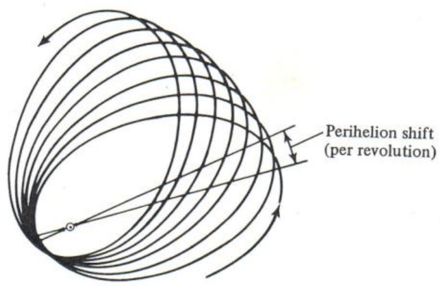


Figure (3). Planetary orbit with perihelion precession.[5]

The *predicted* angular advance of the perihelion per revolution would be $6\pi\,(GM/A)^2$. To Mercury, Venus and Earth they would be, respectively, **43´´, 8.6´´** and **3.8´´** and the *observed* values being: **43´´, 8.4´´** and **5.0´´.**[4-6]

(3.2)Lagrangian Equations.[5]

To get the *timelike geodesic* (massive particle) we may use its proper time τ as an affine parameter to obtain the 4 geodesic equations:[5]

$$d(\partial Ł/\partial \dot{x}^{\mu})d\tau - (\partial Ł/\partial x^{\mu}) = 0 \qquad \textbf{(3.9),}$$

where $x^* = dx/d\tau$ and the Lagrangian $Ł(x^{*\sigma}, x^{\sigma})$ is given by[5]

$$Ł(x^{*\sigma}, x^{\sigma}) = (1/2)\, g_{\mu\nu}\, x^{*\mu}\, x^{*\nu} =$$

$$= (1/2)\{c^2(1-2m/r)t^{*2} - (1-2m/r)^{-1}r^{*2} - r^2(\theta^{*2}+\sin^2\theta\, \phi^{*2})\} \textbf{(3.10)}$$

where the symbol * denotes derivatives with respect to τ, of the coordinates $x^o = t$, $x^1 = x$, $x^2 = \theta$ and $x^3 = \phi$. That is, $t^* = dt/d\tau$, $x^* = dx/d\tau$, $\theta^* = d\theta/d\tau$ and $\phi^* = d\phi/d\tau$.

Due to the spherical symmetry, there is no loss of generality in confining our attention to particles moving in the equatorial plane with $\theta = \pi/2$. With this value the third term ($\mu = 2$) of **Eqs.(3.9)** is satisfied, and the second of these, with $\mu = 1$, reduces to[5]

$$(1-2m/r)^{-1}[d^2r/d\tau^2] + (mc^2/r^2)[dt/d\tau]^2 - (1-2m/r)^{-2}(m/r^2)[dr/d\tau]^2 - r[d\phi/d\tau]^2 = 0 \quad \textbf{(3.11).}$$

As these are cyclic coordinates,[2,5] that is, $\partial Ł/\partial t^* =$ const and $\partial Ł/\partial \phi^* =$ const and taking into account that $\theta = \pi/2$ we can get two integration constants, k and h:[2,3,5] $h = r^2(d\theta/dt) = L^2/m$ and E = orbit energy $= c^2(k^2-1)/h^2$. In this way, from **Eq.(3.11)** we get,

$$(du/d\theta)^2 + u^2 = E + (2GM/h^2)u + (2GM/c^2)u^3 \qquad \textbf{(3.12).}$$

For very small values $2GM/c^2 << 1$ results the Newtonian limit, in agreement with **Section 1,** that is[2]

$$(du/d\theta)^2 + u^2 = E + (2GM/h^2)u \qquad \textbf{(3.13),}$$

giving,

$$u = 1/r = (GM/h^2)[1 + \varepsilon \cos(\theta - \theta_o)] \qquad \textbf{(3.14),}$$

where $\varepsilon^2 = 1 + Eh^4/G^2M^2$. When $2GM/c^2 << 1$ we get,[4-6]

$$1/r = (GM/A^2)\{1 + \varepsilon \cos[\theta - 3(GM/A)^2\theta]\} \qquad \textbf{(3.15),}$$

in agreement with **Eq.(3.7),** describing the perihelion precession.

(4)Restricted 3-body Problem.

The "**Restrict 3-body System"** is a self gravitating system composed by 3 bodies that move in a same plane around circularly around their center of mass (see **Figure 4**) obeying the following conditions:[8]

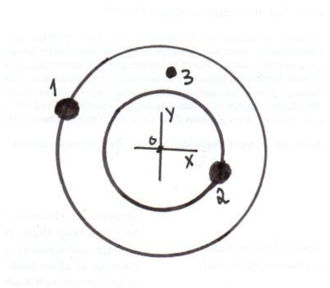


Figure (4). Three gravitating bodies (1, 2 and 3) in the same plane; 1 and 2 moving circularly around the point O. The mass m_3 is much smaller than m_1 and m_2.

(a) The two large bodies (1 and 2), with masses m_1 and m_2, are moving on circular paths about their center mass O.
(b) One of the three masses ("test particle", 3) is so small that its effect on the motion of the other two is negligible.
(c) The 3 bodies move in a same plane

This could be, for instance, the motion of the Moon around the Earth and of a **spacecraft** with mass m_3.**[8]** The objective of the **restrict 3-body problem** is to calculate the dynamics of the test particle 3 as it moves under the gravitational influence of the bodies 1 and 2. The problem is farther simplified by restricting the motion of the two primary masses (Earth and Moon) to circular orbits about their center of mass.[8]

As *Einstein field equations* are nonlinear they cannot in general be solved exactly. This problem was studied by Weinberg.[8] using a *post-Newtonian approximation*. The equations of motion of the test particles, x, y and t, have been obtained using the **Lagrangian formalism**:[5]

$$Ł = (g_{ij}/2)(dx^i/d\tau)(dx^j/d\tau) = (g_{ij}/2)\ x^{i*}\ x^{j*} \qquad \mathbf{(4.1),}$$

taking into account that

$$(d/d\tau)(\partial Ł/\partial x^{\mu}*) - (\partial Ł/\partial x^{\mu}) = 0 \qquad \mathbf{(4.2)}$$

where x^{μ} ($\mu = 0,1,2$) are functions of the proper time τ for an observer on the test particles. The equations of motion $x^{\mu}** = d^2x^{\mu}/d\tau^2$ and $dt/d\tau = t*$, with $\mu = 1, 2$ can be obtained as functions of the time only by **numerical integration**. So, the 3-bodies trajectories, would be estimated by numerical calculations. According to Wanex[9] these would show the chaotic nature of the *3-body restricted motion* in General Relativity. To guarantee that these predictions are correct they ought to be identical to the motion equations of the (*restricted*) 3-body problem obtained in the Newtonian limit($c \rightarrow \infty$). Under this limit they need to match exactly with the equations for the Newtonian (*restricted*) 3-body problem[see Moulton[10]and Szebehely[11]].

(5)Comments on Chaos and Stability in Solar System.

Before to analyze the Solar System stability let us remember the main aspects of the ***Chaos theory***[12] that was summarized by Edward Lorenz as:[12,13] **"When the present determines the future, but the approximate present does not approximately determines the future".** It is an interdisciplinary area of scientific study and branch of mathematics focused on underlying patterns and *deterministic laws* of dynamical systems that are highly sensitive to initial conditions. The *butterfly effect* for instance, an underlying example of *chaos*, describes how a small change in one state of a *deterministic nonlinear system* can result in large differences in a later state, meaning that there is sensitive dependence on initial conditions. Once it was thought to have completely random states of disorder and irregularities. However, this theory states that within the ***apparent randomness*** of chaotic complex systems, there are underlying patterns, interconnection, constant feedback loops, repetition, self-similarity, fractals, and self-organization. In a few words, which seems a contradiction, this theory is "*not completely chaotic"!* The deterministic nature of these systems does not make them predictable. This behavior is known as **deterministic chaos**, or simply **chaos.**[13,14] The theory of *nonlinear dynamical systems* (*chaos theory*), which deals with *deterministic* systems that exhibit a complicated, apparently random-looking behavior, has formed an interdisciplinary area of research and has affected almost every field of science in the last 20 years. Chaotic behavior exists in many natural systems, including fluid flow, heartbeat irregularities, weather, climate, sociology, computer science,..., and Solar System.[15]The **stability** of the Solar System[15] is a subject of much inquiry in astronomy. Though the planets motion, which have been observed for a very long time, seen to be stable, and will be in the short term, their weak gravitational effects on one another can add up in unpredictable ways. For this reason, the Solar System is *chaotic* in the technical sense of the mathematical chaos theory.[14,15] Even the most precise long-term models for orbital motion of the Solar System are not valid over more than a few of millions years.[16]The Solar System is stable in human terms, and far beyond: planets will not collide with each other or be ejected from the system in next billion years[17]and the Earth´s orbit will be relatively stable.

Since Newton´s law of gravitation (1687), mathematicians and astronomers (as Pierre -Symon Laplace, Gauss, Poincaré, Komolgorov, V.Arnold and J. Moser) have searched for stability evidence of the planetary motion and this quest led to many mathematical developments and several successive "proofs" of the Solar System stability.[15]

The planets' orbits are chaotic over longer timescales, in such a way that the whole Solar System possesses a Lyapunov time in the range of 2–230 million years.[15] In all cases, this means that the position of a planet along its orbit ultimately becomes impossible to predict with any certainty. In some cases, the orbits themselves may change dramatically. Such chaos manifests most strongly as changes in eccentricity, with some planets orbits becoming significantly more - or less - elliptical.[15] In calculation, the unknowns include asteroids, the solar quadrupole moment, mass loss from the Sun through radiation and solar wind, drag of solar wind on planetary magnetospheres, galactic tidal forces, and effects from passing stars.[14]

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