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WEAK QUANTIZATION IN A NON-PERTUBATIVE MODEL

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Abstract : The concepts of extended operator convergence and of spectral concentration are used to study rigourously a class of simple models for the tunnel effect and the laser. We compute exactly the asymptotic decay times of the eigenmodes, and we prove their link with the line-width of the corresponding resonances.

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I. Description of the model

A massive quantum particle is restricted to move on the one-dimensional half-space $[0,\infty)$ with a rigid wall at $x = 0$. Its motion is free, except for a "square" potential barrier, starting at $x = \pi$, with height a and width b; for simplicity, we shall first assume that b is independent of a, and we will only later (section IV) indicate the modifications to be brought to the theory when b is allowed to go to zero as a approaches infinity. The Hamiltonian of the system for finite $a \geq 0$ is thus $H_a = H_0 + aV$ where $(Vf)(x) = \chi_{[\pi,\pi+b]}(x)f(x)$ for all f in $\mathfrak{H} = \mathcal{L}^2([0,\infty),dx)$; H_0 is the self-adjoint operator $-\Delta$, where Δ is the Laplacian, with domain [see X.3. in [1]] $\mathcal{D}_0 = \{\phi \in \mathfrak{H} \mid \phi$ and ϕ' absolutely continuous; $\phi'' \in \mathfrak{H}$; and $\phi(0) = 0\}$. Since V is bounded, $\mathcal{D}_0$ is also [see V.4.1 in [2]] the domain of self-adjointness of H. Note that the spectrum of H_a is $[0,\infty)$ and is absolutely continuous with respect to Lebesgue measure; in particular, $H_a \geq 0$ for every finite $a \geq 0$.

This system is thus the simplest possible, and well-known [e.g. Ex. III.3 in [3]], model for the tunnel effect. The purpose of this paper is to present a precise mathematical analysis of the asymptotic behaviour of this system as a tends to infinity.

In the limit of infinitely large a, the physicist's intuition is that the wall decouples the inside region $I = [0,\pi]$ from the outside region $III = [\pi+b,\infty)$; and that the evolution is free in both of these regions, which are then limited by rigid walls at $x = 0$, π and $\pi + b$. The Hilbert space of the system thus becomes $\mathfrak{H}_\infty = \mathfrak{H}^I \oplus \mathfrak{H}^{III}$ with $\mathfrak{H}^I = \mathcal{L}^2([0,\pi],dx)$, $\mathfrak{H}^{III} = \mathcal{L}^2([\pi+b,\infty),dx)$; and the evolution is governed by the self-adjoint operator $H_\infty = H^I \oplus H^{III}$ given by $-\Delta$ in both regions, with respective domains [1]: $\mathcal{D}^I = \{\phi \in \mathfrak{H}^I \mid \phi$ and ϕ' absolutely continuous; $\phi'' \in \mathfrak{H}^I$; and $\phi(0) = 0 = \phi(\pi)\}$ and $\mathcal{D}^{III} = \{\phi \in \mathfrak{H}^{III} \mid \phi$ and ϕ' absolutely continuous; $\phi'' \in \mathfrak{H}^{III}$; and $\phi(\pi+b) = o\}$. Note that H_∞ is the Friedrichs extension [see for instance VI.2.3 in [2)]] in $\mathfrak{H}_\infty$ of the restriction of H_o to the dense domain $\mathcal{D}_1 = \mathcal{D}_o \cap P\mathfrak{H}$, where P is the projector from $\mathfrak{H}$ onto $\mathfrak{H}_\infty$.

H^{III} is clearly unitarily equivalent to H_o, and therefore has absolutely continuous spectrum. H^I on the other hand has purely discrete spectrum $\{ m^2 \mid m = 1,2, \ldots \}$.

Whereas H_a is obviously a perturbation of H_o, it is not a small perturbation away from H_∞. Our aim is to describe how H_∞ is nevertheless the limit of H_a as a tends to infinity, and to control this limit well enough to allow an understanding of the exponential decay which one expects on physical grounds in the tunnel effect.

II. Operator convergence

Before addressing the problem of exponential decay we want in this section to elucidate the sense in which H_∞ is the limit of H_a as a tends to infinity.

Theorem II.1 : For every ϕ in $\mathcal{D}$, a domain of essential self-adjointness of H_∞, there exists $\{ \phi_a \mid a \in (o,\infty)\} \subset \mathcal{D}_o$ such that, as $a\to\infty$: (i) $\| \phi_a - \phi \| = 0\,(a^{-1/2})$

$$\text{(ii)}\quad \| H_a \phi_a - H_\infty \phi \| = 0\,(a^{-1/2})$$

Proof: We can deal with regions I and III separately. Let first $\phi \in \mathcal{D}^{III}$, which we embed in $\mathfrak{H}$ by setting $\phi(x)=o$ for all $x \leqslant X = \pi+b$. If $\phi'(X) = 0$, ϕ belongs to the domain of H_a as well, so that (i) and (ii) are trivially satisfied by $\phi_a = \phi$ for all $a \in (o,\infty)$. We can therefore suppose, without loss of generality, that $\phi'(X) = A \neq o$, and that there exists $\varepsilon > o$ for which ϕ does not vanish in $(X, X+\varepsilon]$. Let ξ and ζ be two non-increasing functions in $\mathcal{E}^{\infty}(-\infty,\infty)$ with $\xi(x) = 1$ for all $x \leqslant o$, $\xi(x) = o$ for all $x \geqslant \pi$; $\zeta(x) = 1$ for all $x \leqslant X$, $\zeta(x) = o$ for all $x \geqslant X+\varepsilon$. We further define, for every $a>o$ and every x in $[o,X]$:

$$\psi_a(x) = A\, a^{-1/2} \exp\left\{ (x-X) a^{1/2} \right\} .$$

One then verifies that an approximating net $\left\{\phi_a \mid a \in (o,\infty)\right\}$, in the sense of the theorem, is obtained by setting $\phi_a(x)$ equal to:

$-A\, a^{-1/2} \exp(-X\, a^{1/2})\,\xi(x) + \psi_a(x)$ for $x \in I = [o,\pi]$

$\psi_a(x)$ for $x \in II = [\pi,\pi+b]$

$A\, a^{-1/2}\, \zeta(x) + \phi(x)$ for $x \in III = [\pi+b,\infty)$.

Hence $\mathcal{D}^{III} \subseteq \mathcal{D}$. A similar argument could be made for region I. We however find it more instructive to construct explicitly one approximating net $\left\{ \phi_a^{(m)} \mid a \in (o,\infty) \right\}$ for each eigenvector $\phi^{(m)}$ ($m = 1,2, \ldots$) of H^I. We chose $\phi^{(m)}(x) = \sin mx$, and embed $\phi^{(m)}$ in $\mathfrak{H}$ by setting $\phi^{(m)}(x) = o$ for all $x \geqslant \pi$. For each m fixed, we define m_a, with $m_a \to m$ as $a \to \infty$, by $m_a^{-1} \tan(m_a \pi) = -a^{-1/2}$. We further introduce $M_a^{(m)} = a^{1/2} \sin(m_a \pi)$. Upon noticing that $(m-m_a)$ and $m-M_a^{(m)}$ are both $O(a^{-1/2})$ as $a \to \infty$, one verifies that an approximating net $\left\{ \phi_a^{(m)} \mid a \in (o,\infty)\right\}$, in the sense of the theorem, is obtained for $\phi^{(m)}$ by setting $\phi_a^{(m)}(x)$ equal to :

$\sin(m_a x)$ for $x \in I$

$M_a^{(m)}\, a^{-1/2} \exp\left\{ (\pi-x) a^{1/2} \right\}$ for $x \in II \cup III$.

Note that $\left\{ \phi^{(m)} \mid m = 1,2, \ldots\right\}$ is an orthogonal basis in $\mathfrak{H}^I$, consisting of eigenvectors of H^I; H^I is thus essentially self-adjoint on the linear span of these vectors. The above argument shows that this manifold is contained in $\mathcal{D}$. We can therefore prove the assertion of the theorem with

$$\mathcal{D} = \text{Span} \left\{ \phi^{(m)} \mid m = 1,2, \ldots \right\} \oplus \mathcal{D}^{III} \qquad \text{q.e.d.}$$

Let now $\left\{ U_a(t) \mid t \in (-\infty,+\infty) \right\}$ (resp. $\left\{ U_\infty(t) \mid t \in (-\infty,+\infty) \right\}$)

be the unitary group on $\mathfrak{H}$ (resp $\mathfrak{H}_\infty$) generated by H_a (resp. H_∞).

Corollary II.2: For every $\phi \in \mathfrak{H}_\infty$ and every $T \in [0,\infty)$

$$\lim_{a\to\infty} \sup_{0\leq t\leq T} \| U_a(t)\phi - U_\infty(t)\phi \| = 0$$

Proof: With $\mathcal{D}$ as in Thm.II.1, we have [see V.3.4 in [2)]] $\{ (H_\infty - iI)\phi \mid \phi \in \mathcal{D} \}$ dense in $\mathfrak{H}_\infty$. The corollary then follows directly from Kurtz' criterion [4)] in his theory of extended operator convergence.

Hence on the Hilbert space $\mathfrak{H}_\infty$ corresponding to the limit of an infinitely high wall, the time-evolution $U_a(t)$ converges strongly to the limiting time-evolution $U_\infty(t)$, uniformily in t on compacts. The latter result (for b independent of a) is not new [5,6)]. As a particular case of these papers, one has indeed, as $a\to\infty$, that the resolvant $R_a(z)$ of H_a converges strongly on $\mathfrak{H}_\infty$ to the resolvant $R_\infty(z)$ of H_∞ for every $z \in \mathbb{C} - [0,\infty)$ (in conformity with Cor. II.2, by a slight modification of the classical argument [see IX.2.5 in [2)]]). Morevoer [6)], it follows from the strong resolvant convergence that, as $a\to\infty$, the semi-group $\{ S_a(t) = \exp(-H_a t) \mid t \in [0,\infty) \}$ converges strongly on $\mathfrak{H}_\infty$ to the semi-group $\{ S_\infty(t) = \exp(-H_\infty t) \mid t \in [0,\infty) \}$. The estimate of Thm. II.1 however is new, and is of some independent interest [see in particular sections III and IV below] .

III. Decay

We saw in section II that the limiting dynamics corresponds to the hard wall condition in the Hamiltonian H_∞. The limiting process has drastically changed the spectrum from continuous (H_a) to discrete (H_∞). We now turn around and think of the initial system as the one with infinitely high walls, and then bring down the wall to a finite, albeit very large, height. In such a scenario the spectrum of the relevant Hamiltonian makes a transition from discrete to continuous, a transition we want to investigate in details. For this purpose we use the spectral transformation (or generalized Fourier transform) of H_a [for these notions the reader may consult [7)]].

In this simple model, we can solve the Schrödinger equation exactly and obtain the eigenfunctions $\{ \psi_\lambda \mid \lambda \in [0,\infty) \}$:

$$\psi_\lambda(x) = \begin{cases} \alpha(k)\ \sin kx & x \in I = [0,\pi] \\ \beta_-(k)\ \exp[-\xi(x-\pi)] + \beta_+(k)\ \exp[\xi(x-\pi)] & x \in II = [\pi, \pi+b] \\ \gamma(k)\ \sin[k(x-\pi-b) + \delta] & x \in III = [\pi+b, \infty), \end{cases}$$

where we have written $k^2 = \lambda$ and $\xi = (a-k^2)^{1/2}$. The coefficients α, β are determined in terms of γ by the requirement that ψ_λ be locally in the domain $\mathcal{D}_o$ of H_a, i.e. ψ_λ and ψ_λ' be locally absolutely continuous. For most of our calculations we shall need the details of only α , the latter turning out to be:

$$\alpha(k)^2 = \gamma(k)^2 \Big[(\sin^2 k\pi - k^2 \xi^{-2} \cos^2 k\pi) + a \{ k^{-1}\eta_-(k) \sin k\pi + \xi^{-1} \eta_+(k) \cos k\pi \}^2 \Big]^{-1} \qquad (1)$$

where $\eta_\pm(k) = [\exp(\xi b) \pm \exp(-\xi b)]/2$. We choose the normalization $\gamma(k) = (\pi k)^{-1/2}$.

We take the initial situation to be the one with infinitely high walls and begin with an eigenmode ϕ_n [$\phi_n(x) = (2/\pi)^{1/2} \sin nx$] of H^I trapped in region I. The wall is then "lowered" from $a=\infty$ to some finite, but large a . We want the asymptotic behaviour, as $a \to \infty$, of the probability $|(\phi_n, \exp[-iH_a t]\ \phi_n)|^2$ that the eigenmode ϕ_n (now evolving under the group $\exp[-iH_a t]$) will remain in the same mode after a time t has elapsed. From section II, $U_a(t)$ converges strongly to $U_\infty(t)$ on $\mathfrak{H}^I$; from this follows that the above probability converges (uniformly in t on compacts) to $||\phi_n||^2 = 1$, as $a\to\infty$. Further information on the rate of this convergence is of physical interest for the description of the tunnel effect. We observe that

$$(\phi_n, \exp\left[-iH_a t\right] \phi_n) = \int_0^\infty d\lambda \exp(-i\lambda t)|\phi_n(\lambda)|^2 \tag{2}$$

where $$\phi_n(\lambda) = \int_0^\pi dx\ \psi_\lambda(x)^* \phi_n(x) \tag{3}$$

is the spectral representative (or generalized Fourier transform) of ϕ_n . We have :

$$|\phi_n(\lambda)|^2 = (2\pi)^{-1}\alpha(k)^2 \left[(k-n)^{-1}\sin(k-n)\pi - (k+n)^{-1}\sin(k+n)\pi \right]^2 \tag{4}$$

The term in square-braket is bounded in k and converges to π^2 as $k\to n$; therefore, the major contribution will come from $\alpha(k)^2$. From (1) one concludes that this contribution originates from the neighbourhoods of the zeros of the a priori larger term (as $a\to\infty$).This leads to the resonance equation (or approximate eigenvalue eqn.):

$$F(k,a) = 0 \quad , \quad 0<k<a^{1/2} \quad \text{where} \tag{5}$$

$$F(k,a) = \left[\eta_-(k)/\eta_+(k) \right] \tan k\pi + \left[k/\xi \right] \tag{6}$$

We observe that F is $\mathcal{C}^\infty$ in a neighbourhodd of (n,∞), with $F(n,\infty) = 0$ and $F_k(n,\infty) \equiv (\partial F/\partial k)\ (n,\infty) \neq 0$. From the implicit function theorem [8)] there exists a positive a_0 , large enough such that for all $a>a_0$, the resonance equation (5) has a unique solution $k = k_n(a)$ $\left[\text{i.e. } \lambda=\lambda(n,a) = k_n(a)^2 \right]$ with $\lambda(n,\infty) = n^2$. An asymptotic expansion of $\lambda(n,a)$ in terms of $a^{-1/2}$ can now be derived:

$$\lambda(n,a) = n^2 - 2\,n^2\,\pi^{-1}\,a^{-1/2} \left[\eta_+/\eta_- \right]_{a=\infty} + O(a^{-1}) \tag{8}$$

We remark that, as $a \to \infty$, $\lambda(n,a)$ approaches the n-th eigenvalue of H^I. Thus the resonance equation (5), by itself, asymptotically selects H^I from the one-dimensional [1] manifold spanned by the self-adjoint extensions of the symmetric operator obtained as the restriction of H_o (or H_a) to $\mathcal{D}_o \cap P^I \mathfrak{H}$.

The phase-shift δ in the eigenfunction in region III is :

$$\tan \delta = k\, \xi^{-1} \left[\tan k\pi + k\, \xi^{-1} (\eta_- / \eta_+) \right] F(k,a)^{-1}. \qquad (9)$$

From this follows that the phase-shift at resonance is $\delta_n = \pi/2$. Moreover

$$\left[d\delta/dk \right] (k = k_n(a)) = \xi_n^2\, k_n^{-2}\, \eta_+(k_n)\, \eta_-(k_n) F_k(k_n, a) \qquad (10)$$

which is a large positive number. This is in conformity with the conventional definition of a resonance. The amplitude at resonance is :

$$\alpha(k_n)^2 = \gamma(k_n)^2\, \eta_+(k_n)^2 \left[\sin k_n \pi \right]^{-2} \qquad (11)$$

From section II, recall that for every $z \in \mathbb{C} - [o,\infty)$, $(z - H_a)^{-1} \phi \to (z - H_\infty)^{-1} \phi$ for all $\phi \in \mathfrak{H}^I$. We thus expect [see VIII.5.2 in [2]] to have a "spectral concentration", expressing that the spectral measure of H_a concentrates, as a becomes large, in some neighbourhoods of the eigenvalues n^2 of H_∞^I. We now want to compute the details of this concentration, i.e. in physical terms, the asymptotic line shape as $a \to \infty$.

Let us denote by $\{ E_a(\lambda) \mid \lambda \in [o,\infty) \}$ the spectral family of H_a. Since H_a is spectrally absolutely continuous, there exist positive, integrable functions $f_a(n,\lambda)$ such that for every real c and d :

$$(\phi_n, E_a([c,d])\phi_n) = \int_c^d d\lambda\, f_a(n,\lambda) \qquad (12)$$

The next theorem states the asymptotic properties of $f_a(n,.)$ as $a\to\infty$.

<u>Theorem III.1</u>: Let $\lambda(n,a)$ and $\{ f_a(n,.) \mid a \in [o,\infty)\}$ be defined as above, and let

$$\Gamma(n,a) = 2\pi^{-1} \lambda(n,a)^{3/2} \left[a - \lambda(n,a) \right]^{-1} \eta_-(k_n(a))^2 \qquad (13)$$

Then $f_a(n,\lambda) = | \phi_n(\lambda) |^2$ [see (4)] , and the functions $g_a(n,.)$, defined on $(-\infty, +\infty)$ by:

$$g_a(n,h) = \Gamma(n,a) f_a(n,\lambda(n,a) + h\,\Gamma(n,a)) \qquad (14)$$

converge as $a\to\infty$, pointwise and in $\mathcal{L}^1$-norm to $g(.)$ with $g(h) = [\pi\ (1 + h^2)]^{-1}$.

Proof: The first assertion follows directly from (12). Upon using (6), we rewrite (1) as :

$$\alpha(k)^2/\gamma(k)^2 = [\sec^2 k\pi][k^2 \xi^{-2} \eta_-^{-2} - 2k \xi^{-1} \eta_+^2 \eta_-^{-2} F$$
$$+ (a k^{-2} \eta_+^2 + \eta_+^2 \eta_-^{-2}) F^2]^{-1} . \qquad (15)$$

Since $F(\lambda(n,a), a) = 0$ and F is a C^∞ function in a neighborhood of (n,∞), we have the Taylor expansion :

$$F(k,a) = [\lambda-\lambda(n,a)] F_\lambda(\lambda(n,a),a) + 2^{-1} [\lambda-\lambda(n,a)]^2 F_{\lambda\lambda}(\lambda,a)$$
$$= (2k)^{-1} F_k(k_n(a),a) \Gamma(n,a)h + 2^{-1} F_{\lambda\lambda}(\lambda,a) \Gamma(n,a)^2 h^2 \qquad (16)$$

where we defined h by :

$$\lambda = \lambda(n,a) + h\,\Gamma(n,a) \qquad (17)$$

From (13), (15) and (16), we see that :

$$\Gamma(n,a)\alpha(k)^2 = 2\pi^{-2} [1-2\Delta(n,a)h + h^2 + 0(\Gamma(n,a))]^{-1} \qquad (18)$$

where $\Delta(n,a) = [\eta_+/\eta_-] a^{-1/2} k_n(a)$. (19)

Since for any fixed real h, we can find a>o large enough so that $\lambda = \lambda(n,a) + \Gamma(n,a)h > 0$ we have, for such h, that $\Gamma(n,a) f_a(n, \lambda(n,a) + \Gamma(n,a)h)$ approches g(h) as a tends to infinity, pointwise in h. On the other hand, upon setting $g_a(n,h)=0$ for $h < -\lambda(n,a)/\Gamma(n,a)$, we have:

$$\int_{-\infty}^{+\infty} dh\, g_a(n,h) = \int_0^{\infty} d\lambda\, f_a(n,\lambda) = 1 \text{ for all } a>o.$$

Since the $\mathcal{L}^1$-norm of g is also 1, we have [9] that $g_a(n,.)$ converges to g(.) in $\mathcal{L}^1$-norm. q.e.d.

The theorem has two corollaries, both of which can be obtained as in [9].

<u>Corollary III.2</u> : For any $h_1 < h_2$ real :

$$\lim_{a\to\infty} (\phi_n, E_a[\lambda(n,a) + \Gamma(n,a)h_1, \lambda(n,a) + \Gamma(n,a)h_2]\,\phi_n)$$

$$= \int_{h_1}^{h_2} dh\, [\pi(1+h^2)]^{-1}$$

This result gives the explicit form of the spectral concentration: for large a , the resonance approaches a Lorentzian, centered around $\lambda(n,a)$, and of width $\Gamma(n,a)$ given by (13).

<u>Corollary III.3</u> : For any $\tau \geqslant 0$:

$$\lim_{a\to\infty} (\phi_n, \exp[-i\{H_a - \lambda(n,a)\}\,\Gamma(n,a)^{-1}\tau]\,\phi_n) = \exp(-\tau),$$

and the convergence is uniform in $0 \leqslant \tau < \infty$.

Consequently, the probability $|(\phi_n, \exp[-iH_a \Gamma(n,a)^{-1}\tau] \phi_n)|^2$ behaves asymptotically as $\exp(-2\tau)$ when $a\to\infty$. Upon reintroducing the unscaled time $t = \Gamma(n,a)^{-1}\tau$, we thus find that, for large a, $|(\phi_n, U_a(t) \phi_n)|^2$ behaves as $\exp[-2 \Gamma(n,a)t]$. In other words, as we "lower" the barrier from infinite height to a finite, but large one, we can interpret $[2 \Gamma(n,a)]^{-1}$ as the half-life of the eigenmode ϕ_n. This confirms the usual relation between the half-line width of a resonance and the half-life time of its decay. The above calculation indeed shows in a precise manner how the scaling in energy is inversely related to the scaling in time. The rescaled time τ is of the order of $\Gamma(n,a)^{-1}$, i.e. $a \exp(a^{1/2}b)$, which is very large; hence the decay of the eigenmodes indeed takes place very slowly; equivalently the resonances are very sharp, with a very small line-width.

At this point it is worth mentioning that since $a^{1/2}\Gamma(n,a) \to 0$ as $a\to\infty$, there is no contradiction between Thm II.1 and Thm III.1 (or Cors.III.2 and 3), thus bypassing the objection raised by Davies [compare indeed these results with conditions (2) and (4-7) in [9]].

A computation, similar to that carried above, can be made for the off-diagonal elements of $\exp[-iH_a\tau/\Gamma(n,a)]$ in $\mathfrak{H}^I$, indicating that the eigenmode n not only decays, but actually leaks out of region I.

IV. Generalization of the model

The generalization consists in allowing $b \to o$ as $a \to \infty$, i.e. more precisely : $b = 0(a^{\nu})$ with $\nu < 0$. If $0 > \nu > -1/2$ (or $a^{1/2} b \to \infty$ as $a \to \infty$), the construction and the proof of Theorem II.1 remain essentially unchanged. In this case however, the concept of extended operator convergence [4] takes full force and goes beyond the case studied in [6]. Also $(\Delta\lambda)(n,a) \equiv \lambda(n,a) - n^2 = -2n^2 \pi^{-1} a^{-1/2} + 0(a^{-1})$ while $\Gamma(n,a) \simeq 8n^3 \pi^{-1} a^{-1} \exp(-2\, a^{1/2} b)$, showing that the half-width still is exponentially small compared to the shift. If $\nu = -1/2$ (i.e. $a^{1/2} b \to \beta > o$), $(\Delta\lambda)(n,a) = -2n^2 \pi^{-1} a^{-1/2} \coth \beta + 0(a^{-1})$ and $\Gamma(n,a) \simeq 2n^3 \pi^{-1} a^{-1} \operatorname{cosec}^2 \beta$. If however $-1/2 > \nu > -1$ (i.e. $ab = \Lambda \to \infty$ as $a \to \infty$), the resonance equation (5.6) has no solution, and it should be modified to read:

$$G(k,a) \equiv a^{1/2} F(k,a) = o \qquad (20)$$

This modified resonance equation has a unique solution $\lambda(n,a)$ in the neighbourhood of n^2, and we have $(\Delta\lambda)(n,a) \simeq -2n^2 \pi^{-1} \Lambda^{-1}$, while $\Gamma(n,a) \simeq 2n^3 \pi^{-1} \Lambda^{-2}$. In all these cases, one has spectral concentration and decay in the sense of section III. Moreover, upon using the estimates of section III, one proves again that $\exp(-iH_a t) f$ converges strongly to $\exp(-iH^I t) f$ as $a \to \infty$, for all f in $\mathfrak{H}^I$. Finally, if $\nu \leq -1$ (i.e. $ab \to$ finite, possibly zero, limit), none of the considerations of section III applies, and even the modified resonance equation (20) fails to have a solution near n^2; in fact, in the extreme case where $b = 0(a^{-2})$, H_a [resp. $U_a(t)$] clearly converges strongly to H_o [resp. $U_o(t)$] : the wall has become completely transparent.

V. Conclusions

The model is non-pertubative by nature. Yet it is simple enough to be exactly solvable, and to allow a precise control of its asymptotic beahviour as a approaches infinity. It is moreover sophisticated enough to exhibit a host of interesting features, both physical and mathematical, which we briefly review on the basis of our analysis.

First of all, the model exhibits exponential decay, although all the Hamiltonians occuring in the problem are uniformly bounded below, namely by zero. This should be contrasted with the situation encountered in non-equilibrium statistical mechanics, where the presence of an infinite bath at finite temperature allows the generator of the time-evolution to have Lebesgue spectrum, covering the whole real line [for general arguments to this effect, as well as for models, see for instance [10,11]] . The exponential decay found in the present model emphasizes the role of the re-scaling in time, which allows to bypass the usual no-go theorems [e.g. 7.3.3 in [10]] by the mechanism described in section III. This mechanism appears to be quite different from that occuring in the van Hove limit of statistical mechanics [10,12].

We might remark here that the exact asymptotic life-time and width found in this model coincide with the value found in the WKB approximation [see for instance [3]] ; a similar feature has been noticed also in [13]. This coincidence with the exact result, found by an unperturbative approach, seems to have a status similar to that of the Born approximation in the master equation theory [see e.g. [12]] .

The decay found in the present model can be related to the phenomenon known in physics as "weak quantization" [see for instance p.251 in [14], or pp. 403-408 in [15]] . The physical picture is given a firm mathematical basis in this model; we indeed saw that the point spectrum, encountered when the inside region I is decoupled from the outside by an infinitely high hard wall, only persists, as the wall is lowered, in the form of Lorentzian resonances : the higher the wall, the sharper the resonances; still for any finite height of the wall, the spectrum of the Hamiltonian remains absolutely continuous with respect to Lebesgue measure. This phenomena is also known in the mathematical literature [e.g. [2]] as "spectral concentration". It should be however noticed that, for $b = 0(a^{\nu})$ with $0 \geqslant \nu > -1/2$, the spectral concentration found in the present model is much stronger than the usual concentration of polynomial type [9),16)].

Whereas the present model describes very well the qualitative features of the tunnel effect, its one-dimensional character should be removed for a realistic theory of α-decay. On the other hand, the model as it stands, presents some instructive analogy with the laser, its finite high wall playing the role of a semi-transparent mirror. Some of the qualitative asymptotic features of the model are also found [17)], upon using the techniques of S-matrix theory, when the semi-transparent mirror is mimicked by a "δ-function potential of strength Λ"; in the latter case, the limit of large Λ plays the role of our limit of large a. Incidentally, the form-sum $H_o + \Lambda\delta_{\pi}$ can be obtained as the form-limit, when $a \to \infty$, of H_a with $b = \Lambda a^{-1} (\Lambda \neq o)$. When $ab \to o$, one finds H_o back. A true theory of the laser would however require two modifications of the present model. Firstly, the Maxwell equation, rather than the Schrödinger equation for a massive particle, should be taken as the starting point; secondly, a second-quantization, rather than first-quantization, formalism should be used. Nevertheless, it seems likely that the phenomenon of "weak quantization", or "spectral concentration", would persist in such a complete theory, and that it could provide a useful basis for its discussion.

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