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ABSTRACT

We propose an exact, factorized, S-Matrix for the
two-dimensional supersymmetric CP^{N-1} model, which is com-
patible with the $\frac{1}{N}$ expansion.

The two-dimensional supersymmetric CP^{N-1} model (SCP^{N-1}) was introduced by the authors of refs. (1,2), whose motivation was the understanding of some fenomena expected to be present also in four-dimensional QCD. The basic advantage is the available $\frac{1}{N}$ - expansion in the SCP^{N-1} case. This model is the natural geometrical generalization of the CP^{N-1} model, whose exact solubility should be expected[3]. In fact recently we have shown that the usual CP^{N-1} model, minimally coupled to massless fermions, posses a factorized S-matrix[4]. One may now hope that the S-matrix of the supersymmetric extension is equally factorized.

In this paper we will construct a factorized S-matrix, which by standart arguments will be shown to belong to the SCP^{N-1} model. In order to present our arguments we will first recall some proprieties of this and related models.

The CP^{N-1} model and, by supersymmetrization, the SCP^{N-1} model have a close relationship to exactly soluble models whose S-matrix are explicitly known[5-7]. Indeed there is a remarkable analogy between the SCP^{N-1} and the supersymmetric generalization[8,9] of the non-linear σ and Gross-Neveu models. I.e., in supersymmetrizing the non-linear σ model[8,9] one gets an expression that can be interpreted equally well as the supersymmetric Gross-Neveu model, since supersymmetry couples both of them[*1]. In the same way supersymmetry couples the CP^{N-1} and the chiral Gross-Neveu[10,11] models so that what we have been calling the supersymmetric CP^{N-1} model can be viewed as the supersymmetric chiral Gross-Neveu model. The last observation leads one to suspect that the SCP^{N-1} model has a factorized S-matrix since it is known that the chiral Gross-Neveu model does[11,12] and one does not expect to destroy this propriety by supersymmetrizing. An-

other interesting property of this model[1] is the existence

of an extended (0(2)) supersymmetry on the Lagrangean level

supported by some arguments given in ref. 2 about the origin

of the (topological) central charges.

To formulate the 2→2 scattering problem subject to

the factorization constraints[13] one will need, in principle,

the explicit knowledge of the super algebra. This was for in-

stance the way the S-matrix of the supersymmetric non linear

σ model was gotten[7]. However, it is possible to solve the

factorization equations (see below) without knowing any super-

symmetry relations between the scattering amplitudes, the on-

ly drawback being the appearance of two free parameters. But

these can be fixed using the available $\frac{1}{N}$ - perturbative expres-

sions[*2].

The $\frac{1}{N}$ - Feynman rules of this model were obtained in

refs (1,2) whereto we refer the reader. To establish some no-

tation we will write the defining functional integral

$$Z = \int |dZ||dZ^*||d\psi||d\overline{\psi}||d\omega||dA_\mu||d\sigma||d\pi||d\xi||d\overline{\xi}| \exp \frac{i}{2\alpha} \int d^2_x \sum_{j=1}^{N}$$

$$\left[(\partial_\mu - iA_\mu) Z^*_j (\partial^\mu + iA^\mu) Z_j + \omega(Z^*_j Z_j - 1) + \overline{\psi}_j i(\partial_\mu - iA_\mu)\gamma^\mu \psi_j \right.$$

$$\left. - \frac{1}{\sqrt{2}} \overline{\psi}_j (\sigma + i\pi\gamma_5)\psi_j + \overline{\xi} \psi_j Z^*_j + \overline{\psi}_j Z_j \xi \right]$$

Using these rules one observes that all particle-

antiparticle reflection amplitudes vanish in order $\frac{1}{N}$ and the

absence of particle production to order $\frac{1}{N^2}$. Thus in construc-

ting the exact S-matrix one will, as in the chiral Gross-Neveu

model[11,12], start with vanishing reflection amplitudes from

the very beginning[*3]. Transmission amplitudes are now defined

introducing the simbols $(\overline{b}_\alpha(\theta_i)) b_\alpha(\theta_i)$ and $(f_\alpha(\theta_i)) f_\alpha(\theta_i)$ to

denote (anti) bosons and (anti) fermions respectively, where

the variables θ_i are related to energy-momentum by $P_o^i = m\,ch\theta^i$, $P_1^i = m\,sh\theta_i$ and $P_1 P_2 = m\,ch\theta$ $(\theta = \theta_1 - \theta_2)$

$$^{out}\!<b_\beta(\theta_1)b_\delta(\theta_2)|b_\alpha(\theta_1)b_\gamma(\theta_2)>^{in} = v_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + v_2(\theta)\delta_{\alpha\delta}\,\delta_{\gamma\beta}$$

$$^{out}\!<f_\beta(\theta_1)f_\delta(\theta_2)|f_\alpha(\theta_1)f_\gamma(\theta_2)>^{in} = u_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + u_2(\theta)\delta_{\alpha\delta}\,\delta_{\gamma\beta}$$

$$^{out}\!<f_\beta(\theta_1)b_\delta(\theta_2)|f_\alpha(\theta_1)b_\gamma(\theta_2)>^{in} = c_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + c_2(\theta)\delta_{\gamma\delta}\,\delta_{\gamma\beta}$$

$$^{out}\!<b_\beta(\theta_1)f_\delta(\theta_2)|f_\alpha(\theta_1)b_\gamma(\theta_2)>^{in} = d_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + d_2(\theta)\delta_{\gamma\delta}\,\delta_{\gamma\beta}$$

$$^{out}\!<b_\beta(\theta_1)\bar{b}_\delta(\theta_2)|b_\alpha(\theta_1)\bar{b}_\gamma(\theta_2)>^{in} = v_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + v_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$^{out}\!<f_\beta(\theta_1)\bar{f}_\delta(\theta_2)|f_\alpha(\theta_1)\bar{f}_\gamma(\theta_2)>^{in} = u_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + u_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$^{out}\!<b_\beta(\theta_1)\bar{b}_\delta(\theta_2)|f_\alpha(\theta_1)\bar{f}_\gamma(\theta_2)>^{in} = d_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + d_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$^{out}\!<b_\beta(\theta_1)\bar{f}_\delta(\theta_2)|b_\alpha(\theta_1)\bar{f}_\gamma(\theta_2)>^{in} = c_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + c_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

Its is straightfoward to get the following amplitudes with the $\frac{1}{N}$ - Feynman rules

$$v_1(\theta) = 1-\frac{i\pi}{N}\coth\frac{\theta}{2}; \quad v_2 = -\frac{2i\pi}{N\theta}; \quad u_2 = \frac{2i\pi}{N\theta}; \quad c_2 = -\frac{2i\pi}{N\theta}; \quad d_1 = -\frac{\pi i}{N\,sh\frac{\theta}{2}};$$

$$d_2 = 0 + 0(\frac{1}{N^2}) .$$

The computation of the transmission amplitudes $c_1(\theta)$ and $u_1(\theta)$ present a problem comming a $\frac{0}{0}$ - indetermination. This arises from the singular behavior of the mixed A_μ-π propagator at zero momentum transfer. We circunvent this problem by inserting an infinitesimal momentum transfer δ at one vertex, making use of the identity $\bar{u}(P_2)\epsilon^{\mu\nu}(P_2-P_1)_\nu\gamma_5 u(P_1) = \frac{(P_1-P_2)^2}{2m}\bar{u}(P_2)\gamma^\mu u(P_1)$ and finally letting δ go to zero. With this prescription we get

$$c_1(\theta) = 1 + 0(\frac{1}{N^2}) \; ; \quad u_1(\theta) = 1 + \frac{i\pi\coth\theta/2}{N} .$$

The calculation of exact scattering amplitudes from the bootstrap approach was done[5-7,14] and reviewed[13] many

times so we will limit ourselves to state the result. Paying careful attention to signs due the statistics and fixing two free parameters by comparison with the $\frac{1}{N}$ - perturbative expressions one gets the following exact amplitudes ($\lambda = \frac{2}{N}$)

$$c_1(i\pi\phi) = \prod_{\ell=0}^{\infty} \frac{\Gamma(\frac{\phi}{2} + \frac{\lambda}{2} + \ell)\Gamma(1 - \frac{\phi}{2} + \ell)}{\Gamma(1 - \frac{\phi}{2} + \frac{\lambda}{2} + \ell)\Gamma(\frac{\phi}{2} + \ell)} \prod_{\ell=0}^{\infty} \frac{\Gamma(\frac{\phi}{2} - \frac{\lambda}{2} + \ell)(1 - \frac{\phi}{2} + \ell)}{\Gamma(\frac{1}{2} - \frac{\phi}{2} - \frac{\lambda}{2} + \ell)\Gamma(\frac{\phi}{2} + \ell)}$$

(1a)

$$v_1(i\pi\phi) = \frac{\sin\frac{\pi}{2}(\phi-\lambda)}{\sin\frac{\pi\phi}{2}} c_1(i\pi\phi)$$ (1b)

$$u_1(i\pi\phi) = \frac{\sin\frac{\pi}{2}(\phi+\lambda)}{\sin\frac{\pi\phi}{2}} c_1(i\pi\phi)$$ (1c)

$$d_1(i\pi\phi) = - \frac{\sin\frac{\pi\lambda}{2}}{\sin\frac{\pi\phi}{2}} c_1(i\pi\phi)$$ (1d)

$$v_2(\theta) = - \lambda\frac{i\pi}{\theta} v_1(\theta) \;;\; u_2(\theta) = \frac{\lambda i\pi}{\theta} u_1(\theta) \;;\; c_2(\theta) = - \frac{\lambda i\pi}{\theta} c_1(\theta) \;;\; d_2(\theta) = - \frac{\lambda i\pi}{\theta} d_1(\theta)$$

(1e)

Let us now discuss some interesting proprieties of the solution (1a - e). The pole contained in $c_1(\theta)$ generates the well known spectrum[15]

$$m_n = m \frac{\sin\frac{n\pi\lambda}{2}}{\sin\frac{\pi\lambda}{2}} \qquad n = 1,2,\ldots, N-1 \quad .$$

In particular we have $m_{N-1} = m$. The fact that the bound state of N-1 particles has the same mass as the original particles is one ingredient in the proof that the symmetry of our solution (1.a-e) is SU(N). In fact there are two bound states of (N-1)

particles with mass m in the $\overline{N}$ channel and one verifies that they have identical scattering amplitudes as the original anti-particles $\overline{b}$ and $\overline{f}$. This entails the following identification of antiparticles with bound states of (N-1) particles: the antiboson is identified with the bound state of (N-1) fermions and the antifermion is the bound state of (N-2) fermions with one boson. Symbolically we may write

$$\overline{b} = f_1 f_2 \cdots f_{N-1}$$

$$\overline{f} = f_1 \cdots f_{N-2} b_{N-1} + f_1 \cdots f_{N-3} b_{N-2} f_{N-1} + \cdots + b_1 f_2 \cdots f_{N-1} .$$

This implies that our fundamental particles obey generalized statistics, which should be treated as in ref. 16.

For N = 2 equs (1a-e) prevent the existence of the elemetary O(3) triplet in the supersymmetric non linear σ model. This is also signaled by the vanishing of the (would be) fermion-boson reflection amplitude in the O(3) case[7]. Hence we expect the O(3) spectrum to consist only of SU(2) kinks with scattering amplitudes given by eq. (1a-e).[*4]

To exhibit the similarity of the supersymmetrization process with the one occurring in the non linear σ-model we rewrite $c_1(\theta)$ as

$$c_1(\theta) = u(\theta,\lambda) \; \hat{Y}_o(\theta,\lambda)$$

where

$$u(i\pi\phi,\lambda) = \frac{\Gamma(\frac{\phi}{2} - \frac{\lambda}{2}) \; \Gamma(1 - \frac{\phi}{2})}{\Gamma(\frac{1}{2} - \frac{\phi}{2} - \frac{\lambda}{2}) \; \Gamma(\frac{\phi}{2})}$$

$$\hat{Y}_o(i\pi\phi,\lambda) = \prod_{\ell=0}^{\infty} \frac{\Gamma(\frac{\phi}{2} + \frac{\lambda}{2} + \ell) \; \Gamma(1 - \frac{\phi}{2} + \ell)}{\Gamma(1 - \frac{\phi}{2} + \frac{\lambda}{2} + \ell) \; \Gamma(\frac{\phi}{2} + \ell)} \; \prod_{\ell=1}^{\infty} \frac{\Gamma(\frac{\phi}{2} - \frac{\lambda}{2} + \ell) \; \Gamma(1 - \frac{\phi}{2} + \ell)}{\Gamma(\frac{1}{2} - \frac{\phi}{2} - \frac{\lambda}{2} + \ell) \; \Gamma(\frac{\phi}{2} + \ell)}$$

Note that the pole is present in $u(\theta,\lambda)$ and $\hat{Y}_o(\theta,\lambda)$ is "mini-mal" in the sense of having no poles in the physical sheet. By the substitution $\lambda \to -\lambda$ the pole "moves" from $u(\phi,\lambda)$ to $\hat{Y}_o(\theta,\lambda)$ leaving $c_1(\theta)$ invariant in the same way as in the super-symmetric non linear σ model[7]. The analogy is now stressed by observing that the amplitude $u(\theta,\lambda)$ (with the pole) belongs to the chiral Gross-Neveu model[11,12] and $u(\phi,-\lambda)$ belongs to the CP^{N-1} model minimally coupled to massless fermions[4]. The function $\hat{Y}_o(\phi,\lambda)$ describes the supersymmetry.

Finally let us mention that it is possible to introduce a supersymmetric $Z(N)$ model that generalizes the one studied in ref (17). The fermion-boson transmission amplitude is the product of the minimal supersymmetry function $\hat{Y}_o$ times the amplitude of the ordinary $Z(N)$ model[17]. This will be presented in a separate paper[18].

FOOT NOTES

*1 As was explained in ref (7) this is even more transparent
    from the S-matrix point of view.

*2 In fact this also happens in the supersymmetric non linear
    σ model where one parameter has to be fixed[7].

*3 In the notation of ref (14) this would correspond to class
    II solution

*4 This expectation may possible be extended to the usual
    O(3) non linear σ-model.

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