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ON-THE-ENERGY-SHELL APPROXIMATION FOR THE HEAVY ION COUPLED-CHANNELS PROBLEMS (I)

Consequences on the Sub-Barrier Elastic Scattering†

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ABSTRACT

Starting with the coupled channels equations describing multiple Coulomb excitations in heavy ion collisions we develop an approximation scheme based on replacing the channel Green's functions by their on-the-energy shell forms, which permits an exact analytic solution for the scattering matrix. We construct the trivially equivalent Coulomb polarization potential valid for strong coupling and small energy loss in the excitation processes. This potential is seen to have a very simple r-dependence. A simple formula for the sub-barrier elastic scattering cross section is then derived both by using the WKB approximation and by summing the Born series for the T-matrix. Comparison of the two forms for the elastic cross section shows that they give almost identical numerical results in the small coupling limit only. We also compare our results with the predictions of the Alder-Winther theory.

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I. INTRODUCTION

Multistep processes are the rule rather than the exception in heavy ion collision phenomena.[1] To deal with these processes one has to perform a coupled channels calculation which becomes prohibitively costly as the number of channels increases. Several approximation schemes have been developed which have in common the basic aim of reducing the computation time. In particular, due to the short wave lengths that characterize these reactions, several forms of semiclassical approximations were developed and put into test recently giving overall reasonable results. Of these theories we mention that developed by the Copenhagen group[2] and the methods based on the work of Miller[3] developed primarily by the Berkeley group.[4] However, the intrinsic difficulty of performing a many coupled channel calculation is not completely overcome by these methods. The Winther-de Boer code for Coulomb excitation can handle few channels; the method developed by the Berkeley group can, in principle, handle a larger number of coupled channels but is restricted to cases where a classical hamiltonian function can be constructed. Most of the numerical results given in Ref.4 were obtained for back angle scattering in which case the resulting geometrical simplification makes possible a speedy calculation. Extension of the methods of Ref. 4 to three dimensions is still pending.[5]

In the present series of papers we try an alternative method which is based on the use of the on-energy-shell form of the channel Green's functions. Although several authors have discussed this method in several contexts,[6] we believe that a study of the full consequence of the OES approximation is called for. Moreover, this approximation scheme has been adopted recently[7] to extend the closed formalism approach developed by Frahn[8] by including

specific multistep processes in heavy ion collision phenomena. An important test is a comparison of the range of applicability of the OES approximation with the already established results of multiple Coulomb excitation theory.[9]

The paper is organized as follows: in Section II we formulate the coupled-channels problem for multiple Coulomb excitation. In Section III we introduce the on-energy-shell approximation for the channel Green's functions and derive an expression for the locally equivalent Coulomb polarization potential (LECPP) in closed form. In Section IV we demonstrate, through numerical calculation, that the cross section for sub-barrier elastic scattering calculated using the WKB approximation and using the LECPP is almost identical to that calculated by summing the Born series for the elastic amplitude without any reference to the LECPP. Discussion of our results as well as suggestions for possible improvements and the conclusions are given in Section V.

II. THE QUANTUM MECHANICAL COUPLED CHANNEL EQUATIONS FOR MULTIPLE COULOMB EXCITATION

We consider the collision of a spherical nucleus 1 on a deformed target nucleus 2, at sub-barrier energies. We study the Coulomb excitation of low-lying states IM of spin I and magnetic quantum number M with excitation energy E_I. We consider nucleus 1 as a point charge, $Z_1 e$. For the total wave function in the center of mass system we use the expansion[9]

$$|\psi\rangle = \sum_{\substack{\ell_o \ell \\ IJN}} \frac{\sqrt{\pi(2\ell_o+1)}}{k_{I_o} r} i^{\ell_o+1} e^{i\sigma_{\ell_o}(\eta_{I_o})} \langle \ell_o\, 0\, I_o M_o | JN\rangle \times \psi^{\ell_o I_o}_{(\ell I)J}(r) |(\ell I)JN\rangle \tag{1}$$

where the channel wave functions are given by

$$|(\ell I)JN\rangle = \sum_{Mm} \langle \ell m\ IM|JN\rangle\ |IM\rangle\ Y_{\ell m}(\hat{r}). \tag{2}$$

Here J is the total conserved channel angular momentum and N its projection on the Z-axis. The relative position of the centers of mass of 1 and 2 is denoted by r while the superindices $\ell_o I_o$ indicate the initial condition. Inserting the wave function (1) into the time-independent Schrodinger equation

$$(H-E)|\psi\rangle = 0. \tag{3}$$

we obtain a system of coupled equations for the radial wave functions

$$\left(\frac{d^2}{dr^2} + k_I^2 - \frac{2\mu}{\hbar^2}\frac{\ell(\ell+1)}{2\mu r^2} - \frac{2\mu}{\hbar^2}\frac{Z_1 Z_2 e^2}{r}\right) \psi^{\ell_o I_o}_{(\ell I)J}(r) = \sum_{\ell' I'} V^J_{\ell I, \ell' I'}(r)\ \psi^{\ell_o I_o}_{(\ell' I')J}(r) \tag{4}$$

where

$$k_I^2 = \frac{2\mu}{\hbar^2}(E-E_I)$$

The coupling potential $V^J_{\ell I, \ell' I'}$ is given by

$$V^J_{\ell I, \ell' I'}(r) = \frac{4\sqrt{\pi} Z_1 e\mu}{\hbar^2} \sum_{\lambda=1} \frac{(2\ell+1)(2\ell'+1)}{(2\lambda+1)} (-1)^{J+I'+\lambda} \tag{5}$$

$$\times \langle I'||M(E\lambda)||I\rangle \begin{pmatrix} \ell & \ell' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J & \ell' & I' \\ \lambda & I & \ell \end{Bmatrix} \frac{1}{r^{\lambda+1}}$$

where $\langle I'||M(E\lambda)||I\rangle$ is the reduced matrix element of the electric λ-pole moment of the target. We shall, in the following, consider the specific case of a quadrupole-deformed even-even target nucleus 2. Assuming a rotational band structure for the low-lying excited state of 2, the coupling

potential matrix $V^{\ell_o}_{\ell I,\ell' I'}(r)$ becomes

$$V^{\ell_o}_{\ell I,\ell' I'}(r)=\sqrt{a_I a_{I'} \eta_I \eta_{I'}}\; q_{I\to I'} \begin{pmatrix} I & I' & 2 \\ 0 & 0 & 0 \end{pmatrix} \sqrt{(2I+1)(2I'+1)}$$

$$\times \sqrt{(2\ell+1)(2\ell'+1)}\,(-1)^{\ell+I'} \begin{pmatrix} \ell & \ell' & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

$$\begin{Bmatrix} \ell & \ell' & I \\ 2 & I & \ell \end{Bmatrix} \frac{1}{r^3} \equiv a_{\ell I \ell' I'} \frac{1}{r^3}$$

where a_I is half the distance of closest approach for head on collision in channel I, $a_I = \dfrac{Z_1 Z_2 e^2}{2(E-E_I)}$, $\eta_I = \dfrac{Z_1 Z_2 e}{\hbar v_I} = k_I a_I$ is the Sommerfeld parameter in channel I and $q_{I\to I'}$ is the symmetrized dimensionless quadrupole strength parameter for the coupling $I\to I'$ and is defined by

$$q_{I\to I'} = \sqrt{\frac{\pi}{5}}\; \frac{\sqrt{\eta_I \eta_{I'}}}{a_I a'_I}\; \frac{\langle I||M(E2)||I'\rangle}{Z_2 e} \times$$

$$\frac{1}{\sqrt{(2I+1)(2I'+1)}\begin{pmatrix} I & I' & 2 \\ 0 & 0 & 0 \end{pmatrix}} \tag{7}$$

In the limit of a pure quadrupole rotational band and zero energy loss in the different excitation processes we have

$$q_{I\to I'} = (-1)^{\frac{I+I'+2}{2}}\; q_{0\to 2} \tag{8}$$

Once the $\psi^{\ell_o I_o}_{(\ell I)J}(r)$ are obtained from Eq. (4) the corresponding T-matrices $T^J_{\ell I,\ell_o I_o}$ are extracted from their asymptotic forms, i.e.,

$$\psi^{\ell_o I_o}_{(\ell I)J}(r) \overset{r\to\infty}{=} \delta_{I I_o}\delta_{\ell \ell_o} e^{-i\phi_{\ell I}(r)} - \sqrt{\frac{k_{I_o}}{k_I}}\; e^{i\phi_{\ell I}(r)}\; T^J_{\ell I,\ell_o I_o} \tag{9}$$

where $\phi_{\ell I}(r) = k_I r - \eta_I \ln(2k_I r) - \frac{\pi}{2}\ell + \sigma_\ell(\eta_I)$

σ_ℓ being the Coulomb phase shift. With the $T^J_{\ell I, \ell_o I_o}$ one can then calculate the amplitudes $f_{I_o M_o \to IM}(\theta,\phi)$ for Coulomb excitation from the ground state $|I_o M_o\rangle$ to the final state $|IM\rangle$ (in coordinate system C of Ref. (10))

$$f_{I_o M_o \to IM}(\theta,\phi) = \frac{i\sqrt{\pi}}{\sqrt{k_I k_{I_o}}} \sum_{\ell \ell_o J} \sqrt{(2\ell_o+1)} \langle \ell_o 0\ I_o M_o | JM_o\rangle$$

$$i^{\ell_o - \ell} \langle \ell m\ IM | JM_o\rangle \{\delta_{II_o}\delta_{\ell\ell_o} - e^{i(\sigma_\ell(\eta_I)+\sigma_{\ell_o}(\eta_{I_o}))}$$

$$* \; T^J_{\ell I, \ell_o I_o}\} \; Y_{\ell m}(\theta,\phi) \qquad (10)$$

The differential cross sections are then calculated as usual.

III. THE ON-ENERGY-SHELL APPROXIMATION AND THE EQUIVALENT LOCAL COULOMB POLARIZATION POTENTIAL

In order to calculate the cross section for inelastic Coulomb scattering one has to solve Eq. (4) with the appropriate boundary condition of an incoming wave present only in the elastic channel. For fuller details we refer the reader to Refs. 9 and 10. It is important to recognize that with an increasing number of coupled channels the solution to Eq. (4) becomes more and more complicated. For our purposes we adopt the on-energy-shell approximation for the channel Coulomb Green's function[11]

$$G^{(+)}_{(\ell I)}(r,r') \tilde{=} -\frac{i}{k_I} F_\ell(k_I r_<)\, F_\ell(k_I r_>) \qquad (11)$$

We shall assess the accuracy of this approximation later. In the above, $F_\ell(kr)$ is the regular Coulomb function and $r_<(r_>)$ corresponds to the smaller (larger) of r and r'.

The solution of Eq. (4) may be written as an integral equation

$$\psi_{(\ell I)\ell_o}(r) = F_{\ell_o}(k_o r)\,\delta_{II_o}\,\delta_{\ell\ell_o} + \sum_{\ell' I'} \int G_{\ell I}(r,r')\, a_{\ell I \ell' I'}\, \frac{1}{r'^3}\, \psi_{(\ell' I')\ell_o}(r')\,dr' \tag{12}$$

We now use the approximation (11) in (12) and obtain

$$\psi_{(\ell I)\ell_o}(r) = F_{\ell_o}(k_o r)\,\delta_{II_o}\,\delta_{\ell\ell_o} + \left(\frac{-i}{k_I}\right) F_\ell(k_I r) \sum_{\ell' I'} \int_0^\infty dr' \tag{13}$$

$$\times\, F_\ell(k_I r')\, a_{\ell I \ell' I'}\, \frac{1}{r'^3}\, \psi_{(\ell' I')\ell_o}(r')$$

Multiplying (12) by $\frac{1}{r^3}\frac{1}{k_{I''}} F_{\ell''}(k_{I''} r)$ and integrating over r we obtain the following relation

$$M_{\ell''\ell} = \frac{1}{k_{I''}} I_{\ell''\ell}(k_{I''},k_I)\,\delta_{II_o}\,\delta_{\ell\ell_o} + \left(\frac{-i}{k_I}\right) \sum_{\ell'} I_{\ell''\ell}(k_{I''},k_I)\, a_{\ell I \ell' I'}\, M_{\ell\ell'} \tag{14}$$

where we have defined the coefficients

$$M_{\ell\ell'} \equiv \frac{1}{k_I} \int dr'\, F_\ell(k_I r')\, \frac{1}{r'^3}\, \psi_{(\ell' I')\ell_o}(r') \tag{15}$$

and the integrals $I_{\ell''\ell}(k''_I,k_I)$ are the usual Coulomb excitation integrals[9]

$$I_{\ell''\ell}(k_{I''},k_I) = \int dr\, F_{\ell''}(k_{I''} r)\, \frac{1}{r^3}\, F_\ell(k_I r) \tag{16}$$

Eq. (14) can easily be solved for $\sum_{\ell' I'} a_{\ell I \ell' I'} M_{\ell \ell'} \equiv Y_{\ell I}$

$$Y_{\ell I} = \sum_{\ell' I'} \{[\underline{1} + i \underline{C}]^{-1}\}_{\ell I \ell' 2} \frac{1}{k_o} a_{\ell' 2 \ell_o 0} I_{\ell' \ell_o}(k_2, k_o) \qquad (17)$$

where we have introduced the coupling matrix C whose matrix elements are given by

$$C_{\ell'' I'', \ell I} = \frac{1}{k_{I''}} a_{\ell'' I'' \ell I} I_{\ell'' \ell}(k_{I''}, k_I) \qquad (18)$$

The matrix $(\underline{1} + i\underline{C})^{-1}$ acts in the full channel space.

Since $a_{\ell' I' \ell_o 0} I_{\ell' \ell_o}$ corresponds to the coupling of the elastic channel to the 2+ channel (due to the quadrupole nature of the coupling only the 2+ can couple directly to the 0^+) the values of ℓ' are completely determined by the 3-j symbols that appear in $a_{\ell' 2 \ell_o 0}$, i.e., $\begin{pmatrix} \ell_o & \ell' & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, ℓ' can be ℓ_o, $\ell_o + 2$ or $\ell_o - 2$. This says that Eq. (17) may be rewritten as (labeling the matrix elements of $[\underline{1} + i \underline{C}]^{-1}$ by the intrinsic spin only)

$$Y_I = [[\underline{1} + i \underline{C}]^{-1}]_{I2} \underline{C}_{20} \qquad (19)$$

The vector Y contains I + 1 components (ℓ and I must always add to $J = \ell_o$) Therefore, the matrix $[\underline{1} + i \underline{C}]^{-1}$ is a $(I + 1) \times 3$ matrix. Once $Y_{\ell I}$ is evaluated, the wave function $\psi_{\ell I}(r)$, which is also a vector with I + 1 components, is then obtained from Eq. (13) vis

$$\psi_{(\ell I)\ell_o}(r) = F_{\ell_o}(k_o r)\, \delta_{I0}\, \delta_{\ell \ell_o} + (-i)\, F_\ell(k_I r) [\underline{1} + i \underline{C}]^{-1}_{I2} \underline{C}_{20} \qquad (20)$$

Eq. (20) can be used to calculate the trivially equivalent local Coulomb polarization potential in the elastic channel as was done in Refs. 11 and 12. Inserting the second term of Eq. (20) in the r.h.s. of Eq. (4) for $\psi_{(\ell_o 0)\ell_o}(r)$ we immediately obtain for the sum

$$\sum_{\ell' I'} V^{\ell_o}_{\ell_o 0 \ell' I'}(r)\ \psi^{\ell_o 0}_{(\ell' I')\ell_o}(r)$$

$$= a_{\ell_o 0 \ell 2}\ \frac{1}{r^3}\left(-i\right) F_\ell(k_2 r)\left[\underline{1} + i\underline{C}(\ell)\right]^{-1}_{22} C_{20}(\ell) \tag{21}$$

$$\equiv \frac{2\mu}{\hbar^2}\ V_{opt}(r)\ F_{\ell_o}(k_o r)$$

$$\text{or}\quad \frac{2\mu}{\hbar^2}\ V_{opt}(r) = -i(F(\ell;r))_{02}\quad \left[\underline{1} + i\ \underline{C}(\ell)\right]^{-1}_{22} C_{20}(\ell) \tag{22}$$

where $F(\ell;r)$ is a 3-component vector given by

$$(F(\ell;r))_{02} = \left(\frac{F_{\ell_o+2}(k_2 r)}{F_{\ell_o}(k_o r) r^3}\ a_{\ell_o 0, \ell_o+2, 2}\ ,\ \frac{F_{\ell_o}(k_2 r)}{F_{\ell_o}(k_2 r) r^3} a_{\ell_o 0, \ell_o 2}\right.$$

$$\left. \frac{F_{\ell_o-2}(k_2 r)}{F_{\ell_o}(k_o r) r^3}\ a_{\ell_o 0, \ell_o-2, 2}\right) \tag{23}$$

Eq. (22) describes the effect of the coupling of the elastic channel to all other inelastic channels. It is a simple realization of the Feshbach theory for the optical potential.[13] Denoting the projection operator that projects out the elastic channel of the full wave function by P and the complementary operator $Q \equiv 1-P$ which projects out all inelastic channels, we can write Eq. (23) as

$$V_{opt} = V_{PQ}[\underline{1} + i\ \underline{C}_Q]^{-1}\ V_{QP} \tag{24}$$

Therefore, the matrix $[\underline{1} + i\ \underline{C}_{QQ}]^{-1}$ is effectively the Q-space propagator.

In our particular case of multiple Coulomb excitation through the quadrupole coupling, V_{QP} is just V_{20} as only the 2^+ channel is coupled directly to the elastic, 0^+, channel. Within the Q-space the 2^+ channel couples directly only to the 4^+. The 4^+ then couples to the 6^+ state and so on. This suggests that the structure of the matrix $[\underline{1} + iC_{QQ}]^{-1}$ is

$$[\underline{1} + i\ \underline{C}_{QQ}]^{-1}_{22} = [\underline{1} + i\ \underline{C}_{22} + \underline{C}_{24}\ [\underline{1} + i\underline{C}_{Q-2\ Q-2}]^{-1}_{44}\underline{C}_{42}]^{-1} \tag{25}$$

where the symbol Q-2 refers to the subspace spanned by states in the Q-space except the 2^+. Eq. (25) can easily be derived by expanding the matrix propagator in $\underline{C}$ and inserting Q whenever calculating products of $\underline{C}$'s, i.e.,

$$\begin{aligned}[\underline{1} + i\ \underline{C}\ ]^{-1}_{22} &= \underline{1} - i\ \ \underline{C}_{22} - (i)^2\ \underline{C}_{2Q}\ Q\ \underline{C}_{Q2} \\ &\quad - (i)^3\ \underline{C}_{2Q}\ Q\ \underline{C}_{QQ}\ Q\ \underline{C}_{Q2} \\ &= \underline{1} - i\ \ \underline{C}_{22} - (i)^2\ (\underline{C}_{22})^2 \\ &\quad - (i)^2\ \underline{C}_{24}\ Q\text{-}2\ \ \underline{C}_{Q2} - (i)^3\ \underline{C}_{2Q}\ Q\text{-}2\ \underline{C}_{QQ}\ Q\text{-}2\ \underline{C}_{Q2} + \ldots\end{aligned} \tag{26}$$

Rearranging and summing the series (26) we obtain (25). It should be emphasized that once the matrix propagator is decomposed into propagation in

smaller subspaces as given in (25) the calculation of V_{opt} becomes quite simple. Realizing that C_{ij} is an (i+1)x(j+1) matrix (in magnetic quantum number space) such a calculation can then be easily made by inverting the matrix propagator as was done in Ref. 14. In particular if all coupling except the C_{20} were made equal to zero we obtain the potential considered in Refs. 12 and 15.

$$V_{opt}^{(2)}(r) = -\frac{i}{r^3}\,(F(\ell;r))_{02}C_{20}(\ell) \tag{27}$$

In the following we assume zero energy loss in the different excitation processes ($k_I = k_o$). We also utilize the simple relations, valid for large ℓ, among $F_\ell(kr)$, $F_{\ell+2}(kr)$ and $F_{\ell-2}(kr)$ obtained in Refs. 11 and 12,

$$\frac{F_{\ell+2}(kr)}{F_\ell(kr)} = -1 + \frac{2}{\ell^2+\eta^2}\left[\eta^2 + \frac{2\eta\,\ell^2}{kr} + \frac{\ell^4}{k^2r^2}\right] = \frac{F_{\ell-2}(kr)}{F_\ell(kr)}\,. \tag{28}$$

Finally, we shall use the large ℓ values of the 3-j and 6-j symbols needed in the calculation and given in Appendix I. With the above approximations and assumptions the expression for $V_{opt}^{(2)}(r)$ given in (27) reduces to the form derived in Refs.12 and 15.

$$V_\ell^{(2)}(r) = -i\,\frac{2}{5}\,\frac{E}{\eta}\,q_{0\to2}^2\,g_{0\to2}(\xi_{0\to2})\,\left(\frac{a}{r}\right)^3\left[\frac{3\bar{\ell}^2+1}{\bar{\ell}^2(\bar{\ell}^2+1)^2} - \frac{1}{\bar{\ell}^3}\arctan\bar{\ell}\right.$$
$$+\frac{4\,\bar{\ell}^2}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right) \tag{29}$$
$$\left.+\frac{2\bar{\ell}^4}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right)^2\right]$$

where we have introduced the quantity $\bar{\ell} \equiv \frac{\ell + 1/2}{\eta}$ and have used the explicit forms of the Coulomb integrals $I_{\ell\ell'}(k_I=k_o)$ as given in Ref. 9,

$$I_{\ell\ell+2} = \frac{\eta^2}{[(\ell+1)^2 + \eta^2][(\ell+2)^2 + \eta^2]} \frac{1}{6a^2} \simeq \frac{1}{6a^2} \frac{1}{\bar{\ell}^2+1} \simeq I_{\ell\ell-2} \tag{30}$$

$$I_{\ell\ell} \simeq \frac{1}{\bar{\ell}^2} \left(1 - \frac{\arctan \bar{\ell}}{\bar{\ell}}\right) \frac{1}{2a^2} \tag{31}$$

We corrected for the $k_I \neq k_o$ approximation by inserting the semiclassical energy loss factors $g_{I\to I'}(\xi_{I\to I'})$, for the process $I \to I'$,[16] where $\xi_{I\to I'}$ is the adiabaticity parameter, which is given by

$$\begin{aligned} \xi_{I\to I'} &\equiv \eta_{I'} - \eta_I = \eta_o \sqrt{E} \left[\frac{1}{(E-E_{I'})^{1/2}} - \frac{1}{(E-E_I)^{1/2}}\right] \\ &\simeq \frac{1}{2} \eta_o \frac{(E_{I'}-E_I)}{E} \end{aligned} \tag{32}$$

where η_o and E are, respectively, the Sommerfeld parameter and the center of mass energy in the elastic channel. E_I is the excitation energy of state I. Numerical values of $g_{I\to I'}(\xi_{I\to I'})$ are given in Ref. 16. It is easily seen that the simple r-dependence of the potential given in (29) also holds for the more exact potential of Eq. (22) if the same assumptions and approximations are made,[11] i.e.,

$$iV_{opt}(r) = \frac{a_\ell}{r^3} + \frac{b_\ell}{r^4} + \frac{c_\ell}{r^5} \tag{33}$$

This is clear since the r-dependence of $V_{opt}(r)$ is contained only in the vector $F(r;\ell)$. The propagator $[\underline{1} + i\ \underline{C}(\ell)]^{-1}$, which is the quantity used to get approximate forms for $V_{opt}(r)$, does not depend on r. In Eq.(33).

the complex coefficients, a_ℓ, b_ℓ and c_ℓ depend on the orbital angular momentum, ℓ, the Sommerfeld parameter, η, the center of mass energy, E, and the quadrupole coupling strength q_{ij}. In Ref. 15 the potential $V_{opt}(r)$ was derived using the semiclassical theory of Coulomb excitation developed by Alder and Winther.[9] The ℓ- and r-dependence of $V_{opt}(r)$ of Eq. (33) and that of Ref. 15 are quite different. One advantage of $V_{opt}(r)$ of Eq. (33) is its simple r-dependence as well as the explicit ℓ-dependence which can be obtained by inverting the matrix $(\underline{1}+i\underline{C}(\ell))$ in Eq. (22). We shall give below the result for the case of the coupling of the 0^+ channel to the 2+ channel including the reorientation of the 2+ to all orders. We shall also present the result when including the coupling to the 4+ channel. In order to perform these calculations we first rewrite Eq. (25) in the equivalent, but more transparent form

$$[\underline{1} + i\ \underline{C}(\ell)]^{-1}_{22} = [1+i\underline{C}_{22}(\ell)+\underline{C}_{24}(\ell)[1+i\underline{C}_{44}(\ell)+\underline{C}_{46}(\ell)[\ldots]\ ]^{-1}\ C_{42}(\ell)]^{-1} \tag{34}$$

In the above each coupling matrix $\underline{C}_{ij\neq i}(\ell)$ contains an appropriate energy loss factor $\sqrt{g_{ij}(\xi_{ij})}$ whereas the reorientation matrices $\underline{C}_{ii}(\ell)$ do not. Therefore, the Coulomb polarization potential for the two channel case with reorientation is given by

$$V^{(2)Reor.}_{opt}(r) = -\frac{i}{r^3}\left(\frac{\hbar^2}{2\mu}\right)(F(r;\ell))_{02}\,[\underline{1} + i\ \underline{C}_{22}(\ell)]^{-1}\ \underline{C}_{20}(\ell) \tag{35}$$

The matrix $\underline{C}_{22}(\ell)$ has the form

$$\underline{C}_{22}(\ell) = \frac{2}{7} q_{2\to 2} \begin{pmatrix} \alpha & \beta & 0 \\ \beta & -\alpha & \beta \\ 0 & \beta & \alpha \end{pmatrix} \tag{36}$$

where we have defined

$$\alpha \equiv -\frac{1}{\bar{\ell}^2}\left(1 - \frac{\tan^{-1}\bar{\ell}}{\bar{\ell}}\right)$$
$$\beta \equiv \frac{1}{\sqrt{6}} \frac{1}{1+\bar{\ell}^2} \tag{37}$$

Since $\underline{C}_{22}(\ell)$ is symmetric and real we could immediately invert the propagator and write down the expressions for the real and imaginary part of $V_{opt}^{(2)Reor}$ utilizing the form for $\underline{C}_{20}(\ell)$

$$\underline{C}_{20}(\ell) = \frac{1}{\sqrt{5}} q_{0\to 2}\, g_{02}(\xi_{0\to 2}) \begin{pmatrix} \beta \\ -\alpha \\ \beta \end{pmatrix} \tag{38}$$

$$\mathrm{Re}\, V_{opt}^{(2)Reor}(r) = -\frac{4}{35}\left(\frac{E}{\eta}\right) q_{0\to 2}^2\, q_{2\to 2} \left(\frac{a}{r}\right)^3 g_{02}(\xi_{0\to 2})\, x$$

$$\left[1 + \frac{4}{49} q_{2\to 2}^2 \left(\frac{1}{\ell^4}\left(1 - \frac{\arctan\bar{\ell}}{\bar{\ell}}\right)^2 + \frac{1}{3}\frac{1}{(1+\bar{\ell}^2)^2}\right)\right]^{-1} x$$

$$x \left[\frac{(1-\arctan\bar{\ell}/\bar{\ell})^2}{\bar{\ell}^4} + 2\,\frac{(1-\arctan\bar{\ell}/\bar{\ell})}{\bar{\ell}^2}\left(\frac{\bar{\ell}^2-1}{(\bar{\ell}^2+1)^2}\right)\right.$$

$$- \frac{1}{3(\bar{\ell}^2+1)^2} - 8\frac{(1-\arctan\bar{\ell}/\bar{\ell})}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right)$$

$$-4\,\frac{\bar{\ell}^2(1-\arctan\bar{\ell}/\bar{\ell})}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right)^2] \qquad (39)$$

and

$$\mathrm{Im}\, V_{opt}^{(2)Reor}(r) = [1 + \frac{4}{49} q_{2\to2}^2 (\frac{1}{\bar{\ell}^4}(1-\arctan\bar{\ell}/\bar{\ell})^2 + \frac{1}{3}\frac{1}{(\bar{\ell}^2+1)^2})]^{-1} V_\ell^{(2)}(r) \qquad (40)$$

where $V_\ell^{(2)}(r)$ is given in Eq. (29). Eq. (40) is the potential given in Refs. 11 and 14. It is interesting to notice that Re $V_{opt}^{(2)Reor}(r)$ becomes identically zero when $q_{2\to2} = 0$, i.e., no reorientation effect. As a matter of fact this result is more general, Re $V_{opt}(r) = 0$ when all reorientation couplings vanish, $q_{ii} = 0$, as can be seen from Eq. (34) and (22). The above observation sheds some light on the results of Ref. 17 where $V_{opt}(r)$ was calculated assuming a harmonic vibrational spectrum for the target nucleus (all $q_{ii} = 0$) and it was found that $V_{opt}(r)$ is purely imaginary.

Including the coupling to the 4^+ state to all orders results in a potential $V_\ell^{(4)}(r)$ which has the form [14]

$$V_\ell^{(4)}(r) = -i\,\underline{F}(r,\ell)\,\frac{1}{1+i\underline{C}_{22}(\ell) + \underline{C}_{24}(\ell)\underline{C}_{42}(\ell)}\,\underline{C}_{20}(\ell)$$

$$= -i\,\frac{2}{5}\frac{E}{\eta} q_{0\to2}^2 \left(\frac{a}{r}\right)^3 g_{02}(\xi_{02})\{(-1+f(r))\;[1+\frac{3}{49} q_{2\to4}^2 g_{24}(\xi_{24})(\alpha^2+\frac{\gamma^2}{3}) - i\,\frac{4}{7} q_{2\to2}\,\alpha]\,\gamma$$

$$- (1 + \frac{6}{49} q_{2\to4}^2 g_{24}(\xi_{24})(\alpha^2+\frac{\gamma^2}{3}))\,\alpha$$

$$+ i\,\frac{2}{7} q_{2\to2}(\frac{\gamma^2}{3} - \alpha^2)\}\;x$$

$$x\,\{(1+\frac{3}{49}\; q_{2\to4}^2\; g_{24}\;(\xi_{24})(\alpha^2+\frac{\gamma^2}{3}))(1 + \frac{36}{245}\; q_{2\to4}^2\; g_{24}(\xi_{24})(\alpha^2+\frac{\gamma^2}{3}))$$

$$+ \frac{4}{49} q_{2\to2}^2 (\alpha^2 + \frac{\gamma^2}{3}) - i \frac{18}{15x49} q_{2\to4}^2 g_{2\to4}(\xi_{24}) q_{2\to2}(\gamma^2-\alpha^2)\alpha\}^{-1} \qquad (41)$$

where f(r) is given by

$$f(r) = \frac{2}{1+\bar{\ell}^2} [1 + 2\bar{\ell}^2(\frac{a}{r}) + \bar{\ell}^4(\frac{a}{r})^2]$$

and $\gamma = \beta\sqrt{6}$. Eq. (41) reduces to (39) and (40) in the limit $q_{2\to4} = 0$. It is clear that one could calculate explicitly $V_{opt}(r)$ to whatever order desired. However, such expressions become more and more complicated as already indicated in Eq. (41). Instead one could simply invert the matrix propagator numerically in conjunction with optical model calculations. In Fig. 1 we show the coefficients a_ℓ, b_ℓ and c_ℓ as function of $\bar{\ell}$ for the different cases studied above as well as for that including the excitation of up to the $I = 16^+$ state. It is clear from Fig. 1 that the imaginary part of $V_{opt}(r)$, determined by Re a_ℓ, Re b_ℓ and Re c_ℓ, behaves basically like r^{-3} for small values of $\bar{\ell}$ whereas for large values of $\bar{\ell}$ it goes as r^{-5}. This fact seems to hold irrespective of the value of the quadrupole coupling strength q. The above result continues to hold when the energy loss is properly included by use of the semiclassical energy loss factors as discussed before. The real part of $V_{opt}(r)$ shows a similar behavior to Im $V_{opt}(r)$ namely it goes as r^{-3} for small $\bar{\ell}$ and as r^{-5} for large $\bar{\ell}$. For intermediate values of $\bar{\ell}$ both the real and the imaginary parts of $V_{opt}(r)$ show the general r-dependence of Eq. (33). The q-dependence of V_{opt} is shown in Fig. 2. The coefficient a, b_ℓ and c_ℓ have a rather smooth dependence on q. Notice that for $\bar{\ell} = 0$ the coefficients $b_{\bar{\ell}=0}$ and $c_{\bar{\ell}=0}$ are identically zero for all q. This corroborates the discussion above about the r-dependence of $V_{opt}(r)$. It is instructive to compare our results for $V_{opt}(r)$ with the potential derived in Ref. 15 where the semiclassical theory

of multiple Coulomb excitation has been used. From the closed expression for the elastic amplitude for $\bar{\ell} = 0$ and $\xi = 0$ (assuming a pure rotational band) obtained by Alder and Winther,[9] one can extract the $V_{opt}(r)$ following the method developed in Ref. 15.

$$\text{Re } V^{AW}_{\ell=0}(r) = -3\frac{E}{\eta}\left[\frac{2}{3}q - \arctan\left(\frac{S_2(2q)}{C_2(2q)}\right)\right]\left(\frac{a}{r}\right)^3 \tag{42}$$

$$\text{Im } V^{AW}_{\ell=0}(r) = -3\frac{E}{\eta}\ln\left[\sqrt{\frac{\pi}{4q}}\left\{[C_2(2q)]^2 + [S_2(2q)]^2\right\}.\right]\left(\frac{a}{r}\right)^3$$

where C and S are the Fresnel integrals.[9]

We have compared our potentials of Eq. (22) with that of Alder and Winther given in (42) and the results are shown in Fig. 2. where we have plotted $V_{\ell=0}$ as a function of the quadrupole strength parameter q_{02}. It is clear that our potential is ~50% smaller than V^{AW}. More importantly we do not reproduce the broad oscillations seen in $\text{Im}V^{AW}_{\ell=0}(q)$. One possible reason for the disagreement can certainly be tracked down to a shortcoming of the on-energy-shell approximation. Another possibility is the fact that the potential given in (42) is extracted from the elastic amplitude, an asymptotic quantity, and therefore is a phase-shift equivalent potential whereas our potential is a wave function equivalent potential being obtained directly from the wave function. We shall address ourselves here to the question of the on-shell approximation and its limitations. Implicit in our calculation is the neglect of all terms involving the irregular Coulomb solution $G_\ell(kr)$. If we were to calculate the potential to first order, as was done in Ref. 12 such an approximation is all right since it amounts to neglecting terms of the type (assuming zero energy loss)

$$\int F_{\ell}(kr)\,\frac{1}{r^3}\,G_{\ell'}(kr)\,dr \qquad (43)$$

with $\ell' = \ell + 2, \ell-2, \ell$. It is clear that for large η, the Sommerfeld parameters, F_ℓ and G_ℓ oscillate out of phase and therefore the above term may be neglected as compared to the dominant term $\int F_{\ell}(kr)\,\frac{1}{r^3}\,F_{\ell'}(kr)dr$. It would seem reasonable to adopt the above approximation, i.e., neglect all integrals involving F_ℓ and G_ℓ even in the calculation of higher order terms in the potential. However, upon an inspection of our series (12), one notices immediately that terms like $\int G_\ell \frac{1}{r^3} G_\ell$ start appearing and one certainly cannot neglect them as they could be as important as the terms involving pairs of F_ℓ's. Short of actually taking full account of these terms we, instead, attached one parameter, α, to all the reorientation matrices $C_{ii}(\ell)$ and another one, ,to all the coupling matrices $C_{ij}(\ell)$ in our expression for $V_{opt}(r)$ and adjusted these two parameters to obtain a best fit to the Alder-Winther potential of Eq. (42). Our results are shown in Fig. 3. The final adjusted values of α and β are 2.23 and 4.46, respectively. It is clearly seen that the real part can be nicely fitted, the imaginary part, however, fits only on the average without exhibiting any oscillations. The fact that the parameters α and β which fit the average behavior of V^{AW} come out to be greater than unity is an indication that the neglected terms if included, would tend to enhance the matrix elements $C_{ii}(\ell)$ and $C_{ij}(\ell)$.

IV. THE SUB-BARRIER ELASTIC CROSS SECTION

Once the optical potential is obtained one may then calculate the sub-barrier elastic scattering cross section either from an exact, one-channel, optical model calculation or simply by using the WKB approximation. Due to the fact that the number of partial waves involved in the heavy ion reactions we are considering is quite large, one may use the usual arguments of replacing the partial wave sum in the elastic amplitude by an integral and perform the integration using the stationary phase method. Insofar as the real part of $V_{opt}(r)$ of Eq. (22) is quite small as compared with the dominant monopole-monopole Coulomb potential one may then use the Coulomb deflection function to relate $\bar{\ell} = \frac{\ell+1/2}{\eta}$ to the c.m. scattering angle θ,

$$\bar{\ell} = \cot \frac{\theta}{2} \tag{44}$$

A numerical calculation was reported in Ref. 14 where the $V_{opt}^{(2)Reor}(r)$ corresponding to two channel coupling with reorientation was incorporated into an optical model code. The deflection function extracted from the resulting phase shifts was found to be very close to a pure Rutherford deflection function, Eq. (44). This gives us confidence in using the WKB formula for $\sigma_{e\ell}$ given in Refs.2 and 18, which adapted to our case, becomes

$$\frac{\sigma_{e\ell}}{\sigma_R}(\theta) = \exp\left[-2\,\frac{2\mu}{\hbar^2}\int_{r_t(\bar{\ell}(\theta))}^{\infty} \frac{\mathrm{Im}\, V_{opt}(r)}{k_{\bar{\ell}(\theta)}(r)}\, dr\right] \tag{45}$$

where $\bar{\ell}$ is given by Eq. (44), and

$$k_{\bar{\ell}}(r) = \left[\frac{2\mu}{\hbar^2} E\left(1 - \bar{\ell}^2 \frac{a^2}{r^2} - 2\frac{a}{r}\right)\right]^{1/2} \tag{46}$$

and $r_t(\bar{\ell})$ is the classical turning point determined by the larger root of $k^2_{\bar{\ell}}(r) = 0$. From the general r-dependence of our potential given in Eq. (33), Eq. (45) may be rewritten in the form

$$\frac{\sigma_e}{\sigma_R}(\theta) = \exp\{-[\mathrm{Re}\, a_{\bar{\ell}(\theta)} I_3(\theta) + \mathrm{Re}\, b_{\bar{\ell}(\theta)} I_4(\theta) + \mathrm{Re}\, c_{\bar{\ell}(\theta)} I_5(\theta)]\} \tag{47}$$

where

$$I_i(\theta) \equiv 2\left(\frac{2\mu}{\hbar^2}\right) \int_{r_t(\bar{\ell})}^{\infty} \frac{1}{(r)^i k_{\bar{\ell}(\theta)}(r)}\, dr \tag{48}$$

and $\mathrm{Re}\, a_{\bar{\ell}(\theta)}$, etc. are functions only of the center of mass angle, the quadrupole strength parameters q_{ij} and the energy loss factors $g_{ij}(\xi_{ij})$. The integrals (48) can be evaluated in closed form and the final expressions for i = 3, 4 and 5 are collected in Appendix II.

The above formula for $\frac{\sigma_{e\ell}(\theta)}{\sigma_R}$ was used in Refs.11 and 12 to calculate the Coulomb damping in the elastic channel. In Ref.12 the Coulomb orbit integrals of Eq. (48) were evaluated in a slightly different way from our expressions given in Appendix II but comparison of our expression for $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$ with the $V_{opt}(r)$ given in Eq. (29), i.e., $V^{(2)}_{\ell}(r)$ and with the $I_i(\theta)$ given in Appendix II and their expression (Eq. (9), Ref. 12) indicate that they are practically identical. In Ref.11 the potential $\mathrm{Im}\, V^{(2)\mathrm{Reor}}_{opt}(r)$ given in Eq. (40), was used in $\frac{\sigma_{e\ell}(\theta)}{\sigma_R}$ and the expression found was (assuming only target excitation)

$$\frac{\sigma_{e\ell}}{\sigma_R}(\theta) = \exp\left[-\frac{16}{45} q^2_{0\to 2}\, g_{02}(\xi_{02})\, \left(1 + \frac{4}{49} q^2_{2\to 2}\, f_1(\theta)\right)^{-1} f(\theta)\right] \tag{49}$$

where

$$f_1(\theta) = [(1 - (\frac{\pi-\theta}{2})\tan\frac{\theta}{2})^2 \tan^4\frac{\theta}{2} + \frac{1}{3}\sin^4\frac{\theta}{2}]$$

and $f(\theta)$ is the universal angle function given in Ref. 11. Again the comparison of Eq. (49) with the corresponding expression obtained from Eq. (47) and using the I_i's given in Appendix II indicate that they give almost identical results. When performing calculations with the general potential of Eq. (35), Eq. (47) for $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$ is more appropriate. In Fig. 4 we present the results for the system $^{20}Ne + ^{152}Sm$ at E_{lab} = 70 MeV. The agreement with the data (from Ref. [19]) is quite good. In Fig. 5 we present the results for $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$ for two values of q (assuming a pure quadrupole rotational band) fixing the potential V_{opt} so that the maximum I included satisfies the inequality

$$I_{max} \geq 2q \tag{50}$$

The above inequality guarantees, for a given value of q, the presence of the effects of the coupling to all channels included in the construction of $V_{opt}(r)$. This follows from the classical relation between the value of q and the maximum angular momentum transferred namely,[9]

$$2q \tilde{=} \langle I \rangle \tag{51}$$

Another way of calculating the elastic scattering cross section is by summing of the Born series for the elastic scattering amplitude obtained directly from Eq. (4) upon replacing the right hand side by $V_{opt}(r)\,\psi(r)$. This results in the following simple expression for $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$

$$\frac{\sigma_{e\ell}}{\sigma_R}(\theta) = \left| \frac{1 - \frac{i}{k}\int_0^\infty F_{\bar{\ell}(\theta)}(kr)\, V_{opt}(r)\, F_{\bar{\ell}(\theta)}(kr)dr}{1 + \frac{i}{k}\int_0^\infty F_{\bar{\ell}(\theta)}(kr)\, V_{opt}(r)\, F_{\bar{\ell}(\theta)}(kr)dr} \right|^2 \equiv |A_{00}(\theta)|^2 \tag{52}$$

Here we have adopted the on-energy-shell approximation for the elastic channel Green's function. The amplitude $A_{00}(\bar{\ell})$ may be written in terms of the coupling matrices $\underline{C}_{ij}(\ell)$ as can be seen from the structure of V_{opt} given in Eq. (22) and the definition of $\underline{C}_{ij}(\ell)$ given in Eq. (18)

$$A_{00}(\bar{\ell}) = \frac{1-i\underline{C}_{02}(\bar{\ell}) \ \langle 2| \ \frac{1}{\underline{1} + i \ \underline{C}(\ell)} \ |2\rangle \ \underline{C}_{20}(\ell)}{1+i\underline{C}_{02}(\ell)\langle 2| \frac{1}{\underline{1} + i \ \underline{C}(\ell)} \ |2\rangle \ \underline{C}_{20}(\ell)} \qquad (53)$$

where the Q-space "propagator" of the 2+ state $\langle 2| \ \frac{1}{\underline{1} + i \ \underline{C}(\ell)} |2\rangle$ is given in Eq. (25).

We have evaluated $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$ as given in Eq. (52) including in the construction of the 2+ propagator states up to $\mathbf{I} = 18$, i.e., including up to 8 members of the rotational band and taking q_{02} to be 2.36 and 9.56. The result is shown as the dashed lines in Fig. 6. It is clear that the WKB expression for $\frac{\sigma_{e\ell}}{\sigma_R}$ given in Eq. (47) and the summed Born series of Eq. (52) give almost identical results for small values of q. However, for large q the two expressions, although give close results at small angles, presents qualitatively different behavior as can be seen in Fig. 5b. We have accounted for the energy loss by inserting the factors $g_{ij}(\xi_{ij})$ in the coupling matrices $C_{ij\neq i}$. The semiclassical coupled channels-calculation of $\frac{\sigma}{\sigma_R}$ for the above system has been reported recently and it shows that σ/σ_R oscillates slightly at angles larger than 60^o.

Finally, we give below the elastic scattering cross section evaluated at $\theta = \pi$ (included in this calculation are the couplings to the 2+ and 4+ states as well as the reorientation of the 2+ state)

$$\frac{\sigma_{e\ell}(\pi)}{\sigma_R(\pi)} = \exp\left[-\frac{16}{45} q^2_{0\to2} \ g_{0\to2}(\xi_{0\to2}) \ \frac{1+0.065 \ q^2_{2\to4} \ g_{2\to4}(\xi_{2\to4})}{0.036 \ q^2_{2\to2} + [1+ 0.065 \ q^2_{2\to4} \ g_{2\to4}(\xi_{2\to4})]^2}\right] \qquad (54)$$

It is interesting to note that at back angles both the WKB and the summed Born series expressions for $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$ become, for a pure rotational band and in the zero-energy loss limit, a function of only one variable, the quadrupole strength parameter $q_{0\to 2}$. This is also the case with the elastic scattering probability calculated by Alder and Winther[9] without resorting to the Coulomb polarization potential.

From the elastic scattering cross section one may calculate the total inelastic scattering cross section using the unitarity condition which the matrix must satisfy,

$$\frac{\sigma_{in\ell}(\theta)}{\sigma_R(\theta)} \equiv \sum_{i=2,4,\ldots} \frac{\sigma_i^{+}(\theta)}{\sigma_R(\theta)} = 1 - \frac{\sigma_{e\ell}(\theta)}{\sigma_R(\theta)} \tag{55}$$

In order to exhibit this property of the S-matrix we notice that a manifestly unitary S-matrix may be constructed from our equation (20) through[21]

$$S = (k)^{+1/2} \tilde{S} (k)^{-1/2} \tag{56}$$

where $\tilde{S}$ is obtained from Eq. (20),

$$\tilde{S} = 1 - 2ik^{-1} \int_0^{\infty} dr\, F\, V\psi^{(+)} \tag{57a}$$

where F and ψ^{+} are matrices in channel space and V is the coupling matrix. From the unitary S-matrix above we can determine the T-matrix which is related to S by

$$1 - 2iT = S$$

or

$$T = k^{1/2}\, k^{-1} \int F V \psi\; k^{-1/2} \tag{57b}$$

$$= k^{-1/2} \int F V \psi\; k^{-1/2}$$

Defining a new coupling matrix $\underline{C}'$ to be $\underline{C}' = k^{1/2}\ k^{-1}\underline{C}\,k^{-1/2} = k^{-1/2}\,\underline{C}\,k^{-1/2}$ we finally obtain

$$T = \frac{k^{-1/2}\,\underline{C}\,k^{-1/2}}{1 + ik^{-1/2}\,\underline{C}\,k^{-1/2}} = \frac{\underline{C}'}{1 + i\underline{C}'} \tag{58}$$

and

$$S = \frac{1 - ik^{-1/2}\,\underline{C}\,k^{-1/2}}{1 + i\,k^{-1/2}\,\underline{C}\,k^{-1/2}} = \frac{1-i\underline{C}'}{1+i\underline{C}'} \ .$$

In the above symmetrized form, T may be used to calculate the different inelastic cross sections. This calculation is reported in the following paper.[22]

V. DISCUSSION AND CONCLUSIONS

In this paper we have explored the consequences of the on-energy-shell approximation for the channel Green's function on the multiple Coulomb excitation coupled channel problem. We have, among other things, derived a closed expression for the Coulomb polarization potential in the limit of zero energy loss accounting partially for the last through the semiclassical energy loss factors. Our general expression for the Coulomb polarization potential reduces to the more approximate expressions given in Refs.11 and 12 as limiting cases. As a check on our results we have compared our potential for $\ell = 0$ with the potential based on the Alder-Winther theory, obtained in Ref.15. The discrepancy found is accounted for partly by the terms in the Born series neglected in the on-energy-shell approximation. We have been able to fit, on the average, our potential to the Alder-Winther one by adjusting two parameters--one attached to the reorientation matrices and the other to the coupling matrices. It is argued that the failure of our

adjusted imaginary potential to reproduce the oscillations found in the Alder-Winther potential could be a consequence of both the fact that the AW potential was extracted from the asymptotic elastic amplitude whereas ours directly from the wave function, as well as the on-energy-shell approximation. Although the methods developed in this paper are not meant to substitute for more exact coupled channels calculations, they do, nevertheless, supply us with a very simple closed form for the sub-barrier elastic cross section appropriate for the case of strong coupling. Furthermore, having obtained the elastic channel Coulomb polarization potential, which contains multiple Coulomb excitation effects, one could use it to simplify an otherwise more complicated coupled channels calculation. As an example, we cite the case of sub-barrier fusion of deformed nuclei.[23] As is known, deformation of the target induces both static as well as dynamic effects on the fusion cross section.[24] If the target is so deformed that multiple Coulomb excitation is important, one needs to perform a calculation involving several strongly coupled channels to account correctly for these effects. Our potential $V_{opt}(r)$, makes it possible to account almost completely for the dynamic effects in a one-channel optical model description. The study of static effects is thus made easier and could be handled following, for example, the ideas in Ref.23 Aside from the above practical aspects of our results we consider our findings as a first step in the direction of exploring simplifying schemes for the coupled channels calculation in the more general case where the nuclear force is included. In such cases the on-energy-shell approximation amounts to replacing the channel Green's function by(Ref. 7)

$$G_{\ell,c}(r,r') \propto \psi_{\ell,c}(r)\ \psi_{\ell,c}(r')/S^{o}_{\ell,c}(k) \qquad (59)$$

Here $\psi_{\ell}(r)$ is the regular solution of the optical model Schrodinger equation describing the elastic scattering in channel c and $S^{o}_{\ell,c}(k)$ is the elastic

S-matrix element in channel c. Due to strong absorption for low ℓ partial waves the factor $(S^{o}_{\ell,c}(k))^{-1}$ could become quite large for small ℓ, resulting in a possible overestimation of the coupled channel effects.[6] However, as demonstrated in Ref. 7, this needs not be a worrying point if the amplitudes were to be evaluated in closed form as another factor of $S^{o}_{\ell}(k)$, which comes from the distorted waves, approximately cancels the dangerous $[S^{o}_{\ell,c}(k)]^{-1}$ factor in the channel Green's function. Finally, the inclusion of the nuclear excitation into our expressions for the inelastic amplitudes, which are calculated without the nuclear effects in the following paper, would make the investigation of the dependence of the nuclear-Coulomb interference effects on the spins of the excited states[25] simple and transparent. These ideas will be explored and developed fully in the third part of this series.[26]

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APPENDIX I

In this appendix are given the approximate, large ℓ limit, values of the 3-j and 6-j symbols needed in the calculation. Throughout we shall use the definitions and convention of Edmonds.[27]

a) 3-j symbols

$$\begin{pmatrix} \ell & \ell & 2 \\ 0 & 0 & 0 \end{pmatrix} = (-)^{\ell+1} \left[\frac{\ell(\ell+1)}{(2\ell+3)(2\ell+1)(2\ell-1)} \right]^{1/2}$$

$$\simeq \frac{(-)^{\ell+1}}{\sqrt{8\ell}} \tag{AI-1}$$

$$\begin{pmatrix} \ell & \ell+2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = (-)^{\ell} \left[\frac{3}{2} \frac{(\ell+2)(\ell+1)}{(2\ell+5)(2\ell+3)(2\ell+1)} \right]^{1/2}$$

$$\simeq (-)^{\ell} \sqrt{\frac{3}{16\ell}} \tag{AI-2}$$

b) 6-j symbols

In the following $\ell >> I$, $\ell >> m$.

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I-2 & \ell+m-2 \end{Bmatrix}$$

$$\simeq \left[\frac{1}{2\ell} \frac{(m+I-3)(m+I-2)(m+I-1)(m+I)}{(2I-3)(2I-2)(2I-1)(2I)(2I+1)} \right]^{1/2} \tag{AI-3}$$

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I-2 & \ell+m \end{Bmatrix}$$

$$\simeq \left[\frac{3}{\ell} \frac{(m+I-1)(m+I)(I-m-1)(I-m)}{(2I-3)(2I-2)(2I-1)(2I+1)(2I)} \right]^{1/2} \tag{AI-4}$$

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I-2 & \ell+m+2 \end{Bmatrix}$$

$$\simeq \left[\frac{1}{2\ell}\ \frac{(I-m-3)(I-m-2)(I-m-1)(I-m)}{(2I-3)(2I-2)(2I-1)(2I)(2I+1)}\right]^{1/2} \qquad \text{(AI-5)}$$

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I & \ell+m+2 \end{Bmatrix}$$

$$\simeq \left[\frac{3}{\ell}\ \frac{(m+I-1)(m+I)(I-m+1)(I-m+2)}{(2I-1)(2I)(2I+1)(2I+2)(2I+3)}\right]^{1/2} \qquad \text{(AI-6)}$$

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I & \ell+m \end{Bmatrix}$$

$$\simeq \frac{(3m^2 - I(I+1))}{[\ \ell I(2I-1)(2I+1)(2I+2)(2I+3)\ ]1/2} \qquad \text{(AI-7)}$$

APPENDIX II

The Coulomb orbit integrals defined in Eq. (45) are easily evaluated following the methods described in Ref. 28. We obtain for $I_3(\theta)$, $I_4(\theta)$ and $I_5(\theta)$, the following expressions

$$I_3(\theta) = \frac{1}{\bar{\ell}^2}\left[1 - \frac{\arctan\ \bar{\ell}}{\bar{\ell}}\right]\frac{2\eta}{Ea^3}$$

$$= \tan^2\frac{\theta}{2}\left[1-\left(\frac{\pi-\theta}{2}\right)\tan\frac{\theta}{2}\right]\frac{2\eta}{Ea^3} \qquad \text{(AII-1)}$$

$$I_4(\theta) = \frac{1}{2\bar{\ell}^4}\left[(\bar{\ell}^2 + 3)\frac{1}{\bar{\ell}}\arctan\bar{\ell} - 3\right]\frac{2\eta}{Ea^4}$$

$$= 2\tan^4\frac{\theta}{2}\left[(\cot^2\frac{\theta}{2}+3)(\frac{\pi-\theta}{2})\tan\frac{\theta}{2} - 3\right]\frac{2\eta}{Ea^4} \qquad \text{(AII-2)}$$

$$I_5(\theta) = \frac{1}{2\bar{\ell}^6}\left[(\frac{4\bar{\ell}^2}{3} + 5) - (3\bar{\ell}^2+5)\frac{1}{\bar{\ell}}\arctan\bar{\ell}\right]\frac{2\eta}{Ea^5}$$

$$= 2\tan^6\frac{\theta}{2}\left[(\frac{4}{3}\cot^2\frac{\theta}{2}+5) - (3\cot^2\frac{\theta}{2}+5)(\frac{\pi-\theta}{2})\tan\frac{\theta}{2}\right]\frac{2\eta}{Ea^5} \qquad \text{(AII-3)}$$

In obtaining the final angular dependence of $I_i(\theta)$ we have used the Rutherford relation $\bar{\ell} = \cot\frac{\theta}{2}$ (Eq. (41)).

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FIGURE CAPTIONS

Figure 1 The coefficient a_ℓ, b_ℓ and c_ℓ plotted as functions of $\bar{\ell} \equiv \frac{\ell + 1/2}{\eta}$ for several values of the quadrupole coupling parameter q. A factor $\frac{\eta}{E}$ was taken out of the coefficients in order to present the result in as general a form as possible.

Figure 2 The coefficient, a_ℓ, b_ℓ and c_ℓ plotted as functions of the quadrupole coupling parameter q for several values of $\bar{\ell} \equiv \frac{\ell + 1/2}{\eta}$. A factor $\frac{\eta}{E}$ was taken out of the coefficients (see caption to Fig. 1).

Figure 3 The potential $V_{\bar{\ell}=0}$ plotted as a function of q and adjusted to the Alder-Winther potential (dashed line). The adjusted values of the parameters α and β used to fit the potential to the A-W potential were 2.3 and 4.6, respectively (see text).

Figure 4 The sub-barrier elastic cross section normalized to the Rutherford cross section, plotted vs. the center of mass angle for the system $^{20}Ne + ^{152}Sm$ (E_{lab} = 70 MeV). Included in the calculation is the coupling to the 2^+ state as well as the reorientation of the 2^+ to all orders in both target and projectile. The data are from Ref. 19

Figure 5 The ratio $\frac{\sigma_{e\ell}}{\sigma_R}$ calculated both from the WKB expression Eq. (47) (solid lines) as well as from the summed Born series Eq. (52) (dashed lines). a) q = 2.36; b) q = 9.56. Energy loss was accounted for by inserting the appropriate semiclassical energy loss factors (see text).

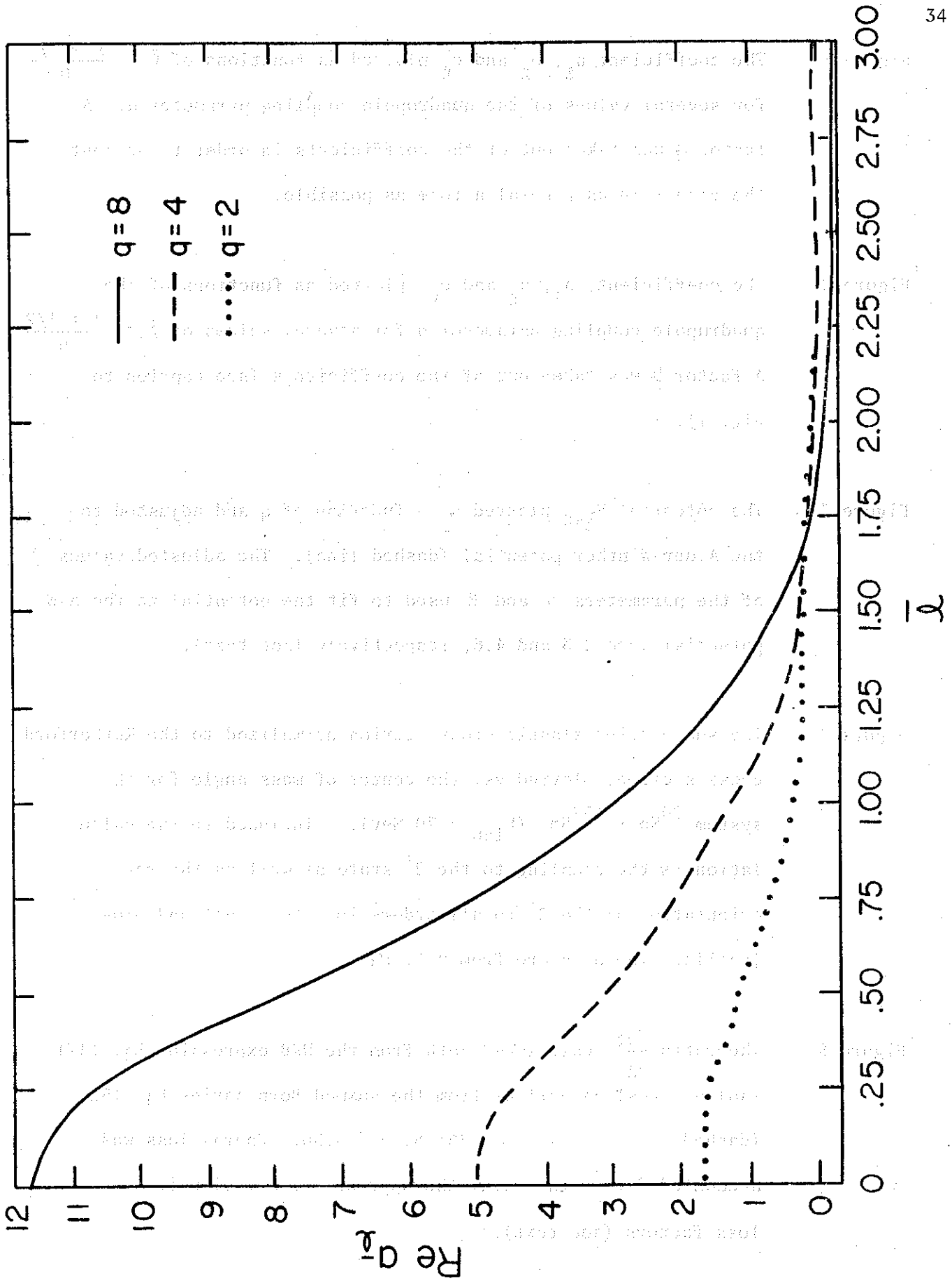
q=8
q=4
q=2
Re $Q_{\bar{k}}$
$\bar{k}$
0
1
2
3
4
5
6
7
8
9
10
11
12
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00

Fig. 1a

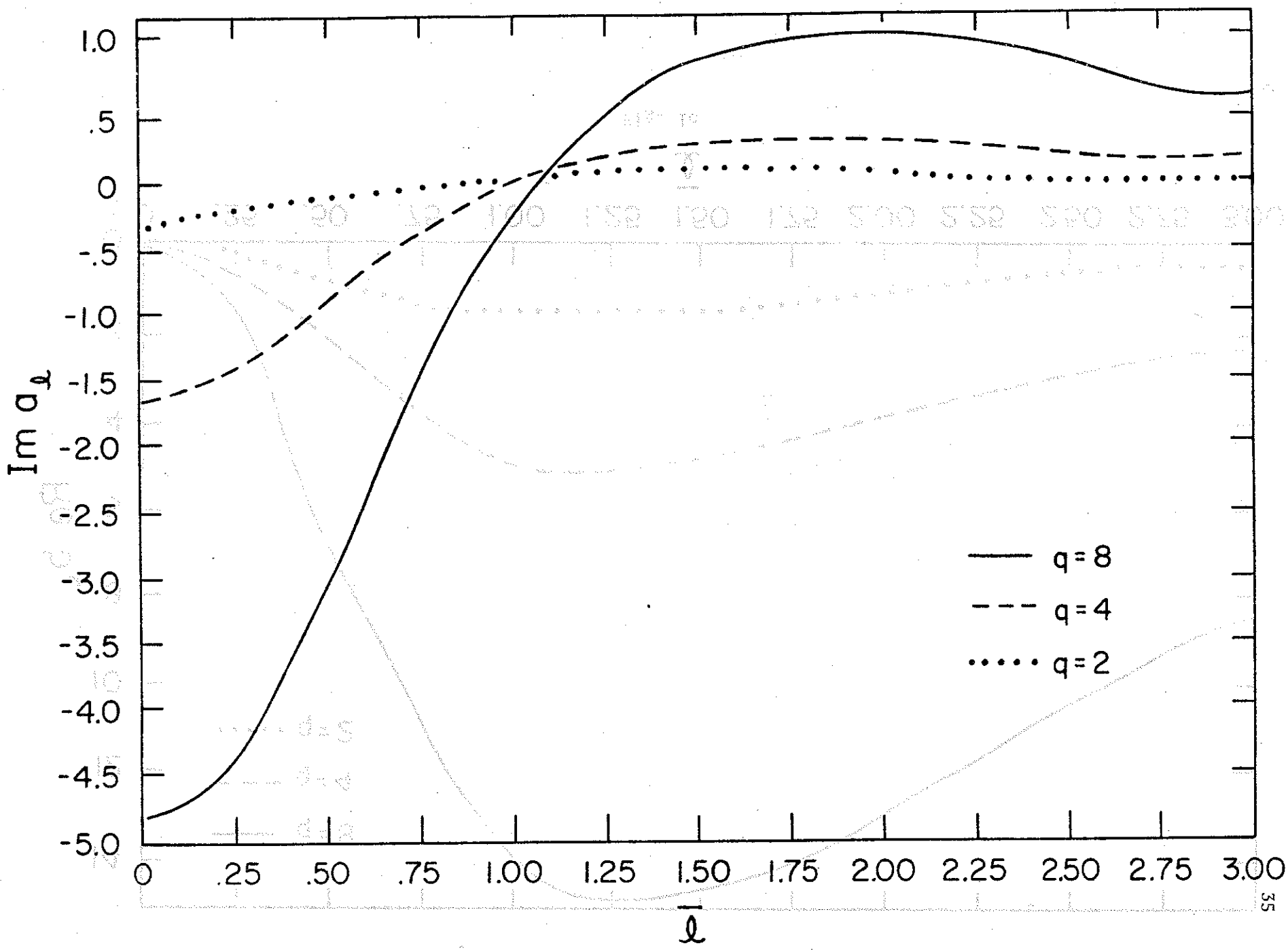
1.0
.5
0
-.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
Im a_ℓ
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
ℓ
q = 8
q = 4
q = 2

Fig. 1b

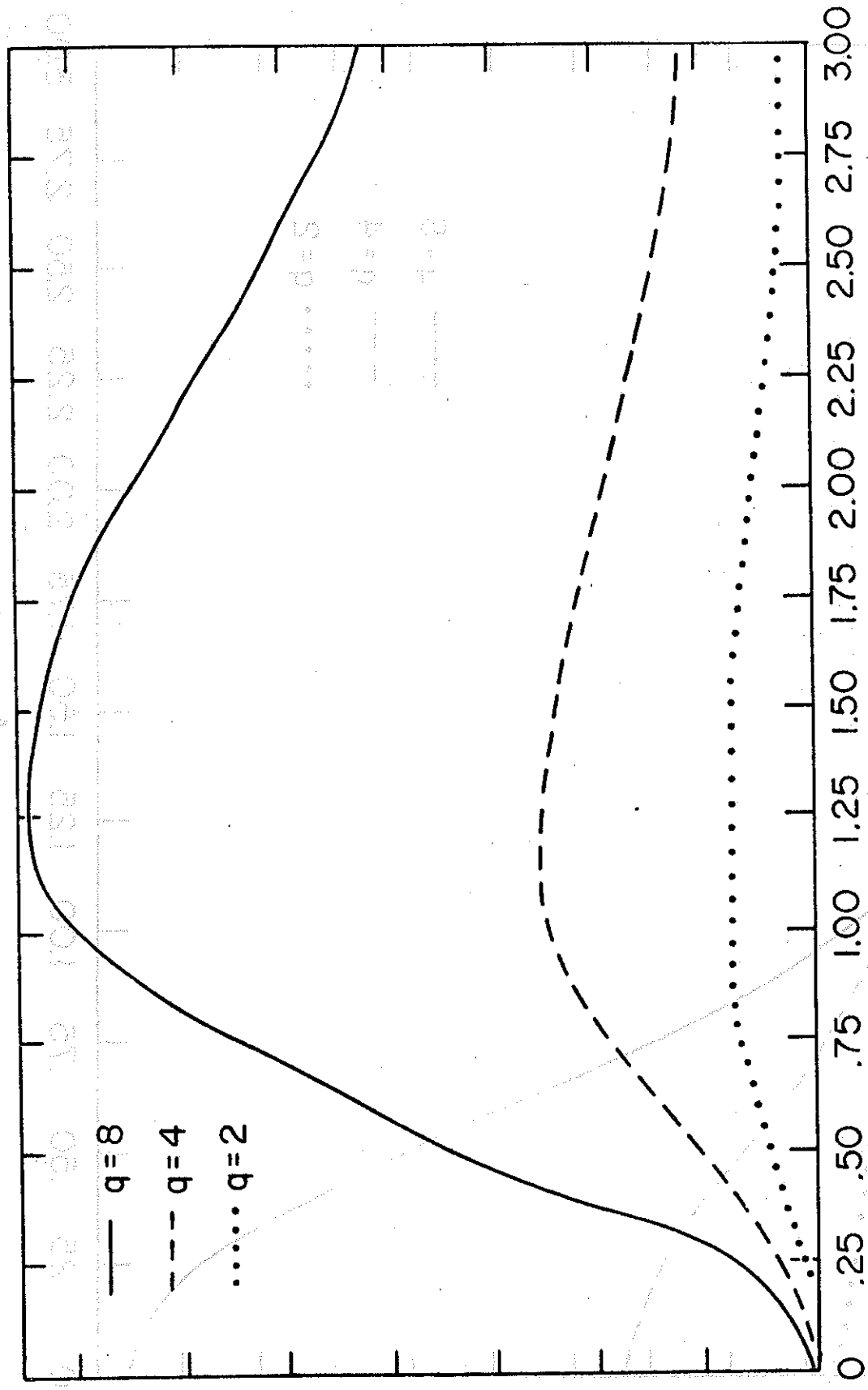
Re b_λ
0
2
4
6
8
10
12
14
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
$\overline{\lambda}$
q=8
q=4
q=2

Fig. 1c

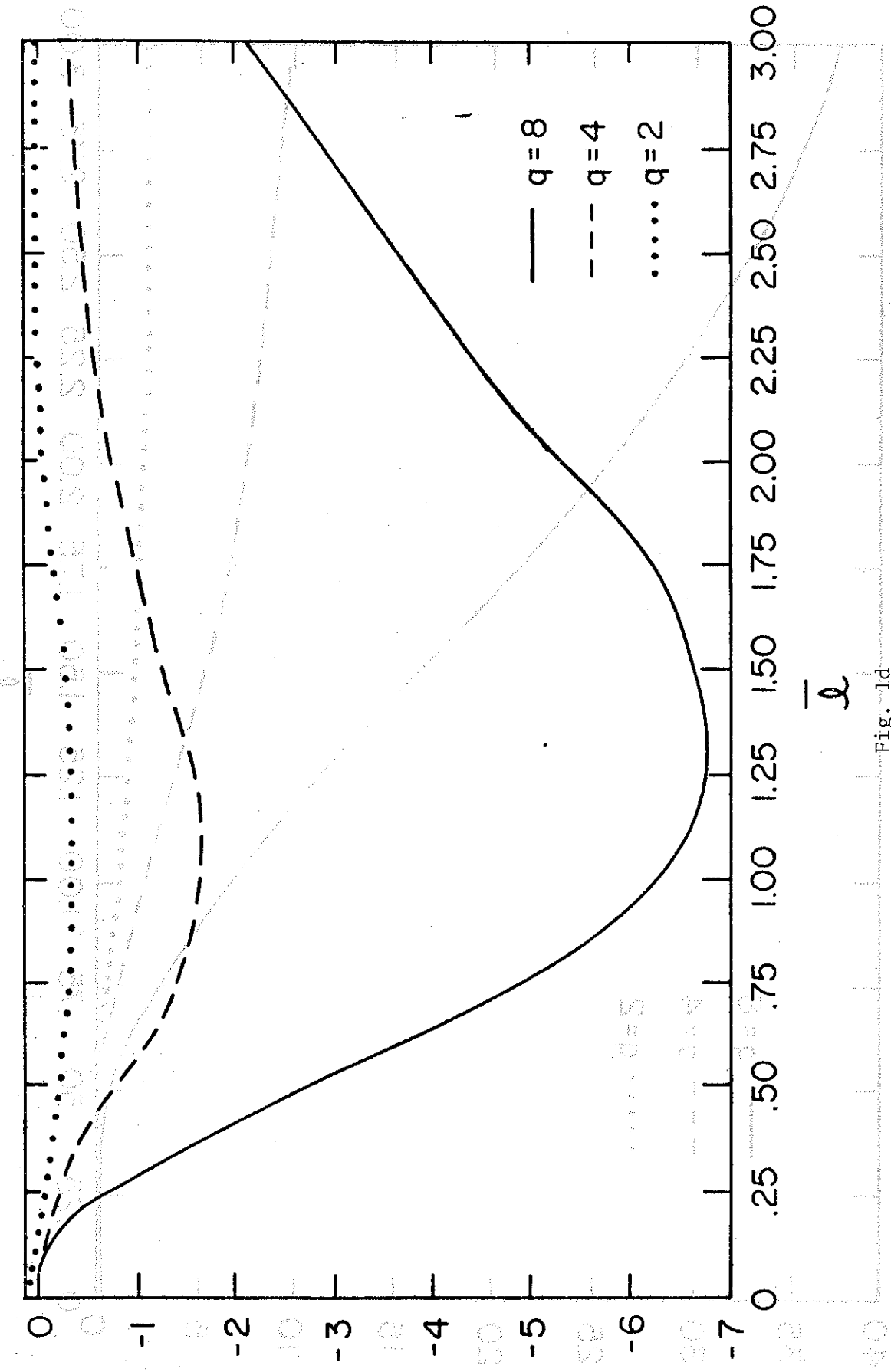
Im b_ℓ
0
-1
-2
-3
-4
-5
-6
-7
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
ℓ
q=8
q=4
q=2

Fig. 1d

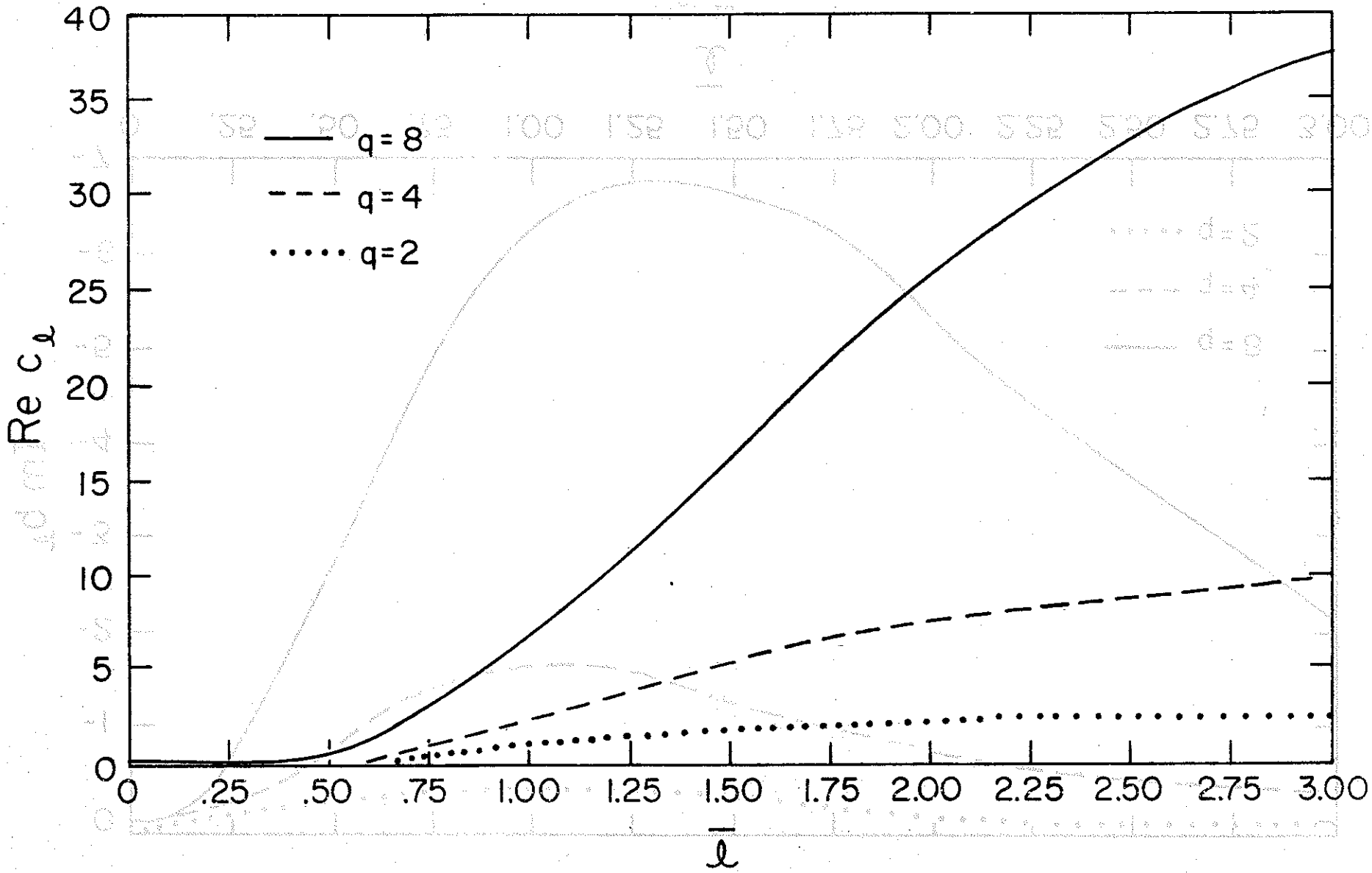
40
35
30
25
20
15
10
5
0
Re $c_{\bar{l}}$
q = 8
q = 4
q = 2
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
$\bar{l}$

Fig. 1e

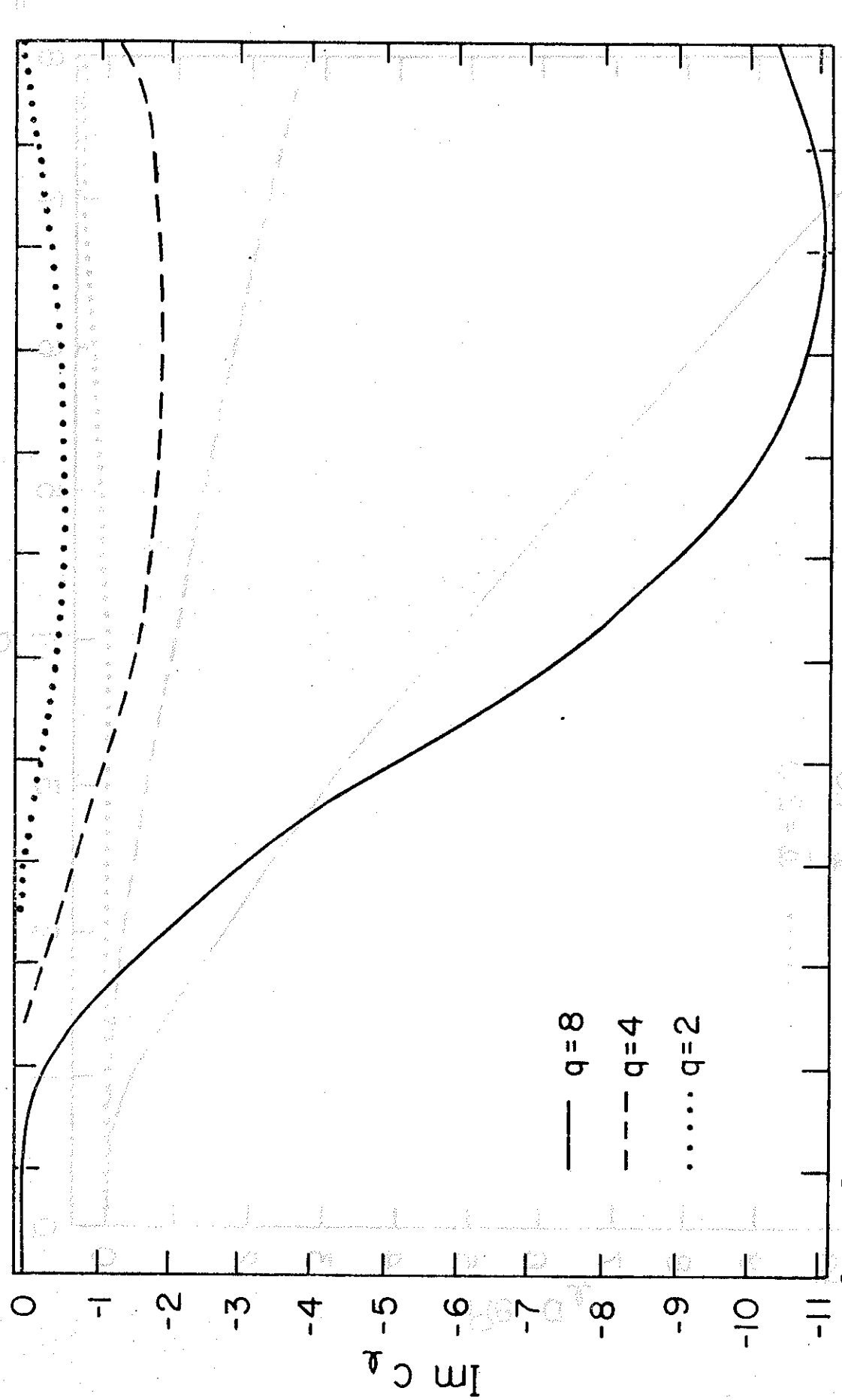
Im c_a
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
$\overline{a}$
q=8
q=4
q=2

Fig. 1f

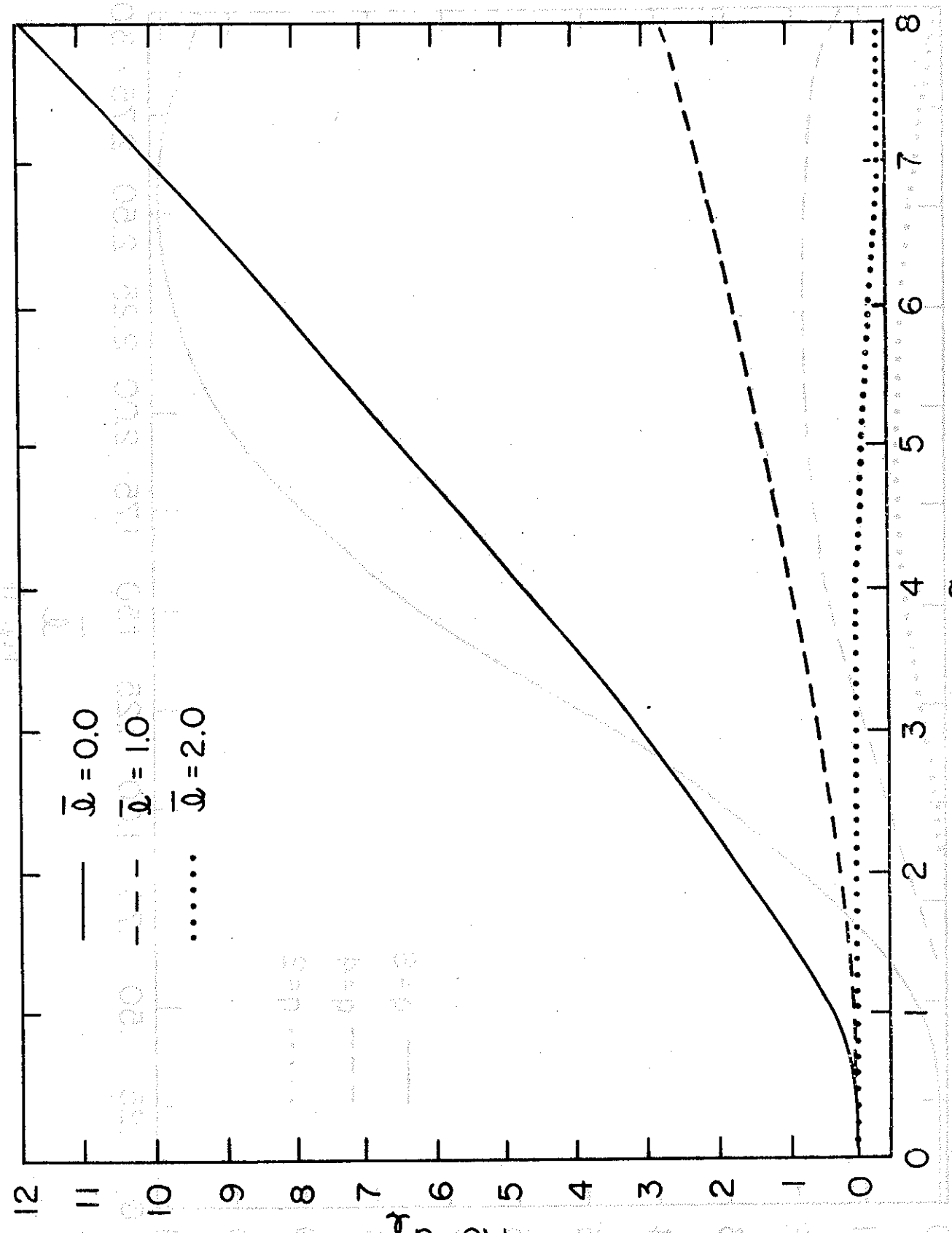
$\bar{\vartheta} = 0.0$
$\bar{\vartheta} = 1.0$
$\bar{\vartheta} = 2.0$
$Re\, Q_r$
q
0
1
2
3
4
5
6
7
8
9
10
11
12

Fig. 2a

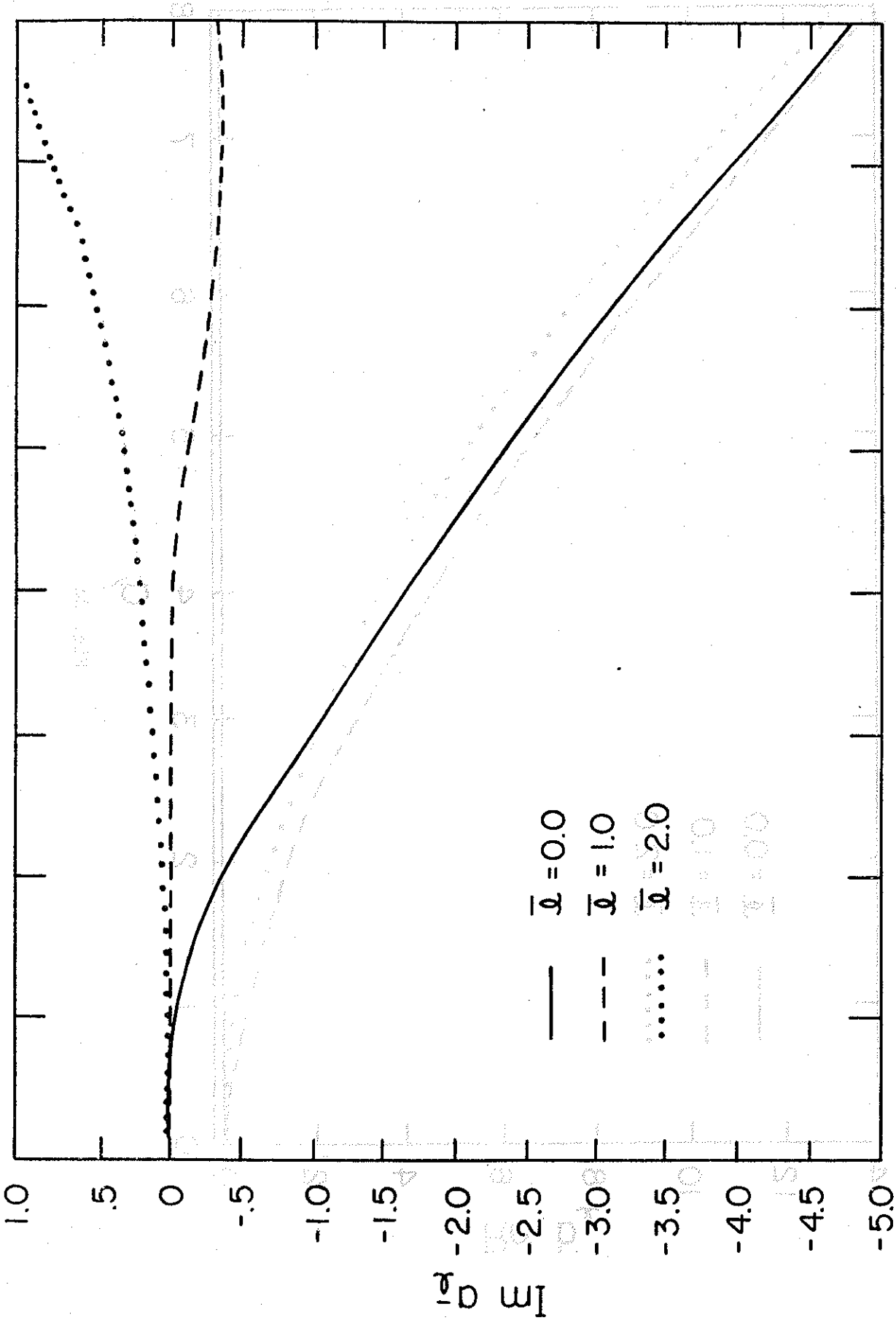
Im $a_{\bar{a}}$
1.0
.5
0
-.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
0
1
2
3
4
5
6
7
8
q
$\bar{a}$ = 0.0
$\bar{a}$ = 1.0
$\bar{a}$ = 2.0

Fig. 2b

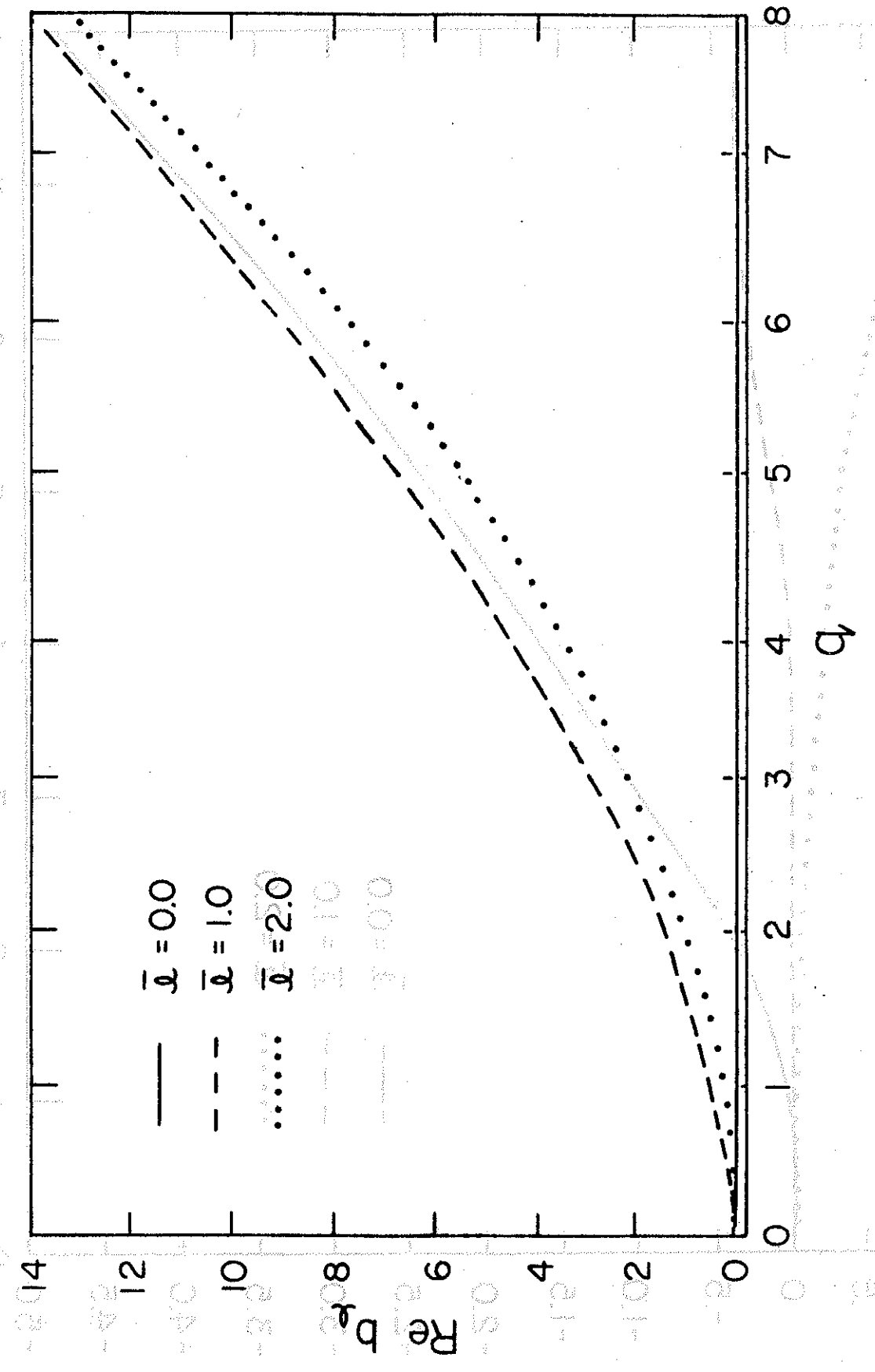
Re b_k
q
$\bar{\alpha}$ = 0.0
$\bar{\alpha}$ = 1.0
$\bar{\alpha}$ = 2.0
0
2
4
6
8
10
12
14
1
3
5
7

Fig. 2c

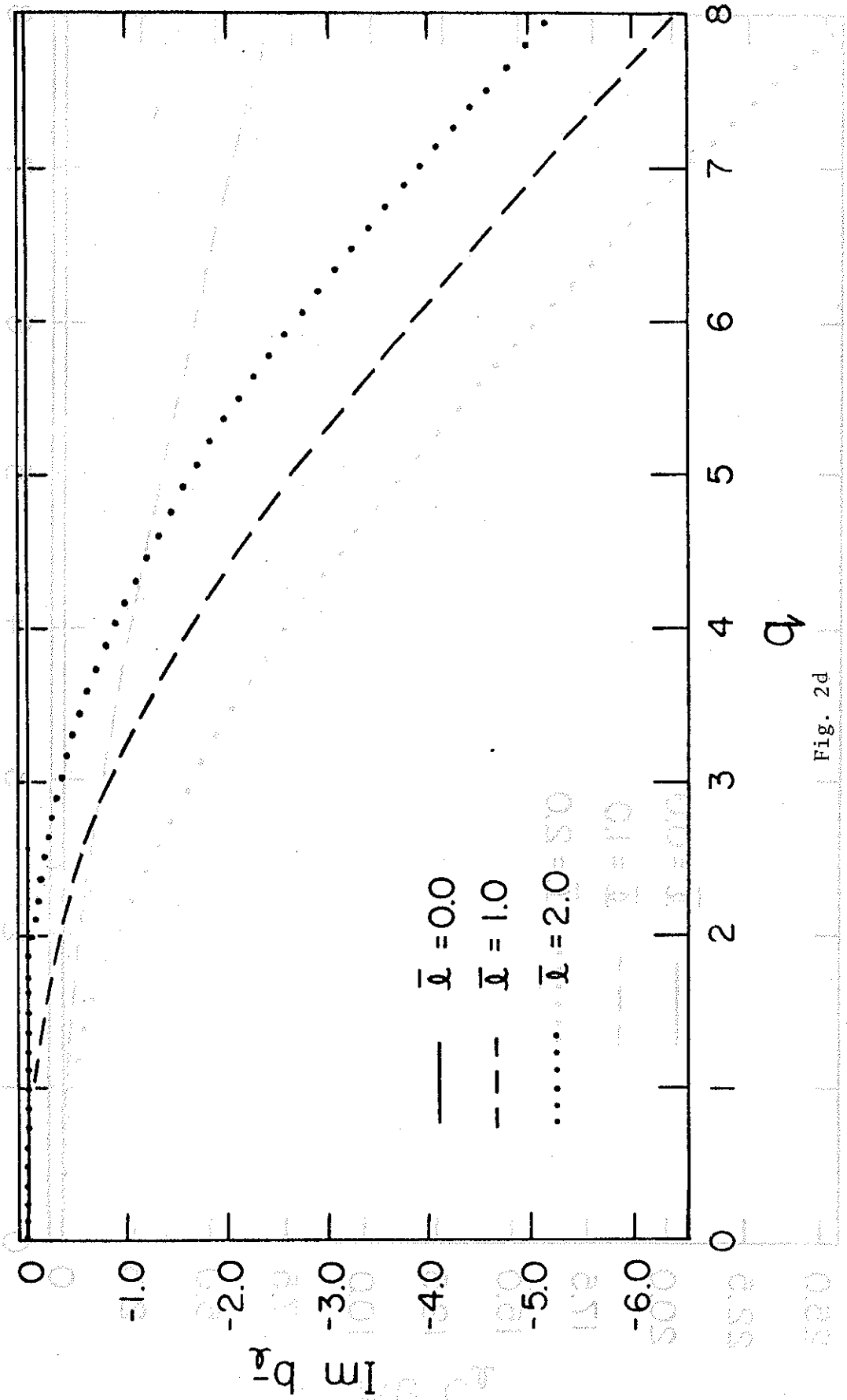
Im $b_{\bar{\alpha}}$
0
-1.0
-2.0
-3.0
-4.0
-5.0
-6.0
0
1
2
3
4
5
6
7
8
q
$\bar{\alpha}$ = 0.0
$\bar{\alpha}$ = 1.0
$\bar{\alpha}$ = 2.0

Fig. 2d

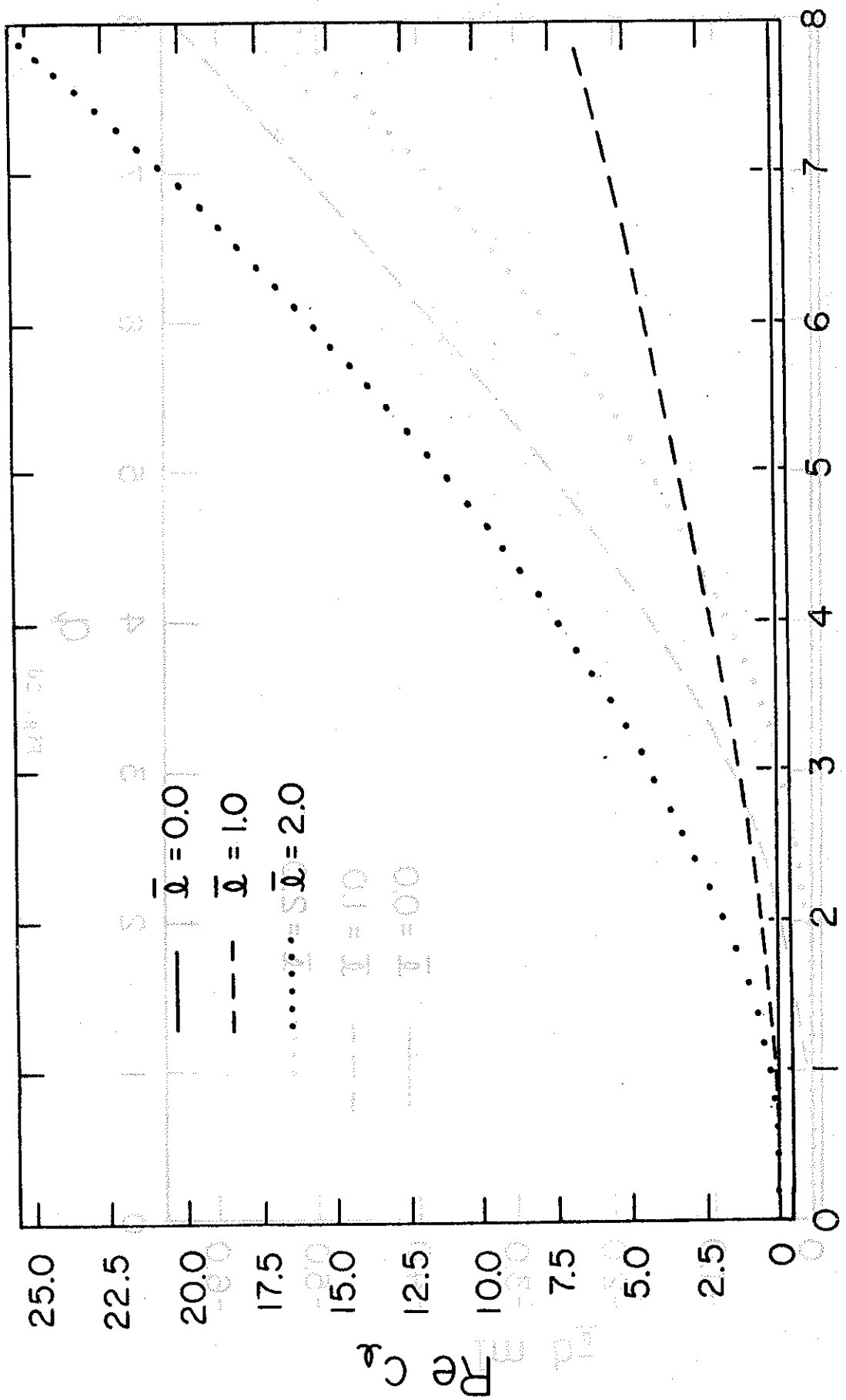
Re C_a
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0
0
1
2
3
4
5
6
7
8
q
$\bar{\alpha} = 0.0$
$\bar{\alpha} = 1.0$
$\bar{\alpha} = 2.0$

Fig. 2e

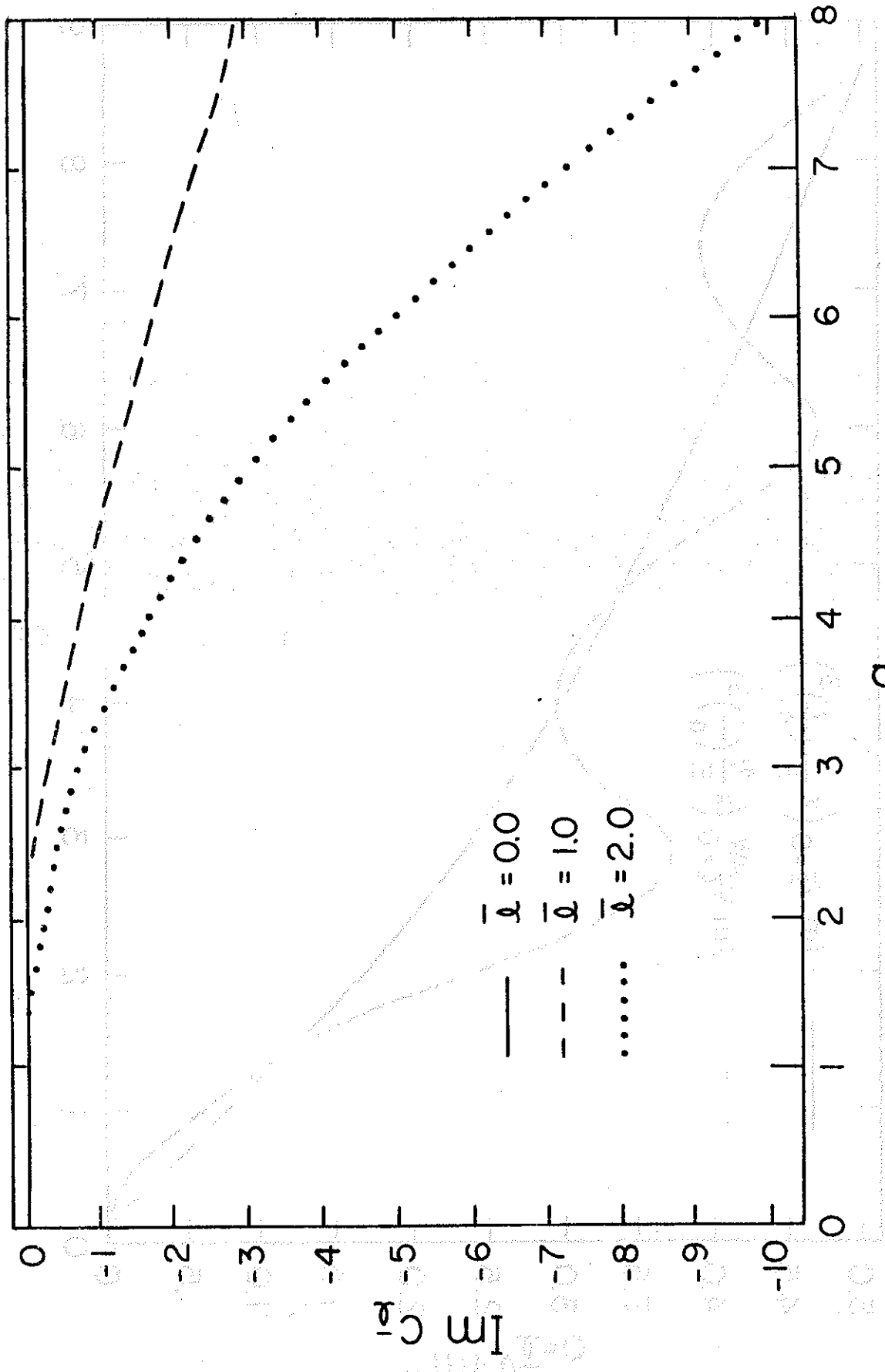

Fig. 2f

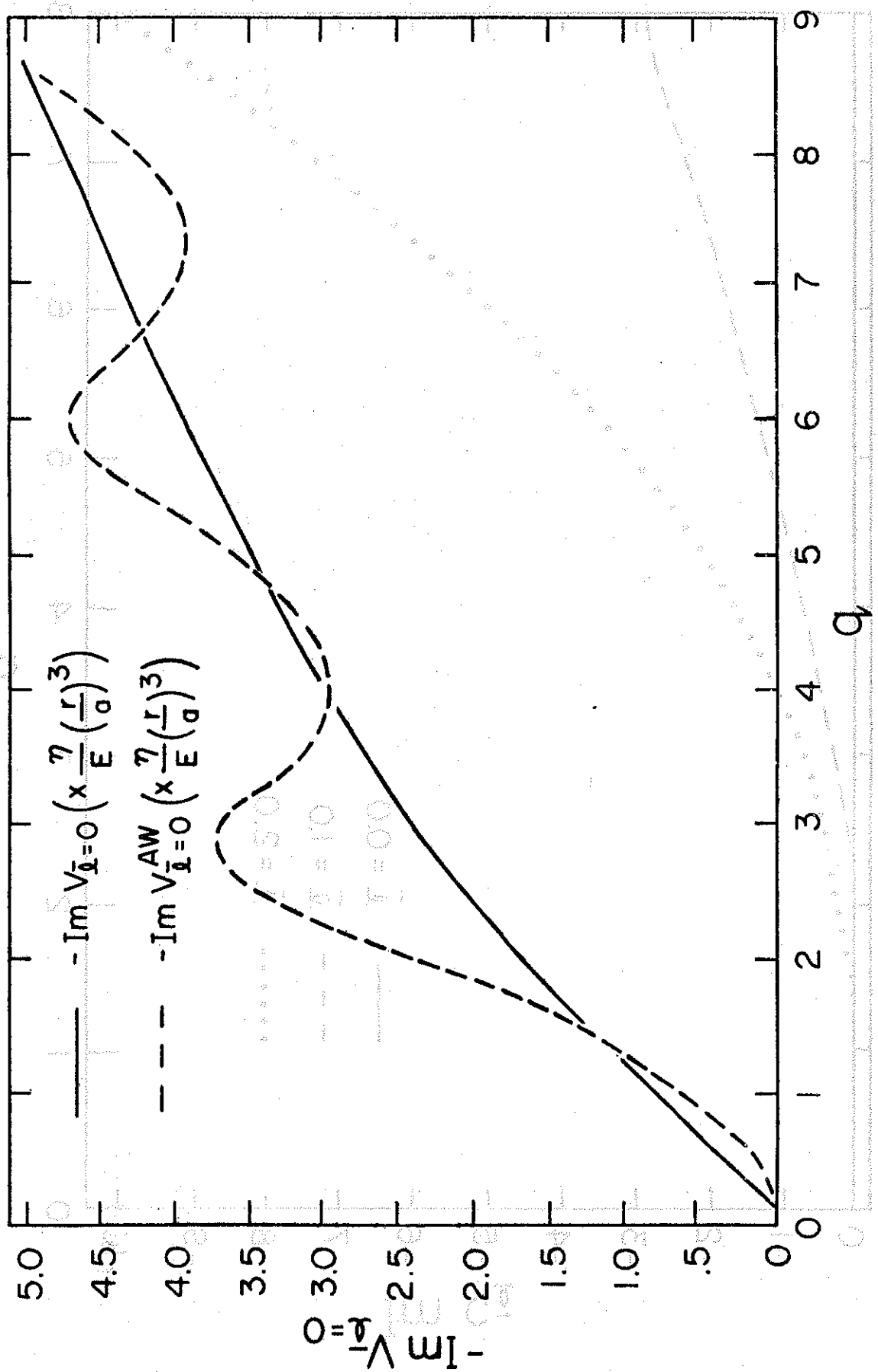
$-\mathrm{Im}\, V_{\vec{k}=0}$
$-\mathrm{Im}\, V_{\vec{k}=0}\left(x\frac{\eta}{E}\left(\frac{r}{a}\right)^3\right)$
$-\mathrm{Im}\, V^{AW}_{\vec{k}=0}\left(x\frac{\eta}{E}\left(\frac{r}{a}\right)^3\right)$
q
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
1
2
3
4
5
6
7
8
9

Fig. 3a

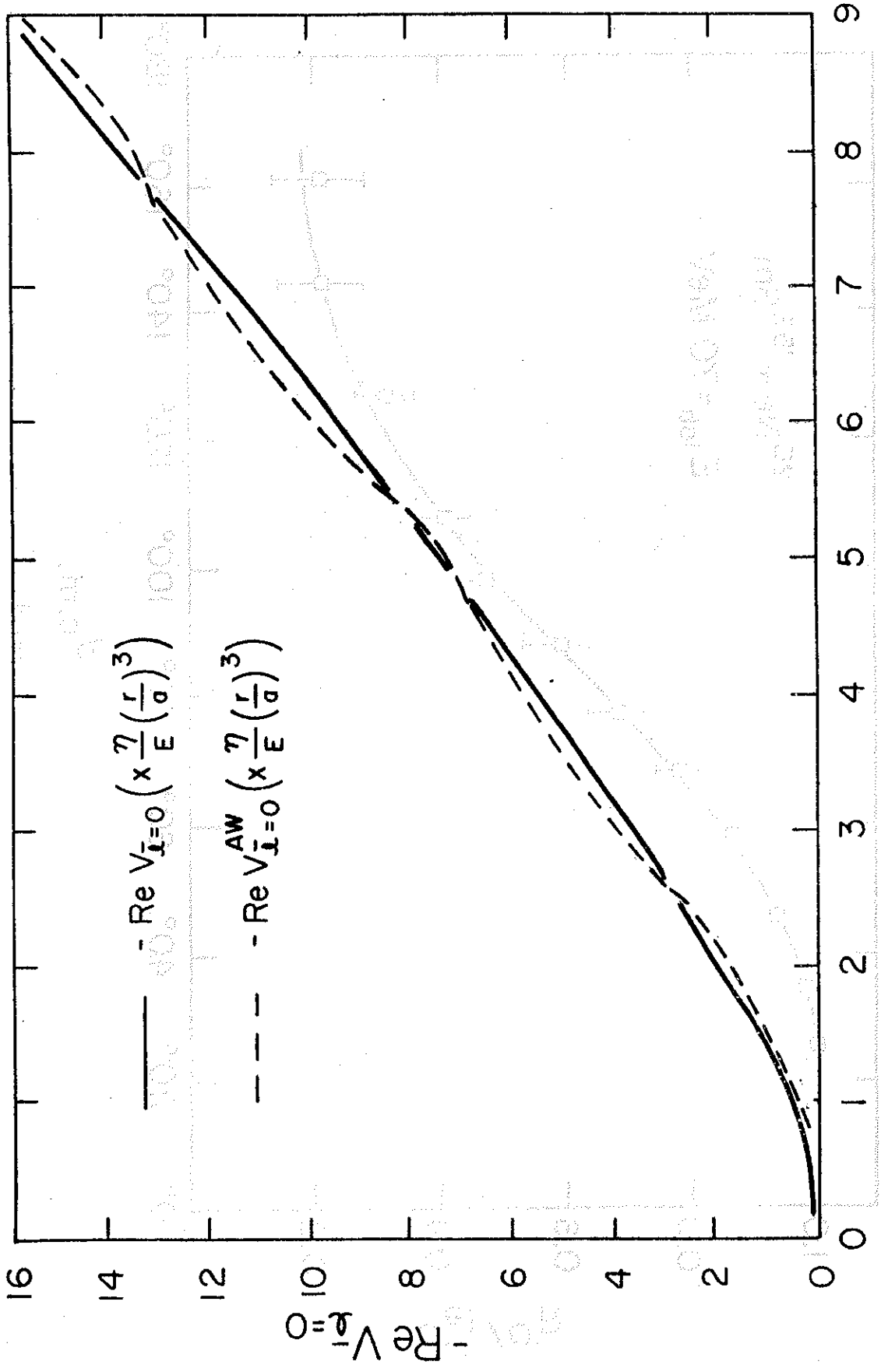
- Re $V_{\perp=0}$
16
14
12
10
8
6
4
2
0
0
1
2
3
4
5
6
7
8
9
q
— Re $V_{\perp=0}\left(x\frac{\eta}{E}\left(\frac{r}{a}\right)^3\right)$
— — — Re $V^{AW}_{\perp=0}\left(x\frac{\eta}{E}\left(\frac{r}{a}\right)^3\right)$

Fig. 3b

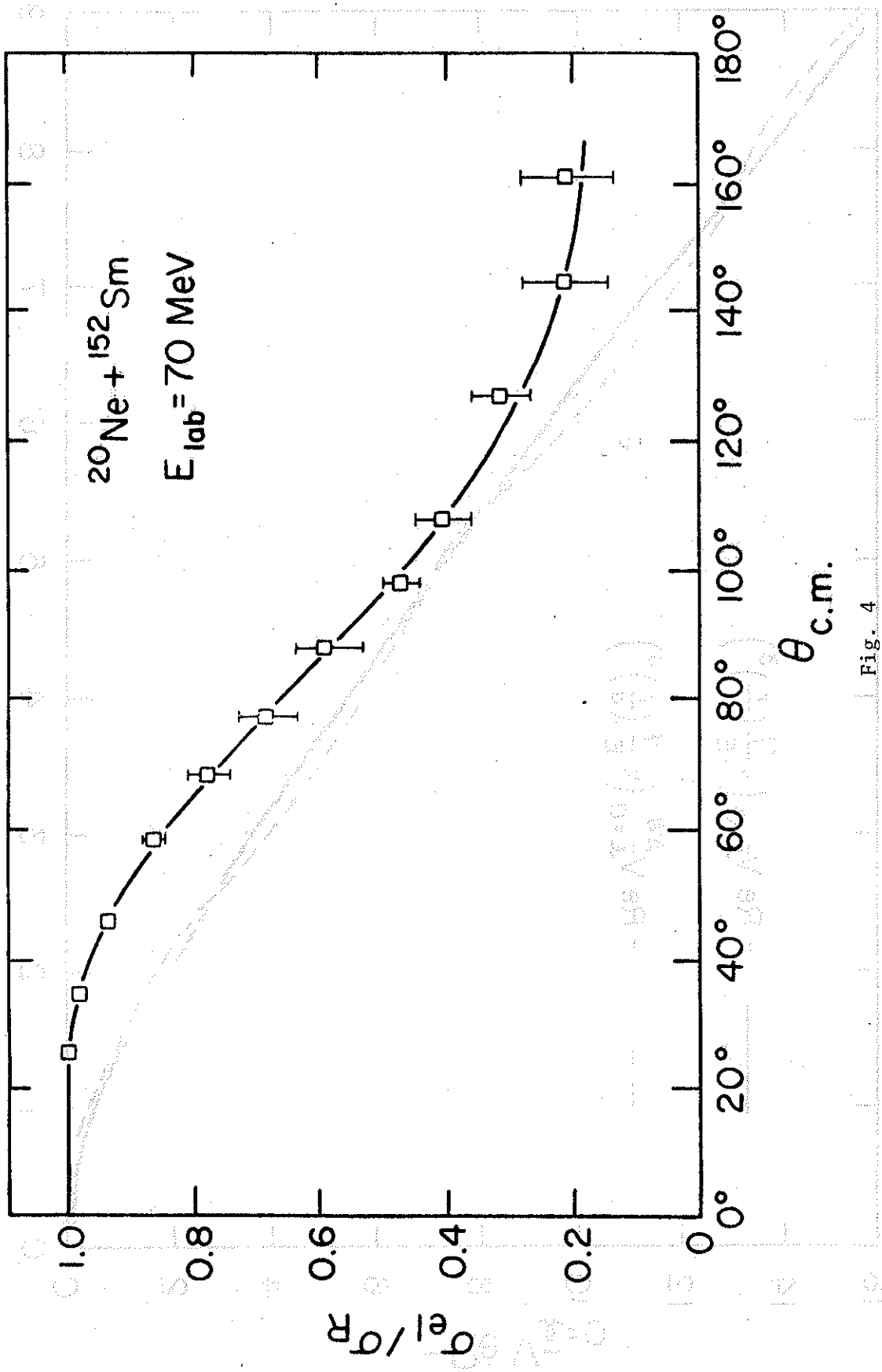
$^{20}Ne + ^{152}Sm$
$E_{lab} = 70$ MeV
σ_{el}/σ_R
$\theta_{c.m.}$
0
0.2
0.4
0.6
0.8
1.0
0°
20°
40°
60°
80°
100°
120°
140°
160°
180°

Fig. 4

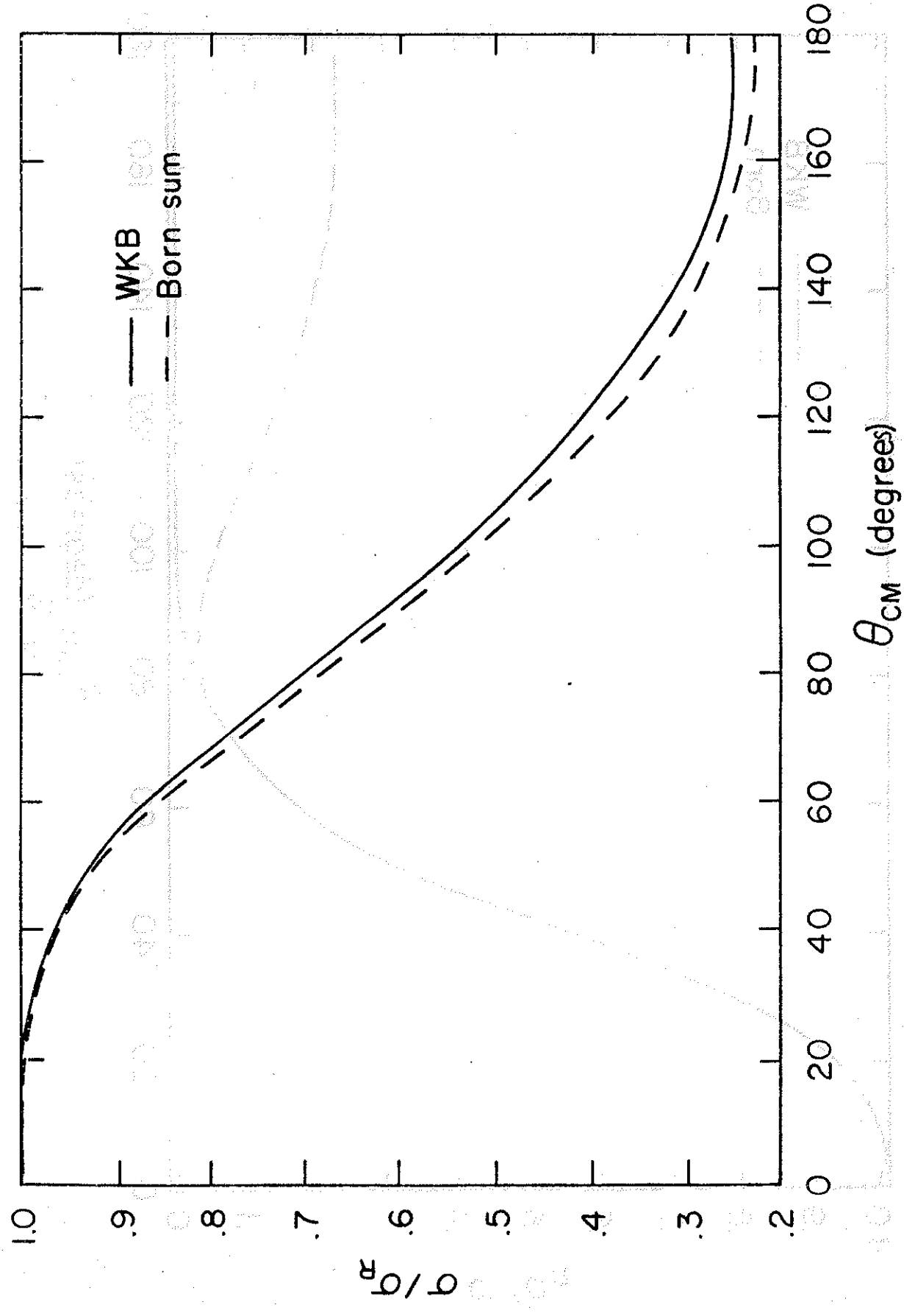
WKB
Born sum
σ/σR
θCM (degrees)
1.0
.9
.8
.7
.6
.5
.4
.3
.2
0
20
40
60
80
100
120
140
160
180

Fig. 5a

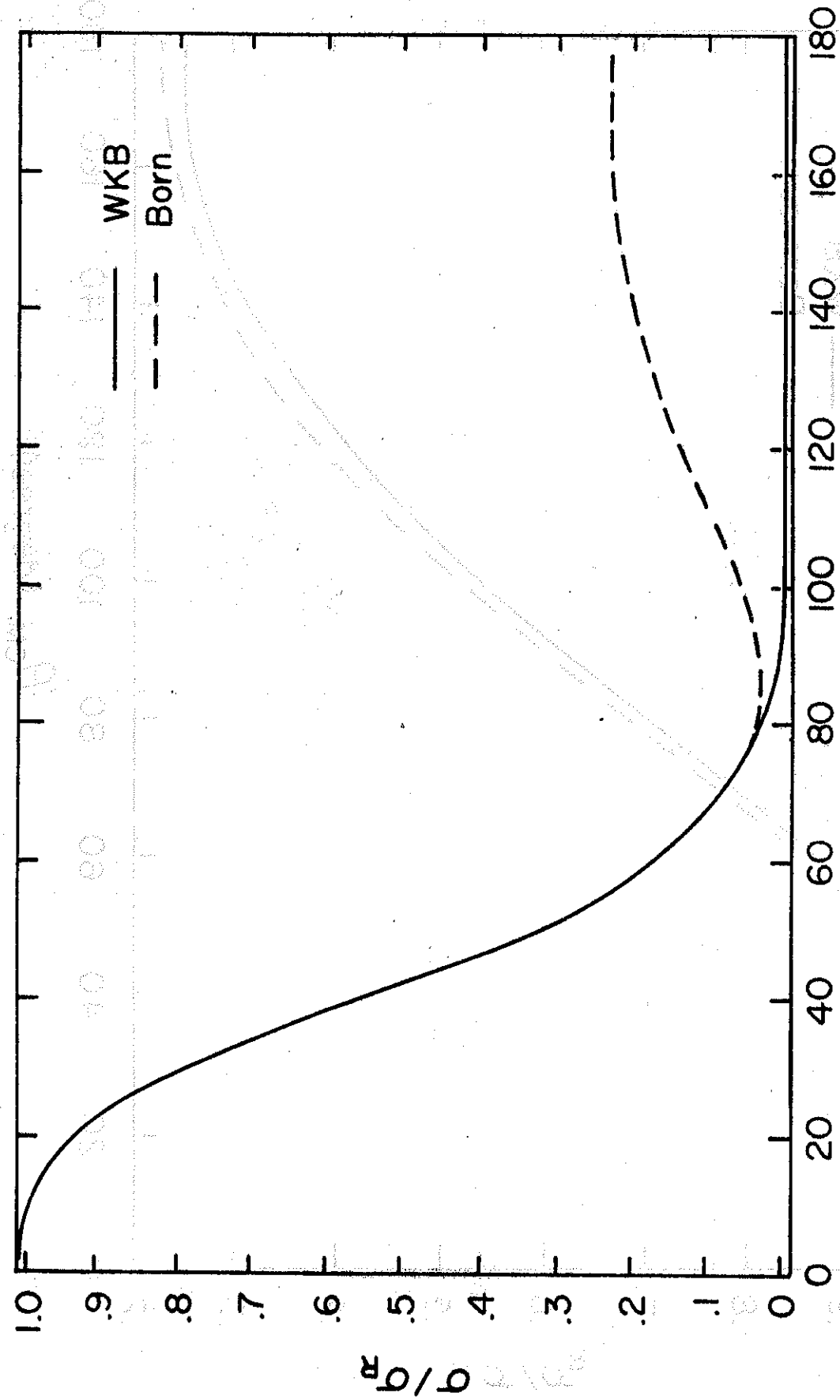
WKB
Born
σ/σR
θc.m. (degrees)
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
20
40
60
80
100
120
140
160
180

Fig. 5b