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ABSTRACT

We prove a form of reflection positivity in planes containing sites for a class of quantum lattice systems. Two applications to typical models are given: a proof of phase transition of ferromagnetic type by the method of infrared bounds for the Fisher-stabilized Ising antiferromagnet in an external magnetic field with parallel and transverse components, and a proof of a phase transition of antiferromagnetic type for the same model with no stabilization by a suitable version of the Peierls argument. We also discuss the spherical model in an appendix.

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1. INTRODUCTION AND SUMMARY

In a pioneer paper ([1]), Fröhlich, Simon and Spencer proved for the first time the existence of phase transitions for classical lattice systems with continuous symmetry. Their method was further generalized to include a class of quantum lattice systems by Dyson, Lieb and Simon ([2]) and later abstracted and generalized to include proofs of phase transitions for both classical and quantum lattice systems by Fröhlich, Israel, Lieb and Simon ([4]). In the latter, the property of reflection positivity (RP) was most clearly isolated as a central element of both the proofs employing the method of infrared bounds ([1], [4], [5]), as well as those which involve generalized versions of the Peierls argument ([3], [4], [5]).

In many applications, RP is required to be in planes between lattice sites ([1], [2]). In references ([3]) and especially ([5]) applications were considered which require RP in planes containing sites. As remarked in [5], this seems to pose the unfortunate limitation that quantum systems are not allowed. In this paper we consider, however, a class of quantum lattice systems requiring RP in planes containing sites, which involve typically a transverse magnetic field. Although somewhat restricted, this class illustrates a method which might be of wider range of application, and some of the results obtained seem to be of interest, in particular the characterization of intervals of variation of the several parameters involved where the phase transition is of ferromagnetic (sect. 3) or antiferromagnetic (sect. 4) type.

The paper is organized as follows. In sect. 2 we prove our main result, which is a form of RP in planes containing (or not) lattice sites for a class of models (theorem 2.1). The method of proof may be very roughly described as rendering the system "classical" by use of the Trotter product formula, together with a choice of convenient intermediate states (alternatively, path space methods similar to [6] could have been used).

In sections 3 and 4 we illustrate the results through a typical model, namely, the Fisher-stabilized Ising antiferromagnet ([5]) in an external magnetic field with both parallel and transverse components. In sect. 3 we employ the method of infrared bounds, which gives us conditions on the various parameters such that the phase transitions be of ferromagnetic type (proposition 3-1 and corollaries). In this section we also employ the "classical version" of the model for the purpose of proving some inequalities along the lines of ref. [6] which are necessary for the proof. The motivation for inequalities of this type stems from the similar structure of the spherical model with external parallel field, which is discussed for completeness in an appendix. In sect. 4 we sketch the proof of a phase transition of antiferromagnetic type for the model without stabilization, using a version of the Peierls argument developped in ([3]) and ([5]).

2. REFLECTION POSITIVITY OF PLANES CONTAINING SITES: REAL QUANTUM SYSTEMS

The notation and terminology of this section follows with minor modifications the one adopted in [4]. Let $\mathfrak{A}$ be a real algebra with unit which in our applications will be typically non-abelian, and let $\hat{\mathfrak{A}}$ be an <u>abelian</u> sub-algebra of $\mathfrak{A}$. Given a linear functional on $\mathfrak{A}$: $A \longrightarrow \langle A \rangle_0$ with $\langle \mathbb{1} \rangle_0 = 1$ and $H \in \mathfrak{A}$ we define:

$$\langle A \rangle_H = \langle A e^{-H} \rangle_0 \, / \, \langle e^{-H} \rangle_0 \tag{2.1}$$

we suppose $\mathfrak{A}$ contains two sub-algebras $\mathfrak{A}_+$ and $\mathfrak{A}_-$ and a real linear morphism θ on $\mathfrak{A}_+ \cup \mathfrak{A}_-$ (the smallest sub-algebra of $\mathfrak{A}$

containing both $\mathfrak{A}_+$ and $\mathfrak{A}_-$) such that:

a) $\theta(\mathfrak{A}_+) = \mathfrak{A}_-$

b) $\theta^2 = \mathbb{1}$

c) $\langle \theta A \rangle_0 = \langle A \rangle_0 \quad \forall A \in \mathfrak{A}$

d) $\theta(\hat{\mathfrak{A}}_+) = \hat{\mathfrak{A}}_-$ where
$\hat{\mathfrak{A}}_\pm \equiv \hat{\mathfrak{A}} \cap \mathfrak{A}_\pm$

<u>Definition 2-1</u>: A real linear functional $\langle \cdot \rangle$ on $\mathfrak{A}$ in called $\hat{\mathfrak{A}}$ - reflection positive iff

$$\langle A\, \theta(A) \rangle \geqslant 0 \tag{2.2}$$

for all $A \in \hat{\mathfrak{A}}_+$. $\langle \cdot \rangle$ is called $\hat{\mathfrak{A}}$-generalized reflection positive iff

$$\langle A_1 \theta(A_1) \; \ldots \; A_n \theta(A_n) \rangle \geqslant 0 \tag{2.3}$$

for all $A_1, \ldots, A_n \in \hat{\mathfrak{A}}_+$ □

<u>Remark 2-1</u>:

1) It is important to notice that we do not assume $\mathfrak{A}_+$ and $\mathfrak{A}_-$ to commute with each other and this is the reason why we consider this restricted form of reflection positivity.

2) Since $\hat{\mathfrak{A}}$ is abelian, $\langle \cdot \rangle$ is $\hat{\mathfrak{A}}$-reflection position iff $\langle \cdot \rangle$ is $\hat{\mathfrak{A}}$-generalized reflection positive. □

It follows from the above definitions ([4]) that if $-H = B + \theta B + \sum_{i=1}^{K} C_i\, \theta(C_i)$ with $B, C_i \in \hat{\mathfrak{A}}_+$ them $\langle \cdot \rangle_H$ is $\hat{\mathfrak{A}}$-reflection positive. The aim of the following discussion is to extend this result, allowing B to be certain operators in $\mathfrak{A}_+$ rather than in $\hat{\mathfrak{A}}_+$.

We will consider the case where $\mathfrak{A}$ is the algebra of observables of a quantum system composed of three "parts" that is, its Hilbert space of states $\mathcal{H}$ is given by $\mathcal{H} = \mathcal{H}_- \otimes \mathcal{H}_0 \otimes \mathcal{H}_+$ where $\mathcal{H}_+$ and $\mathcal{H}_-$ are isomorphic, and $\mathcal{H}_\pm$, $\mathcal{H}_0$ are all finite dimensional. $\mathfrak{A}$ is the algebra of all real operators on $\mathcal{H}$.

$\mathfrak{A}_+$ is the algebra generated by all operators in $\mathfrak{A}$ of the form $\mathbb{1} \otimes A \otimes B$ (under the decomposition $\mathcal{H}_- \otimes \mathcal{H}_0 \otimes \mathcal{H}_+$). Since $\mathfrak{A}$ is the linear span of operators of the form $A \otimes B \otimes C$, Θ is will defined by

$$\theta(A \otimes B \otimes C) = C \otimes B \otimes A \tag{2.4}$$

If $\langle A \rangle_0 = \mathrm{Tr}_{\mathcal{H}} A / \mathrm{Tr}_{\mathcal{H}} \mathbb{1}$, then properties a), b) and c) listed above are trivially verified. Let $\hat{\mathfrak{A}}_+$ be a commutative sub-algebra of $\mathfrak{A}_+$, $\hat{\mathfrak{A}}_- = \Theta \hat{\mathfrak{A}}_+$ and $\hat{\mathfrak{A}} = \hat{\mathfrak{A}}_+ \cup \hat{\mathfrak{A}}_-$. Then property d) is also fulfilled.

In order to state the main result of this section we introduce further the sub-algebra $\mathcal{B}_+$ as the set of elements in $\mathfrak{A}_+$ of the form $\mathbb{1} \otimes \mathbb{1} \otimes A$, $\mathcal{B}_- = \Theta \mathcal{B}_+$ and $\mathcal{B}_0$ as the set of operators in $\mathfrak{A}$ of the form $I \otimes A \otimes \mathbb{1}$. Then $\hat{\mathcal{B}}_0 = \mathcal{B}_0 \cap \hat{\mathfrak{A}}$.

A bounded operator A on a Hilbert space $\mathcal{H}$ is called positivity preserving with respect to a basis $\{\varphi_n\}_{n \geq 1}$ in $\mathcal{H}$ iff $(\varphi_n, A\varphi_m) \geq 0$ for all $n, m \geq 1$.

<u>Theorem 2.1</u>: Let $B \in \mathcal{B}_+$, $B_0 \in \mathcal{B}_0$ and e^{tB_0} for all $t > 0$ be a positivity preserving operator in $\mathcal{H}_0$ with respect to a basis $\{\varphi_n^0\}_{n \geq 1}$ wich diagonalizes $\hat{\mathcal{B}}_0$. If

$$-H = B + \Theta B + B_0 + \sum_{i=1}^{N} C_i \Theta C_i + D + \Theta D \tag{2.5}$$

with $C_i, D \in \hat{\mathfrak{A}}_+$, then $\langle \ \rangle_H$ is $\hat{\mathfrak{A}}$-reflection positive.

Proof: Let $\left\{ \Psi_{\underline{n}} = \varphi_{n_-} \otimes \varphi^0_{n_0} \otimes \varphi_{n_+},\ \underline{n} \equiv (n_-, n_0, n_+) \right\}$ be a basis which diagonalizes $\hat{\mathfrak{A}}$. We first notice that if $V = B + \Theta B + B_o$ then

$$(\Psi_{\underline{n}} , e^{tV} \Psi_{\underline{m}}) = b^t (n_-, m_-)\, b_o^t(n_o, m_o)\, b^t (n_+, m_+) \tag{2.6}$$

whith $b^t(n,m) = (\varphi_n , e^{tB} \varphi_m)$ and $b_o^t (n,m) = (\varphi^0_n , e^{tB_o} \varphi^0_m)$.
Moreover if $-H_o = \sum C_i \Theta C_i + D + \Theta D$ then

$$(\Psi_{\underline{n}} , e^{-tH_o} \Psi_{\underline{m}}) = \delta_{\underline{n}\,\underline{m}} \, F^t (\underline{n}) \tag{2.7}$$

where the function $F_{\underline{n}}$ can be written in the form

$$F_{\underline{n}}(t) = \sum f_i^t (n_o, n_-)\, f_i^t (n_o, n_+) \tag{2.8}$$

We may further suppose the matrix elements $(\Psi_n, A\Psi_m)$ to be real for all $A \in \mathfrak{A}$, since $\mathfrak{A}$ is an algebra of real operators.

Using Trotter product formula we have then

$$\langle A\, \Theta(A) \rangle_H \langle e^{-H} \rangle = \mathrm{Tr}\ A\, \Theta(A)\, e^{-H} =$$

$$= \lim_{k\to\infty} \mathrm{Tr}\ A\, \Theta(A) \left[e^{-H_o/k}\, e^{-V/k} \right]^k \tag{2.9}$$

From (2.6), (2.7) and (2.8) we get

$$\mathrm{Tr}\ A\, \Theta(A) \left[e^{-H_o/k}\, e^{-V/k} \right]^k =$$

$$= \sum_{\underline{n}^1} \langle \Psi_{\underline{n}^1} , A\, \Theta(A) \left[e^{-H_o/k}\, e^{-V/k} \right]^k \Psi_{\underline{n}^1} \rangle =$$

$$= \sum_{\underline{n}^1, \underline{n}^2, \ldots, \underline{n}^k} a(n_o^1, n_-^1)\, a(n_o^1, n_+^1)\, F^t(\underline{n}^1)\, b^t(n_-^1, n_-^2) \,.$$

$$.\ b_o^t (n_o^1, n_o^2)\, b^t(n_+^1, n_+^2) \ldots F^t(\underline{n}^k)\, b^t(n_-^k, n_-^1) \,.$$

$$.\ b_o^t (n_o^k, n_o^1)\, b^t(n_+^k, n_+^1) =$$

$$= \sum_{\underline{i}} \sum_{n_o^1, n_o^2, \ldots, n_o^k} b_o^t (n_o^1, n_o^2) \ldots b_o^t(n_o^k, n_o^1) \left[g_{\underline{i}}(n_o^1, \ldots, n_o^k) \right]^2 \geq 0 \tag{2.10}$$

where the first summation sign $\sum_{\underline{i}}$ refers to the sums we get by using k times formula (2.8) and the $g_{\underline{i}}$ are functions of the type

$$\sum_{n^1, \ldots, n^k} a(n_0^1, n^1)\, f_{i_1}^{t}(n_0^1, n^1)\, b^{t}(n^1, n^2)\, f_{i_2}(n_0^2, n^2)\, b^{t}(n^2, n^3) \ldots$$
$$\ldots f_{i_k}^{t}(n_0^k, n^k)\, b^{t}(n^k, n^1)$$

In the above expressions $t = \frac{1}{k}$ and

$$a(n_0, n) = (\varphi_{n_0}^{\circ} \otimes \varphi_n ,\ A\ \ \varphi_{n_0}^{\circ} \otimes \varphi_n)_{\mathcal{H}_0 \otimes \mathcal{H}_+}$$

By taking the limit $k \to \infty$ we get

$$\langle (\theta A) A \rangle \quad \geqslant 0$$ ■

Remark 2-2:

1) The assumptions of the theorem imply the possibility of having a path space formulation for the abelian algebra $\hat{\mathfrak{O}\mathfrak{l}}$ (see reference [11]) which is implicit in the proof through the use of Trotter's formula. Therefore the systems considered are under some aspects "classical" ones.

2) The possibility of having reflection positivity for sub-algebras of quantum systems is mentioned in [3] and [5]. □

Remark 2.3: The typical application of the above result for quantum spin systems is the following. Let the Hamiltonian of quantum spin system be of the form

$$H_\Lambda = H_{0,\Lambda} + V_\Lambda \tag{2.11a}$$

where
$$H_{0,\Lambda} = \sum_{R \subset \Lambda} J(R)\ S_3(R) \tag{2.11b}$$

and
$$V_\Lambda = -a \sum_{x \in \Lambda} S_1(x) \tag{2.11c}$$

In the above $S_i(x)$, $i = 1, 2, 3$, $x \in \Lambda \subset \mathbb{Z}^\nu$ are spin operator:

$$\vec{S}^2(x) = \sum_{i=1}^{3} S_i(x)^2 = S(S+1)$$

$$[S_i(x), S_j(y)] = i\, \varepsilon_{ijk}\, S_k(x)\, \delta_{x,y}$$

and, for $R \subset \Lambda$

$$S_i(R) = \prod_{x \in R} S_i(x)$$

If $\hat{\mathfrak{A}}$ is the algebra generated by $S_3(R)$, $R \subset \Lambda$ and $\langle \cdot \rangle_{H_0}$ is $\hat{\mathfrak{A}}$ -reflection positive with reflections on a plane Π_0 containing (or not) sites of the lattice then $\langle \cdot \rangle_H$ is also $\hat{\mathfrak{A}}$ -reflection positive with respect to the same reflection operation. (Notice that for planes not containing sites our result would follow from the general theory of reflection positivity as developed in [4] without having to restrict to the abelian sub-algebra $\hat{\mathfrak{A}}$). As examples of the above structure we mention antiferromagnets of the type discussed in this paper, Pirogov-Sinai model ([8]) with a transverse external field, Ising model in triangular lattices with a transverse external field.

Our results also apply to a quantum version of the anharmonic crystal extending the results of [3] for the classical version. This will be the sybject of a subsequent paper. $\square$

3. INFRARED BOUNDS

As a typical example requiring the theory of section 2, we shall treat in this section and the next the Fisher-stabilized Ising antiferromagnet ([5]) in a magnetic field with both a parallel and a transverse component. In contrast to ([5]), the "next-nearest neighbour" interaction is taken for simplicity to be along lattice lines. The Hamiltonian is given by (2-11), with

$$J(R) = \begin{cases} +J > 0 & \text{if } R = \{x,y\}, \ x,y \text{ nearest neighbours } (|x-y|=1) \\ -\varepsilon < 0 & \text{if } R = \{x,y\}, \ x,y \text{ next-nearest neighbours along a lattice line } (|x-y|=2) \\ -h < 0 & \text{if } R = \{x\} \\ \text{zero otherwise} & \end{cases} \tag{3.1}$$

The sign of a is irrelevant, and we shall take a>0 in (2-11c). Due to the nonzero parallel field h, $\hat{\mathfrak{A}}$-generalized reflection positivity (henceforth RP for short) holds only in planes containing sites. The proof in theorem 2.1 through the Trotter product formula is used here in a two-fold way, both through the results of section 2 and for the purpose of proving certain inequalities. The latter are used to obtain bounds on expectation-values of certain operators, which seem difficult to get by other means. (See lemma 3-1). In the present model, and in the Pirogov-Sinai with transverse field mentioned in remark 2-3, these expectation values are not identically zero, due to the absence of the symmetry $S_3(x) \to -S_3(x)$. The symmetry-breaking interactions in these models are just those responsible for the lack of RP in planes between sites, so that these features are intimately related.

By theorem 2.1 and the methods of ([2]) and ([4]) we obtain the infrared bound

$$\left(\hat{S}_3(p)^*, \hat{S}_3(p) \right)_D \leq \frac{1}{2\beta\varepsilon E_2(p)} \qquad p \in \Lambda^*, \ p \neq 0 \tag{3.2}$$

where Λ^* is the lattice dual to Λ (see, e.g., [2]),

$$E_2(p) \equiv \sum_{j=1}^{\nu} (1 - \cos 2p_j)$$

$$\hat{S}_3(p) \equiv \frac{1}{\sqrt{\Lambda}} \sum_{x \in \Lambda} e^{-ipx} S_3(x)$$

and the Duhamel two-point function is defined by

$$(A,B)_D \equiv \frac{1}{\mathrm{Tr}\; e^{-\beta H}} \int_0^1 dx \; \mathrm{Tr}\left(e^{x\beta H} A\, e^{-(1-x)\beta H} B\right)$$

(with $H=H_\Lambda$). We have, on the other hand, the sum rule

$$\frac{1}{\Lambda} \sum_{p\in\Lambda^*} \langle \hat{S}_3^*(p)\, S_3(p) \rangle = 1 \qquad (3.3)$$

The connection between (3.2) and (3.3) is realized by the Bruch-Falk inequality ([7]) (rediscovered in [2]):

$$\frac{(A^*,A)_D}{\frac{1}{2}\langle A^*A + AA^* \rangle} \geq f\left[\frac{\beta \langle [A^*,[H,A]] \rangle}{4\;\frac{1}{2}\langle A^*A + AA^* \rangle}\right]$$

where f is the function from $[0,\infty)$ to $[0,1)$ defined implicitly by

$$f(x \tanh x) = \frac{\tanh x}{x}$$

The function f is monotone decreasing ([2]). An easy consequence of the latter property (theorem 2.2 of [2]) is the fact that if $b \geq g\; f\;(c/4g)$ with $b,g,c \geq 0$ and $b \leq b_o$, $c \leq c_o$, then $g \leq g_o$, where

$$g_0 = \frac{1}{2}(c_o b_o)^{1/2} \coth\left(\frac{c_o}{4b_o}\right)^{1/2} \qquad (3.4)$$

Now,

$$[\hat{S}_3^*(p),[H_\Lambda,\hat{S}_3(p)]] = \frac{4a}{\Lambda}\sum_{x\in\Lambda} S_1(x)$$

and so

$$C \equiv \beta\langle[\hat{S}_3^*(p),[H_\Lambda,\hat{S}_3(p)]]\rangle \leq 4a\beta \equiv C_o \qquad (3.5)$$

Therefore, from (3.2), (3.4), (3.5) and the Bruch-Falk inequality we have

$$\left\langle \hat{S}_3(p)^* \, S_3(p) \right\rangle \leq \left[\frac{a}{2\varepsilon E_2(p)} \right]^{1/2} \coth\left[\beta \left(2a\varepsilon E_2(p)\right)^{1/2}\right] \tag{3.6}$$

$$p \in \Lambda^*, \; p \neq 0$$

If we set

$$\varrho_0^\Lambda \equiv \frac{1}{\Lambda} \left\langle \hat{S}_3(0)^* \hat{S}_3(0) \right\rangle = \frac{1}{\Lambda} \left\langle \hat{S}_3(0)^2 \right\rangle$$

we obtain from (3.3) and (3.6) the inequality

$$\varrho_0^\Lambda \geqslant 1 - \frac{1}{\Lambda} \sum_{\substack{p \neq 0 \\ p \in \Lambda^*}} \left[\frac{a}{2\varepsilon E_2(p)} \right]^{1/2} \coth\left[\beta \left(2a\varepsilon E_2(p)\right)^{1/2}\right] \tag{3.7}$$

Define, now, the quantities

$$\tilde{\varrho}_0^\Lambda = \frac{1}{\Lambda} \left\langle \left(\hat{S}_3(0) - \left\langle \hat{S}_3(0) \right\rangle\right)^2 \right\rangle =$$

$$= \varrho_0^\Lambda - \frac{1}{\Lambda} \left\langle \hat{S}_3(0) \right\rangle^2 \tag{3.8}$$

$$g(\beta, h, a) \equiv \left| \frac{h}{\sqrt{a^2 + h^2}} \tanh\left(\beta \sqrt{a^2 + h^2}\right) \right|^2 \tag{3.9}$$

$$I(\nu) \equiv \frac{1}{(2\pi)^\nu} \int_{B_\nu} d^\nu p \; \frac{1}{2\, E_2(p)} = \frac{1}{(2\pi)^\nu} \int_{B_\nu} d^\nu p \; \frac{1}{2 \sum_{j=1}^{\nu} (1 - \cos p_j)} \tag{3.10}$$

where $B_\nu \equiv [-\pi, \pi]^\nu$

$$\tilde{\rho}_0 \equiv \lim_{\Lambda \to \infty} \tilde{\rho}_0^{\Lambda} \tag{3.11}$$

Lemma 3.1: $$\frac{\langle \hat{S}_3(0) \rangle^2}{\Lambda} \leq g(\beta, h, a)$$

if $\varepsilon \leq J$

We shall prove this lemma at the end of this section. Assuming it for the moment, we are ready to prove the main result of this section:

Proposition 3.1: Model defined by (2.11) and (3.1) has a phase transition characterized by

$$\tilde{\rho}_0 > 0 \tag{3.12}$$

in the region of parameters (β, a, h, ε) defined by the inequalities

$$\frac{1}{(2\pi)^\nu} \int_{B_\nu} d^\nu p \left[\frac{a}{2\varepsilon E_2(p)} \right]^{1/2} \coth \left[\beta \left(2a\varepsilon E_2(p) \right)^{1/2} \right] <$$

$$< 1 - g(\beta, h, a) \tag{3.13}$$

$$0 < \varepsilon \leq J \tag{3.14}$$

Proof: It follows from (3.7), (3.8) and lemma 3.1 (which is true provided (3.14) holds) that

$$\tilde{\varsigma}_0^{\Lambda} \geqslant 1 - g(\beta, h, a) - \frac{1}{\Lambda} \sum_{\substack{p \neq 0 \\ p \in \Lambda^*}} \left[\frac{a}{2\varepsilon E_2(p)} \right]^{1/2} \cdot \coth\left[\beta (2a\varepsilon E_2(p))^{1/2}\right]$$

Taking the limit $\Lambda \to \infty$ in the above inequality, we see that (3.12) will be satisfied if (3.13) is assumed. $\blacksquare$

Remark 3.1: The integral in the l.h.s. of (3.13) is finite if $\nu \geqslant 3$ as the inequality ([2])

$$\coth x \leq \frac{1}{x} + 1 \tag{3.15}$$

shows. $\square$

Remark 3.2: The same estimate (3.13), with g=0, $\varepsilon \to J$, $E_2(p) \to E_1(p) = J \sum_{i=1}^{\nu} (1 - \cos p_j)$ may be applied to the Ising model with transverse field and nearest-neighbour interactions of strength J considered in [6], leading to an improvement of the estimates found there. $\square$

Corollary 3.1: Inequality (3.12) is true if $\beta > \tilde{\beta}_c$, where $\beta = \tilde{\beta}_c$ is the unique solution of

$$1 - \alpha - \frac{1}{(2\pi)^\nu} \int_{B_\nu} d^\nu p \left[\frac{a}{2\varepsilon E_2(p)} \right]^{1/2} \coth\left[\beta (2a\varepsilon E_2(p))^{1/2}\right] = 0 \tag{3.16}$$

provided (3.14) holds and, in addition:

$$g(\beta, h, a) \leq \alpha \tag{3.17}$$

and

$$I(a, \varepsilon) \equiv \frac{1}{(2\pi)^\nu} \int_{B_\nu} d^\nu p \left[\frac{a}{2\varepsilon E_2(p)} \right]^{1/2} < 1 \tag{3.18}$$

Above, α is an arbitrary number such that $0 < \alpha < 1$.

Proof: Using inequality (3.15) together with the dominated convergence theorem (for $\nu \geqslant 3$) we see that the l.h.s. of (3.16) increases monotonically from $-\infty$ to $(1 - \alpha - I(a, \varepsilon))$ as β varies from 0 to ∞. Hence, there will be a unique solution of (3.16) if and only if and only if (3.18) holds. The final assertion follows then from (3.13) and (3.17). ■

<u>Corollary 3.2</u>: Inequality (3.12) holds if, in addition to (3.14) and (3.17), the following inequality holds:

$$\sqrt{\frac{a}{\varepsilon}}\,\sqrt{I(\nu)} + \frac{1}{\beta\varepsilon}\,I(\nu) < 1-\alpha \tag{3.19}$$

<u>Proof</u>: By (3.13), (3.17) and (3.15), (3.12) holds if

$$\frac{1}{(2\pi)^\nu}\int_{B_\nu} d^\nu p \left[\frac{a}{2\varepsilon\, E_2(p)}\right]^{1/2}\left[\frac{1}{\beta\,(2a\varepsilon\, E_2(p))^{1/2}} + 1\right] < 1-\alpha \tag{3.20}$$

By the Schwartz inequality

$$\frac{1}{(2\pi)^\nu}\int_{B_\nu} d^\nu p \left[\frac{a}{2\varepsilon\, E_2(p)}\right]^{1/2} \leq \sqrt{\frac{a}{\varepsilon}}\,\sqrt{I(\nu)} \tag{3.21}$$

and (3.19) follows from (3.20) and (3.21). ■

<u>Remark 3.3</u>: Condition (3.17) is satisfied in particular if

$$a \geqslant \sqrt{\frac{1-\alpha}{\alpha}}\; h$$

independently of β. □

<u>Remark 3.4</u>: Proposition 3.1 is of interest because it provides conditions on the $(\beta, a, h, \varepsilon)$ such that the phase transition be of <u>ferramagnetic</u> type. Indeed, (3.12) involves fluctuations of the <u>magnetization</u> and not of the <u>staggered magnetization</u>, as we should expect for an antiferromagnetic system. It is therefore in some sense complementary to the result obtained in the next section by the Peierls argument (which also holds for $\varepsilon = 0$). □

We now prove lemma (3.1). The proof is based on inequalities introduced in ref. [6] and the FKG inequality ([9]).

Hamiltonian (3.1) may be written

$$H_\Lambda = \frac{1}{2} J \sum_{|x-y|=1} S_3(x) S_3(y) - h \sum_{x\in\Lambda} S_3(x) - - \frac{\varepsilon}{2} \sum_{|x-y|=2} S_3(x) S_3(y) - a \sum_{x\in\Lambda} S_1(x)$$

Let A be the sublattice containing $\{0\}$ and B its complement with respect to $\mathbb{Z}^\nu$. By a rotation of π around the 1-axis of the spins in $B\cap\Lambda$, H_Λ is transformed to

$$H'_\Lambda \equiv -\tfrac{1}{2} J \sum_{|x-y|=1} S_3(x) S_3(y) - \sum_{x\in\Lambda} h(x) S_3(x) - - \frac{\varepsilon}{2} \sum_{|x-y|=2} S_3(x) S_3(y) - a \sum_{x\in\Lambda} S_1(x)$$

where h(.) is an alternating (staggered) magnetic field:

$$h(x) = \begin{cases} h & \text{if } x \in A \\ -h & \text{if } x \in B \end{cases}$$

Let now

$$\tilde{H}_\Lambda(\lambda) \equiv H'_\Lambda + \lambda J \sum_{|x|=1} S_3(0) S_3(x) + + \lambda \varepsilon \sum_{|x|=2} S_3(0) S_3(x)$$

and $\langle\cdot\rangle'_\lambda$ denote the expectation value in the Gibbs state defined by $\tilde{H}_\Lambda(\lambda)$.

In particular, $\langle\cdot\rangle'_{\lambda=0}$ is the Gibbs state defined by H'_Λ and $\langle\cdot\rangle'_{\lambda=1}$ is the state defined by $\tilde{H}_\Lambda(1)$, where the spin at x=0 is "decoupled" from its neighbours. By theorem 2.1,

$$\langle S_3(0)\rangle'_\lambda = \lim_{k\to\infty} \langle S(0,1)\rangle^{(k)}_\lambda \tag{3.21}$$

where $\langle S(0,1)\rangle^{(k)}_\lambda$ is the expectation value of the "classical" spin variable S(0,1) corresponding to $S_3(0)$ in the Ising model in

$(\nu + 1)$ dimensions which results from the proof of theorem 2.1 for this case (k being the index counting the number of iteractions in the Trotter product formula, as in (2.9)). For explicit formulas, see, e.g., ref. [6] . The only explicit result we shall need concerns the signof the coupling constant in the $(\nu + 1)$-th dimension ([6]):

$$\Gamma_k = -\tfrac{1}{2} \log \tanh (\beta a / k) \tag{3.22}$$

where k is the same index above.

Lemma 3.2: a) $\operatorname{sgn} \langle S_3(x) \rangle_\lambda^1 = \operatorname{sgn} h(x) \quad \forall \lambda \in [0,1]$

b) If $\varepsilon \leq J$,

$$\langle S_3(x) \rangle_{\lambda=0}^1 \leq \langle S_3(x) \rangle_{\lambda=1}^1 \tag{3.23a}$$

if $x \in A \cap \Lambda$,

$$\langle S_3(x) \rangle_{\lambda=0}^1 \geq \langle S_3(x) \rangle_{\lambda=1}^1 \tag{3.23b}$$

if $x \in B \cap \Lambda$

Further

$$\langle S_3(x) \rangle_{\lambda=1}^1 = \frac{h(x)}{\sqrt{a^2 + h(x)^2}} \tanh \left(\beta \sqrt{a^2 + h(x)^2} \right) =$$
$$= \operatorname{sgn} h(x) \times g(\beta, h, a) \tag{3.24}$$

Proof: Part a) follows from (3.21) and well-known results for the classical Ising model.

As for part b) we have, $\forall \lambda, k$

$$\frac{d\langle s(0,1)\rangle_\lambda^{(k)}}{d\lambda} = \beta J \sum_{|x|=1} \left[\langle s(0,1)^2 s(x,1)\rangle_\lambda^{(k)} - \langle s(0,1)\rangle_\lambda^{(k)} \langle s(0,1)\, s(x,1)\rangle_\lambda^{(k)} \right] +$$

$$+ \beta \mathcal{E} \sum_{|x|=2} \left[\langle s(0,1)^2 s(x,1)\rangle_\lambda^{(k)} - \langle s(0,1)\rangle_\lambda^{(k)} \cdot \langle s(0,1)\, s(x,1)\rangle_\lambda^{(k)} \right] \tag{3.25}$$

For any $\lambda \in [0,1]$, the N-body interactions in the above Ising model ($N \geqslant 2$) are ferromagnetic for sufficiently large k, because of (3.22). The FKG inequality ([9]) therefore applies and we obtain

$$\langle s(0,1)\, s(x,1)\rangle_\lambda^{(k)} \geqslant \langle s(0,1)\rangle_\lambda^{(k)} \langle s(x,1)\rangle_\lambda^{(k)} \tag{3.26}$$

and by part a)

$$\langle s(0,1)\rangle \geqslant 0 \tag{3.27}$$

Putting (3.26) and (3.27) into (3.25) yields

$$\frac{d\langle s(0,1)\rangle_\lambda^{(k)}}{d\lambda} \leq \beta J \left\{ 1 - \left[\langle s(0,1)\rangle_\lambda^{(k)}\right]^2 \right\} \cdot z_\nu \cdot \langle s(x,1)\rangle_{x\in B\cap\Lambda} + \beta \mathcal{E} \left\{ 1 - \left[\langle s(0,1)\rangle_\lambda^{(k)}\right]^2 \right\} \cdot z_\nu \cdot \langle s(x,1)\rangle_{x\in A\cap\Lambda} \tag{3.28}$$

where z_ν is the number of nearest (and next nearest) neighbours of a lattice point in ν dimensions. By part a) and translation invariance (with simultaneous change of sgnh(x))

$$\langle s(x,1)\rangle_{x\in A\cap\Lambda} = -\langle S(x,1)\rangle_{x\in B\cap\Lambda} \equiv S > 0 \tag{3.29}$$

By (3.28) and (3.29):

$$\frac{d\langle s(0,1)\rangle_\lambda^{(k)}}{d\lambda} \leq -z_\nu \beta (J-\varepsilon)\cdot(1-S^2)\, S$$
$$\leq 0$$

if $\varepsilon \leq J$. This holds for any (sufficiently large) k, hence also in the limit $k\to\infty$. The proof of (3.23b) is identical. ∎

Proof of lemma 3.1

$$\frac{1}{\Lambda}\langle \hat{S}_3(0)\rangle^2 = \left[\frac{1}{\Lambda}\langle \sum_{x\in\Lambda} S_3(x)\rangle\right]^2 =$$
$$= \frac{1}{\Lambda^2}\left[\sum_{x\in A\cap\Lambda}\langle S_3(x)\rangle'_{\lambda=0} - \sum_{x\in B\cap\Lambda}\langle S_3(x)\rangle'_{\lambda=0}\right] \leq$$
$$< g(\beta, h, a)$$

by (3.23) and (3.24). ∎

Remark 3.5: As remarked in the introduction, inequalities of the above type were suggested to us by inspection of the similar structure of the spherical model, which we discuss for completeness in an appendix. □

4. PEIERLS ARGUMENT

This section is very descriptive, because we only verify the assumptions necessary to apply the general results of ([3]) and ([5]) We consider as typical example the same model (III-1) but with

$\mathcal{E} \equiv 0$. Apart from a constant, the Hamiltonian may be written:

$$H_\Lambda \equiv J/2 \sum_{|x-y|=1} S_3(x)\, S_3(y) \;-\; h \sum_{x\in\Lambda} S_3(x) \;-$$

$$- a \sum_{x\in\Lambda} S_1(x) \tag{4.1}$$

At each site $x \in \mathbb{Z}^{\nu=2}$ let $P^{\pm}(x)$ be the orthogonal projection operators, which project onto the subspaces of states $|\pm)_x$ at x with $S_3(x)|\pm)_x = \pm|\pm)_x$. By combining the methods of references ([3]) and ([5]), the following result may be proved:

Proposition 4.1: Let $|h| < J$ Then there exist $0 < \beta_c(J) < \infty$ and $0 < a_0(J) < \infty$ such that, if $\beta > \beta_c(J)$ and $a < a_0(J)$ the following inequality holds:

$$\langle P^{\pm}(x)\; P^{\mp(-1)^{|y|}}(x+y) \rangle < 1/4 \tag{4.2}$$

<u>Remark</u>: As in ([5]), (4.2) implies the existence of more than one equilibrium state.

<u>Proof</u>: The proof follows from the method of ref. [3] (as applied to the quantum antiferromagnet) together with the general RP result of section 2, and the following remarks. As in ([3]) we use a contour argument but now draw contours between nearest neighbour spins if they have the same sign. The relevant "universal projection" P_Λ is of the form

+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+

(N=8, M=4)

or (+ → −)

(compare ([3]), pg 241). Let $e_0(a)$ be the ground state energy of H_Λ, and define (in close analogy to [3], pg. 256)

$$B_\Lambda \equiv a \sum_{x \in \Lambda} S_1(x)$$
$$H_\Lambda^z \equiv H_\Lambda - B_\Lambda$$
$$A_\Lambda \equiv H_\Lambda^z - e_0(a=1)$$

Again here $\pm B_\Lambda \leq a A_\Lambda$ by the variational principle. In analogy to ([3], (3.24), pg. 256), define

$$\varsigma \equiv e_0^z - e_0(a=1) + n \Delta \Lambda$$

where $e_0^z = e_0(a=0)$. Then ([3], pg.257)

$$\mathcal{E}^z(P_\Lambda) \equiv \inf \operatorname{spec}(P_\Lambda H_\Lambda^z P_\Lambda) - e_0(a=1)$$

is the minimal A_Λ-energy of any state in $P_\Lambda \mathcal{H}$ and satisfies:

$$\mathcal{E}^z(P_\Lambda) - \varsigma \geqslant (2J - n\Delta)\Lambda. \tag{4.3}$$

To prove (4.3), it suffices to recall ([5]) that, if $|h| < J$ the ground state of H_Λ^z is doubly degenerate, obtained by periodizing the block $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$, and also by translating the resulting state by one unit. With the above result, the proof is straight-forward along the lines of [3]. $\blacksquare$

<u>Remark 4.1</u>: In contrast with Proposition 3.1, the result of the previous proposition is typical of an <u>antiferromagnetic</u> phase transition (see remark 3.2). $\square$

APPENDIX

The Spherical Model With Staggered External Field.

A Motivating Exemples

Some results of section 3 (Lemma 3.1 for instance) were motivated by the following analogy of the spherical model in the presence of a staggered external field.

In a finite volume $\Lambda \subset \mathbb{Z}^\nu$ we consider a classical "spin" variable $\phi(x) \in \mathbb{R}$ at each site $x \in \Lambda$. For simplicity we take Λ to be the hypercube $\Lambda = \{-L+1, \ldots, 0, \ldots, L\}^\nu$. The energy $H_\Lambda(\phi)$ of a configuration $\phi: \Lambda \to \mathbb{R}$ is given by:

$$H_\Lambda(\phi) = \left(\phi, \left[-\frac{\Delta}{2} - \mu\right]\phi\right) - (h, \phi) \qquad \text{(A.1)}$$

where

a) the "lattice laplacean" Δ is given by

$$(-\Delta\phi)(x) = 2\phi(x) - \sum_{i=1}^{\nu}\left[\phi(x+e_i) - \phi(x-e_i)\right] \qquad \text{(A.2)}$$

The e_i, $i=1, \ldots, \nu$, being the unit vectors in the i-th direction of $\mathbb{Z}^\nu$, with translations defined by periodicity in Λ.

b) the scalar product $(.,.)$ is defined by

$$(f, g) \equiv \sum_{x \in \Lambda} \overline{f(x)}\, g(x) \qquad \text{(A.3)}$$

for any $f, g: \Lambda \longrightarrow \mathbb{C}$

c) $h: \Lambda \to \mathbb{R}$ is the external field:

$$h(x) = \begin{cases} +h & \text{if } x \in \Lambda_e, \text{ i.e., } \sum_{i=1}^{\nu} x_i \text{ is even} \\ -h & \text{if } x \in \Lambda_o, \text{ i.e., } \sum_{i=1}^{\nu} x_i \text{ is odd} \end{cases} \qquad \text{(A.4)}$$

d) the "chemical potential" $\mu = \mu_\Lambda(\beta, h) < 0$ is introduced in order to handle the spherical constraint, i.e. $\mu_\Lambda(\beta, h)$ solves the equation

$$\frac{1}{\Lambda} \langle (\phi, \phi) \rangle_\Lambda = 1 \tag{A.5}$$

where $\langle \quad \rangle_\Lambda$ refers to the expectation value in the Gibbs state defined by H_Λ at inverse temperature β.

For $f\ \Lambda \to \mathbb{C}$ we define its Fourier transform

$$\hat{f}(p) \equiv \frac{1}{\sqrt{\Lambda}} \sum_{x \in \Lambda} e^{-ipx} f(x) \tag{A.6}$$

for $p \in \Lambda^* = \{ p = \frac{x\pi}{L} , x \in \Lambda \}$

The hamiltonian H_Λ reads then:

$$H_\Lambda(\phi) = \sum_{k \in \Lambda^*} [\omega(k) - \mu] \hat{\phi}(k)^* \hat{\phi}(k) + \tag{A.7}$$
$$+ h \sqrt{\Lambda} \hat{\phi}(\pi)$$

where a) $\omega(k) = \sum_{i=1}^{\nu} (1 - \cos k_i)$ (A.8)

and b) $\hat{\phi}(k)^*$ denotes the complex conjugate of $\hat{\phi}(k)$

and so $\hat{\phi}(k)^* = \hat{\phi}(-k)$.

Since only gaussian integrations are involved the correlation functions can be obtained explicitly from the two-point functions:

$$\langle \hat{\phi}(k)^* \hat{\phi}(k) \rangle_\Lambda = \frac{1}{2\beta [\omega(k) - \mu]} \qquad k \neq \pi \tag{A.9}$$

$$\langle \hat{\phi}(\pi)^* \hat{\phi}(\pi) \rangle_\Lambda = \frac{1}{2\beta} \frac{1}{\omega(\pi) - \mu} + \frac{h^2 \Lambda}{4[\omega(\pi) - \mu]^2}$$

The sum rule A.5 can then be written as:

$$\frac{1}{\Lambda} \langle \hat{\phi}(0)^* \hat{\phi}(0) \rangle_\Lambda = 1 - \left[\frac{h^2}{4[\omega(\pi)-\mu]^2} + \right.$$

$$\left. + \frac{1}{\beta} \frac{1}{\Lambda} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \frac{1}{2[\omega(k)-\mu]} \right] \tag{A.10}$$

Therefore in the thermodynamic limit

$$\lim_{\Lambda \to \infty} \frac{1}{\Lambda} \langle \hat{\phi}(0)^* \hat{\phi}(0) \rangle_\Lambda > 0 \tag{A.11}$$

iff $$\frac{|h|}{2\,\omega(\pi)} = \frac{|h|}{4\nu J} < 1 \tag{A.12}$$

and $$\beta > I(\nu) \equiv \frac{1}{(2\pi)^\nu} \int_{B_\nu} \frac{d^\nu p}{2\,\omega(p)}$$

Since

$$\langle \hat{\phi}(0) \rangle = 0 \tag{A.13}$$

(A.12) implies long-range order. The existence of spontaneous (uniform) magnetization in this case can be obtained explicitly as for instance in [12] or from the general theorems of [3].

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