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THEOREM IN STATISTICAL MECHANICS

by

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ABSTRACT : We prove a Goldstone type theorem for a wide class of lattice and continuum quantum systems, both for the ground state and at non-zero temperature. For the ground state (T=0) spontaneous breakdown of a continuous symmetry implies no energy gap. For non-zero temperature, spontaneous symmetry breakdown implies slow clustering (no L^1 clustering). The methods apply also to non-zero temperature classical systems.

Key words : Energy Gap, Clustering, Goldstone Theorem

1. INTRODUCTION

Given a physical system with short range forces and a conti-
nuous symmetry, if the ground state is not invariant under the
symmetry the Goldstone theorem states that the system possesses
excitations of arbitrarily low energy ([1], [2]). In the case of
the ground state (vacuum) of local quantum field theory, the
existence of an energy gap is equivalent to exponential clustering
([3]). In this framework the Goldstone theorem was proved in
([4] and ([5]). For general ground states of nonrelativistic
systems, the two properties (energy gap and clustering) are how-
ever independent and, in particular, the assumption that the ground
state is the unique vector invariant under time translations does
not necessarily follow from the assumption of space-like cluste-
ring, as remarked in ([6]). This point was not taken into account
in the assumptions of ([7]) and ([8]). Another related aspect,
of greater relevance to our discussion, is the fact that the rate
of clustering is not expected to be related to symmetry break-
down and absence of an energy gap, since for example the ground
state of the Heisenberg ferromagnet ([6]) has a broken symmetry
and no energy gap, but is exponentially clustering (for the
ground state is a product state of spins pointing in a fixed
direction). On the other hand, for T>0 no energy gap is expected
to occur, at least under general time-like clustering assumptions
([12] , proposition 3); these assumptions may be verified for the
free Bose gas ([13]).

At non-zero temperature it is the cluster properties that are important in connection with symmetry breakdown. At non-zero temperature we may then formulate the Goldstone theorem as follows. Given a system with short range forces and a continuous symmetry, if the equilibrium state is not invariant under the symmetry, then the system does not possess exponential clustering.

It is our purpose to explore the validity of the Goldstone theorem for a wide class of spin systems and many-body systems, both for the ground state and at non-zero temperature. The main tool we will use at non-zero temperature is the Bogoliubov ine - quality, which is valid for both classical and quantum systems (see, for example, ([9]). We shall however present the discussion in the framework of quantum statistical mechanics. At zero temperature our method is related to that of ([7]), where a version of the theorem was proved, valid for one space dimension. A different proof, valid for the ferromagnetic Heisenberg Hamiltonian of finite range, and in greater analogy to the quantum field theory proofs of ([4]) and ([5]), was given in ([6]), and generalized in ([8]) to quantum spin systems of finite range.

Our results apply to states which are invariant with respect to spatial translations by some discrete set which is suffi - ciently dense. (For lattice systems this could be a sub-lattice and for continuum systems, a lattice imbedded in the continuum). More precisely we require the following condition.

Condition $\mathcal{L}$. There is a constant ℓ such that for all sufficiently large cubes Λ,

$$\frac{|\Lambda_{\mathcal{L}}|}{|\Lambda|} \geqslant \ell$$

where $|\Lambda|$ is the volume of Λ, $|\Lambda_{\mathcal{L}}|$ is the number of points in $\Lambda_{\mathcal{L}} = \Lambda \cap \mathcal{L}$.

We will prove that for interactions which are not too long range (see sections 3 and 4 for examples), for the ground state (T=0) spontaneous breakdown of a continuous symmetry implies no energy gap. For non-zero temperature (T > 0) spontaneous symmetry breakdown implies no exponential clustering (in fact no L^1 clustering).

For continuous system our results cover the case of the break donw of translational invariance. However at T=0 and non-zero den sities there is never an energy gap due to the breakdown of Galilean Invariance as remarked in ([11]).

Finally, we should like to stress that, although we present an informal treatment of the continuum case, our results for quantum spin systems are complete and rigorous.

2. GENERAL FRAMEWORK

The state of the system is described by the vector Ω in some Hilbert space. There is a symmetric Hamiltonian operator H and $H\Omega = 0$. To each cube $\Lambda \subset \mathbb{R}^d$ there is a set of observables $\mathcal{O}_\Lambda$ such that $[A, B] = 0$ if $A \in \mathcal{O}_{\Lambda_1}$, $B \in \mathcal{O}_{\Lambda_2}$ and Λ_1, Λ_2 are disjoint. The set of all observables is $\mathcal{O} \bigcup_\Lambda \mathcal{O}_\Lambda$ We define $\hat{A} = A - (\Omega, A\Omega)$. We suppose $A\Omega$ is in the domain of H for all $A \in \mathcal{O}$.

Let τ_x denote spatial translation by x. (In the continuum case $x \in \mathbb{R}^d$ in the lattice case $x \in \mathbb{Z}^d$). The state Ω is invariant under translations in the discrete set $\mathcal{L}$, satisfying condition $\mathcal{L}$ of the introduction.

Thus $(\Omega, \tau_i A\Omega) = (\Omega, A\Omega) \quad \forall A \in \mathcal{O}, \quad \forall i \in \mathcal{L}$

There is a one-parameter group of symmetry transformations σ_s

of $\mathcal{O}L$ commuting with the Hamiltonian and with all spatial translations :

$$\sigma_s \, H \, A \, \Omega \; = \; H \, \sigma_s \, A \, \Omega$$

$$\sigma_s \, \tau_x \; = \; \tau_x \, \sigma_s \qquad \forall x \in \mathbb{R}^d \; (\forall x \in \mathbb{Z}^d) \tag{2.1}$$

We suppose the symmetry σ_s is generated by a current

$$J_x = \tau_x \, J_0$$

where $J_0 \in \mathcal{O}L_0$ (lattice case);

$\qquad J_0 \in \mathcal{O}L_\Delta$ (continuum case); Δ is a cube of side δ .

(In the continuum case we may suppose J_x is smooth in x by first averaging the current over the small cube Δ).

Thus, if $A \in \mathcal{O}L_\Lambda$

$$\frac{d}{ds}\bigg|_{s=0} \, (\Omega, \, \sigma_s \, A \, \Omega) \; = \; i \, (\Omega, \, [J_\Lambda, A] \, \Omega)$$

where $\quad J_\Lambda = \sum_{i \in \Lambda} J_i \quad$ (lattice case)

$\quad J_\Lambda = \int_{\Lambda_\delta} d^d x \, J_x \quad$ (continuum case) Λ_δ = set of points within

$\qquad\qquad\qquad\qquad\qquad$ distance δ from Λ

By the group property, the invariance of Ω under the symmetry σ_s follows from

$$(\Omega, \, [J_\Lambda, A] \, \Omega) = 0 \qquad \text{for all } A \in \mathcal{O}L_\Lambda, \text{ and all cubes } \Lambda$$

The equilibrium property of Ω is given by

a) T=0 : Ω is a ground state; i.e. $(\psi, H \psi) \geqslant 0$

$\qquad$ for all ψ in the domain of H.

b) T>0 : Ω satisfies the Bogoliubov inequality; i.e. for all A,

B $\in \mathcal{O}L$

$$\left| (\Omega, \, [B,A] \, \Omega) \right|^2 \; \leq \; \beta \;\; (\Omega, \, \tfrac{1}{2}(A^+ A + A A^+)\Omega)\tfrac{1}{i}(\Omega, \, [B, \dot{B}]\Omega)$$

where $\dot{B} = i\,[H,B]$. Note that $\frac{1}{i}\left(\Omega, [B^\dagger, \dot{B}]\Omega\right)$ may be written in the form $\left(B\Omega, HB\Omega\right) + \left(B^\dagger\Omega, HB^\dagger\Omega\right)$.

The basic hypothesis about the state Ω which leads to the absence of symmetry breaking is

a) T=0 : there is an energy gap $\varepsilon > 0$, i.e.

$$\left(\psi, H\psi\right) \geqslant \varepsilon \qquad \text{for all } \psi \text{ in the domain of H, orthogonal}$$

to Ω, $\|\psi\| = 1$

b) T>0 : there is L^1 clustering; i.e. for each observable A,

$$\sum_{i \in \mathcal{L}} \left| \left(\Omega, A^\dagger \tau_i A\,\Omega\right) - \left(\Omega, A^\dagger \Omega\right)\left(\Omega, \tau_i A \Omega\right) \right| < \infty$$

The basic strategy is as follows. For each self-adjoint observable $A \in \mathcal{O}\mathcal{L}_{\Lambda_0}$ define $\gamma = \left.\frac{d}{ds}\right|_{s=0} \left(\Omega, \sigma_s A\,\Omega\right)$
We must show $\gamma = 0$.

Now by the translation invariance of Ω with respect to $\mathcal{L}$ we have for each cube Λ (and $\sigma_s \tau_i = \tau_i \sigma_s$)

$$\gamma = \frac{1}{|\Lambda_{\mathcal{L}}|} \left.\frac{d}{ds}\right|_{s=0} \left(\Omega, \sigma_s \sum_{i \in \Lambda_{\mathcal{L}}} \tau_i A\,\Omega\right)$$

where $\Lambda_{\mathcal{L}} = \Lambda \cap \mathcal{L}$ and $|\Lambda_{\mathcal{L}}|$ is the number of points in $\mathcal{L}$.

Thus

$$\gamma = \frac{1}{|\Lambda_{\mathcal{L}}|}\, i\, \left(\Omega, \left[\hat{J}_{\tilde{\Lambda}}, \sum_{j \in \Lambda_{\mathcal{L}}} \tau_j \hat{A}\right]\Omega\right)$$

where $\tilde{\Lambda} = \bigcup_{i \in \Lambda} \tau_i \Lambda_0$

We estimate γ as follows.

a) T=0 :

$$|\gamma|^2 \leq \frac{4}{|\Lambda_{\mathcal{L}}|^2} \left(\hat{J}_{\tilde{\Lambda}}\Omega, \hat{J}_{\tilde{\Lambda}}\Omega\right)\left(\sum_{i \in \Lambda_{\mathcal{L}}} \tau_i \hat{A}\,\Omega, \sum_{j \in \Lambda_{\mathcal{L}}} \tau_j \hat{A}\,\Omega\right)$$

Now for any observable B, $(\Omega, \hat{B}\Omega) = 0$. Thus the assumption of an energy gap $\varepsilon > 0$ implies

$$(\hat{B}\Omega, \hat{B}\Omega) \leq \frac{1}{\varepsilon}(\hat{B}\Omega, H\hat{B}\Omega) \leq$$

$$\leq \frac{1}{\varepsilon}[(\hat{B}\Omega, H\hat{B}\Omega) + (\hat{B}^\dagger\Omega, H\hat{B}^\dagger\Omega)] =$$

$$= \frac{1}{\varepsilon}(\Omega, [B^\dagger, [H, B]]\Omega) = \frac{1}{i\varepsilon}(\Omega, [B^\dagger, \dot{B}]\Omega)$$

Thus

$$|\gamma|^2 \leq \frac{4}{\varepsilon}\left[\frac{1}{|\Lambda_\varepsilon|}\frac{1}{i}(\Omega, [J_{\tilde\Lambda}, \dot{J}_{\tilde\Lambda}]\Omega)\right] \cdot \frac{1}{|\Lambda_\varepsilon|}\left\|\sum_{j\in\Lambda_\varepsilon}\tau_j\hat{A}\Omega\right\|^2 \tag{*}$$

b) $T > 0$: using the Bogoliubov inequality,

$$|\gamma|^2 \leq \beta\left[\frac{1}{|\Lambda_\varepsilon|}\frac{1}{i}(\Omega, [J_{\tilde\Lambda}, \dot{J}_{\tilde\Lambda}]\Omega)\right]\frac{1}{|\Lambda_\varepsilon|}\left\|\sum_{j\in\Lambda_\varepsilon}\tau_j\hat{A}\Omega\right\|^2 \tag{**}$$

Notice the similarity of inequalities (*) and (**).

In one case the coefficient involves β, the inverse temperature. In the other case the coefficient involves $\frac{1}{\varepsilon}$, the inverse gap.

To prove absence of symmetry breakdown we will show in both cases (T=0 and T>0):

I) $\frac{1}{|\Lambda_\varepsilon|}(\Omega, [J_{\tilde\Lambda}, \dot{J}_{\tilde\Lambda}]\Omega) \to 0$ as $\Lambda \nearrow \mathbb{R}^d$

and

II) $\frac{1}{|\Lambda_\varepsilon|}\left\|\sum_{j\in\Lambda_\varepsilon}\tau_j\hat{A}\Omega\right\|^2 \leq C$ uniformly in Λ.

I)   follows essentially from properties of the Hamiltonian.

II)   follows from L^1 clustering ($T>0$) or from properties of the

Hamiltonian and the energy gap (T=0). Indeed from L^1 clus-

tering, and invariance of Ω under τ_j , $j \in \mathcal{L}$.

$$\frac{1}{|\Lambda_{\mathcal{L}}|} \left\| \sum_{j \in \Lambda_{\mathcal{L}}} \tau_j \hat{A} \Omega \right\|^2 \leq \sum_{j \in \mathcal{L}} \left| (\Omega, A \tau_j A \Omega) - (\Omega, A \Omega)(\Omega, A \Omega) \right| < \infty$$

In the case T=0, from the energy gap $\mathcal{E}$, we have

$$\frac{1}{|\Lambda_{\mathcal{L}}|} \left\| \sum_{\tau \in \Lambda_{\mathcal{L}}} \tau_j \hat{A} \Omega \right\|^2 \leq \frac{1}{i\mathcal{E}} \frac{1}{|\Lambda_{\mathcal{L}}|} \left(\Omega, \left[\sum_{j \in \Lambda_{\mathcal{L}}} \tau_j A , \sum_{k \in \Lambda_{\mathcal{L}}} \tau_k \dot{A} \right] \Omega \right) \leq$$

$$\leq \frac{1}{i\mathcal{E}} \sum_{k \in \mathcal{L}} \left| (\Omega, [A, \tau_k \dot{A}] \Omega) \right|$$

The finiteness of the sum over $\mathcal{L}$ will follow from properties

of the Hamiltonian.

The system is said to have property G_T if

$$\sup_{j \in \mathbb{Z}^d} \sum_{\substack{k \in \mathbb{Z}^d \\ |k-j| \geq D}} \left| (\Omega, [J_j , \dot{J}_k] \Omega) \right| \longrightarrow 0 \text{ as } D \nearrow \infty$$

$$\sup_{x \in \mathbb{R}^d} \int_{|y-x| \geq D} d^d y \ \left| (\Omega, [J_x, \dot{J}_y] \Omega) \right| \rightarrow 0 \text{ as } D \nearrow \infty$$

The system is said to have property G_0 if property G_T holds

and for each selfadjoint observable A,

$$\sum_{k \in \mathcal{L}} \left| (\Omega, [A, \tau_k \dot{A}] \Omega) \right| < \infty$$

We may now state Goldstone's theorem in the form

THEOREM 1.

    a) $T=0$: If the system possesses an energy gap and property

       G_0 then there is no spontaneous symmetry breakdown.

    b) $T>0$: If the system possesses L^1 clustering and property

       G_T then there is no spontaneous symmetry breakdown.

Proof : We must show that I) follows from

$$\sup_{x \in \mathbb{R}^d} \int_{|y-x| \geq D} d^d y \; |(\Omega, [J_x, \dot{J}_y]\Omega)| \xrightarrow[D \nearrow \infty]{} 0$$

(The lattice case is analogous).

Write $\quad \mathcal{g}(x,y) = (\Omega, [J_x, \dot{J}_y]\Omega)$

Then

$$\int d^d y \; |\mathcal{g}(x,y)| < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} \int_{|y-x| \geq D} d^d y \; |\mathcal{g}(x,y)| \xrightarrow[D \nearrow \infty]{} 0$$

Now

$$\int d^d x \; \mathcal{g}(x,y) = \frac{d}{ds}\Big|_{s=0} (\Omega, \sigma_s H J_y, \Omega) = \frac{d}{ds}\Big|_{s=0} (\Omega, H \sigma_s J_y \Omega) =$$

$$= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad (***)$$

Also by the Jacobi identity $\quad \mathcal{g}(x,y) = \mathcal{g}(y,x)$

Then since $\quad |\Lambda_\ell| \geq \ell|\Lambda| \geq \frac{\ell}{|\Lambda_0|}|\tilde{\Lambda}| \quad$ we estimate

$$\frac{1}{|\tilde{\Lambda}|}(\Omega, [J_{\tilde{\Lambda}}, \dot{J}_{\tilde{\Lambda}}]\Omega) = \frac{1}{|\tilde{\Lambda}|}\int d^d x \, d^d y \; \chi_{\tilde{\Lambda}_\delta}(x) \chi_{\tilde{\Lambda}_\delta}(y) \, \mathcal{g}(x,y) =$$

$$= -\frac{1}{2|\tilde{\Lambda}|}\int d^d x \, d^d y \; |\chi_{\tilde{\Lambda}_\delta}(x) - \chi_{\tilde{\Lambda}_\delta}(y)|^2 \, \mathcal{g}(x,y)$$

where $\quad \chi_\Lambda \quad$ is the characteristic function of Λ , using $(***)$.

Thus

$$\left| \frac{1}{|\tilde{\Lambda}|} (\Omega, [J_{\tilde{\Lambda}}, \dot{J}_{\tilde{\Lambda}}]\Omega) \right| \leq$$

$$\leq \frac{1}{|\tilde{\Lambda}|} \int\limits_{\tilde{\Lambda}_\delta} d^d x \int\limits_{\tilde{\Lambda}_\delta^c} d^d y \, |\vartheta(x,y)| \equiv \quad (2.2)$$

We write $\displaystyle\int\limits_{\tilde{\Lambda}_\delta} d^d x = \int\limits_{\Lambda_1} d^d x + \int\limits_{\Lambda_2} d^d x$

where Λ_2 is the set of points in $\tilde{\Lambda}_\delta$ within distance D of the boundary and $\Lambda_1 = \tilde{\Lambda}_\delta - \Lambda_2$

Then

$$(2.2) \leq \sup_x \int\limits_{|x-y| \geq D} d^d y \, |\vartheta(x,y)| + \frac{|\Lambda_2|}{|\tilde{\Lambda}|} \sup_x \int d^d y \, |\vartheta(x,y)|$$

The first term goes to zero by property G_T and the second term goes to zero since $\dfrac{|\Lambda_2|}{|\tilde{\Lambda}|} \xrightarrow[\tilde{\Lambda} \nearrow \mathbb{R}^d]{} 0$.

The theorem gains content by analyzing when property G_T or G_0 holds. This will be done in the following sections.

3. QUANTUM SPIN SYSTEMS

The interaction Φ is determined by specifying for each finite $X \in \mathbb{Z}^d$ the "connected" X - body interaction $\Phi(X) \in \mathcal{O}_X$. The Hamiltonian H is then defined by

$$H A \Omega = \sum_{X \cap \Lambda_0 \neq \emptyset} {}'[\Phi(X), A]\Omega$$

for $A \in \mathcal{O}_{\Lambda_0}$. Let $D(X)$ denote the diameter of X :

$$D(X) \equiv \sup_{i,j \in X} |i - j|$$

THEOREM 2.

The system satisfies G_T and G_0 if $\displaystyle\sup_i \sum_{X \ni i} |X| \, \|\Phi(X)\| < \infty$

and

$$\sup_i \sum_{\substack{X \ni i \\ D(X) \geq d}} |X| \, \|\Phi(X)\| \to 0 \quad \text{as} \quad d \nearrow \infty.$$

Note that if the interaction is translation invariant $\left(\Phi(X+i)=\tau_i\,\Phi(X)\right)$ then the above follows from

$$\sum_{X \ni 0} |X|\,\|\Phi(X)\| < \infty$$

If furthermore the interaction is at most N-body $\left(\Phi(X)=0 \text{ if } |X|>N\right)$ then the above is equivalent to $\sum_{X \ni 0} \|\Phi(X)\| < \infty$.

In particular for the Heisenberg Hamiltonian $\sum J(i-j)\,\sigma_i \cdot \sigma_j$ we require $\sum_{i \in \mathbb{Z}^d} |J(i)| < \infty$.

Proof :

a) We must show

$$\sup_{j \in \mathbb{Z}^d} \sum_{\substack{k \in \mathbb{Z}^d \\ |k-j| \geqslant D}} \left|\left(\Omega, [J_j, \dot{J}_k]\,\Omega\right)\right| \xrightarrow[D \nearrow \infty]{} 0$$

Now

$$\sum_{\substack{k \\ |k-j| \geqslant D}} \|[J_j, \dot{J}_k]\| \leq \sum_{\substack{k \\ |k-j| \geqslant D}} \sum_{X \ni k,j} \|[J_j, [\Phi(X), J_k]]\| \leq$$

$$\leq 4\,\|J_0\|^2 \sum_{\substack{k \\ |k-j| \geqslant D}} \sum_{X \ni k,j} \|\Phi(X)\| \leq 4\,\|J_0\|^2 \sum_{\substack{X \ni j \\ D(X) \geqslant D}} |X|\,\|\Phi(X)\|$$

which goes to zero as $D \nearrow \infty$ by the hypothesis of the theorem.

b) We will show $\sum_{j \in \mathbb{Z}^d} \left|\left(\Omega, [A, \tau_j \dot{A}]\,\Omega\right)\right| < \infty$

for each observable A

The proof is similar to a). Let $A \in \mathcal{O\!l}_{\Lambda_0}$.

$$\sum_{j \in \mathbb{Z}^d} \|[A, \tau_j \dot{A}]\| \leq 4\,\|A\|^2 \sum_{j \in \mathbb{Z}^d} \sum_{\substack{X \cap \tau_j \Lambda_0 \neq \emptyset \\ X \cap \Lambda_0 \neq \emptyset}} \|\Phi(X)\| +$$

$$+ \ 4 \|A\|^2 \sum_{\substack{j \\ \Lambda_0 \cap T_j \Lambda_0 \neq \emptyset}} \sum_{\substack{X \cap T_j \Lambda_0 \neq \emptyset}} \| \phi(X) \| \ \leq$$

$$\leq \ 4 \|A\|^2 \ |\Lambda_0| \ \sup_i \sum_{X \ni i} |X| \ \| \phi(X) \| \ +$$

$$+ \ 4 \|A\|^2 \ |\Lambda_0|^2 \ |\Lambda_0| \ \sup_i \sum_{X \ni i} \| \phi(X) \| \ \leqq$$

$$\leq \ 8 \ \|A\|^2 \ |\Lambda_0|^3 \ \sup_i \sum_{X \ni i} |X| \ \| \phi(X) \| \ .$$

4. CONTINUUM SYSTEMS

In this section we will proceed in an informal way, emphasizing the procedure and type of estimates. In the continuum case un-bounded operators will arise and, having constructed a particular equilibrium state, it would be necessary to verify that the cor-relation functions are indeed well-defined.

The Goldstone theorem is applicable to local quantum fields at zero and non-zero temperature. Indeed, properties G_T and G_0 are immediate consequences of the finite propagation speed.

Non-relativistic many body systems may also be treated. We consider first the breakdown of an internal symmetry in a trans-lation invariant system at non-zero temperature. The bose and fermi cases are treated in the same way. A particle with M in-ternal degrees of freedom is described by the field operators

$$\Psi_j (x) \ , \quad j = 1, 2, \ldots, M$$

which satisfy

$$[\Psi_j (x), \Psi_k^\dagger (y)]_\pm \ = \ \delta_{jk} \ \delta(x-y)$$

Let $\sigma^1, \ldots, \sigma^g$ be the selfadjoint $M \times M$ matrices of a representation of the Lie algebra of a compact semi-simple Lie group $\mathcal{G}$ with (totally antisymmetric ([10]) structure constants:

$$[\sigma^\alpha, \sigma^\beta] = i \; \Gamma^{\alpha\beta}_\gamma \; \sigma^\gamma$$

Define $\quad \varsigma^\alpha(x) = \sum_{jk} \psi^\dagger_j(x) \sigma^\alpha_{jk} \psi_k(x)$

We also write this as $\quad \psi^\dagger(x) \sigma^\alpha \psi(x)$

The operators $\quad \varsigma^\alpha(x) \quad$ are the local generators of the Lie group transformations on the fields $\quad \psi_k(x)$.

The Hamiltonian has the form $\quad H = H_0 + V - \mu N \quad$, where

$$H_0 = \frac{1}{2m} \sum_j \int d^n x \; \underline{\nabla} \psi^\dagger_j(x) \cdot \underline{\nabla} \psi_j(x)$$

$$V = \sum_\alpha \int d^n x \, d^n y \; : \varsigma^\alpha(x)\, \varsigma^\alpha(y) : \; V(x-y)$$

$$N = \sum_j \int d^n x \; \psi^\dagger_j(x) \, \psi_j(x)$$

If the internal symmetry $\mathcal{Y}$ is spontaneously broken, then the absence of L^1-clustering would follow from

$$\int dy \, |(\Omega, [J_0, \dot{J}_y] \Omega)| \; < \infty$$

where $J_y = \int d^n x \; h(y-x) \varsigma^\alpha(x)$ and h is a smooth function of compact support and $\int d^n x \; h(x) = 1$.

With

$$\underset{\sim}{J}^\alpha(x) = \frac{1}{2i} \left(\psi^\dagger \sigma^\alpha \underline{\nabla} \psi - \underline{\nabla} \psi^\dagger \sigma^\alpha \psi \right)$$

we have the following algebraic relations which hold in both the bose and fermi case :

$$[\varsigma^\alpha(x), \varsigma^\beta(y)] = i \, \delta(x-y) \, \Gamma^{\alpha\beta}_\gamma \, \varsigma^\gamma(y)$$

$$i \left[H_o , \varrho^\alpha (x) \right] = - \underline{\nabla} \cdot \underline{J}^\alpha (x)$$

$$\left[N , \varrho^\alpha (x) \right] = 0$$

$$\left[\varrho^\alpha (x) , \underline{J}^\beta (y) \right] = \frac{1}{i} \underline{\nabla}_x \left[\psi^\dagger (x) \frac{\left[\sigma^\alpha , \sigma^\beta \right]_+}{2} \psi (x) \delta(x-y) \right]_+$$

$$+ i \, \Gamma^{\alpha\beta}_\gamma \, \underline{J}^\gamma (y) \, \delta(x-y)$$

$$\left[\varrho^\alpha (y) , \left[\varrho^\alpha (x) , V \right] \right] = \delta(y-x) \sum_{\beta\gamma\delta} \Gamma^{\alpha\beta}_\gamma \Gamma^{\alpha\beta}_\delta \, .$$

$$\cdot \int d_{\mathcal{Z}} \, V(x-\mathcal{Z}) : \left[\varrho^\delta (x) , \varrho^\gamma (\mathcal{Z}) \right]_+ : \; -$$

$$- \sum_{\beta\gamma\delta} \Gamma^{\alpha\beta}_\gamma \Gamma^{\alpha\beta}_\delta \, V(x-y) : \left[\varrho^\gamma (x) , \varrho^\delta (y) \right]_+ :$$

where we have used

$$: \varrho^\alpha (x) \, \varrho^\beta (y) : \; = \; \varrho^\alpha (x) \varrho^\beta (y) - \delta(x-y) \psi^\dagger (y) \sigma^\alpha \sigma^\beta \psi (y)$$

To show $\int dy \left| (\Omega , \left[J_o , \dot{J}_y \right] \Omega) \right| < \infty$ we need only consider
the non-local term

$$\sum_{\beta\gamma\delta} \Gamma^{\alpha\beta}_\gamma \Gamma^{\alpha\beta}_\delta \, V(x-y) \, (\Omega , : \varrho^\gamma (x) \, \varrho^\delta (y) : \Omega)$$

So if $\int dy |V(x-y)| \, \left| (\Omega , : \varrho^\gamma (x) \, \varrho^\delta (y) : \Omega) \right| < \infty$
the result follows.

We note that there is a qualitative difference between abelian
and non-abelian groups since in the abelian case $\Gamma^{\alpha\beta}_\gamma = 0$ and

only local terms occur. (A similar analysis applies to the T=0
case, although here the distinction between abelian and non-abelian
groups does not arise because the term $\left(\Omega, [A, \tau_j \cdot \dot{A}] \Omega\right)$ will in
general have a term with $V(x-y)$ even in the abelian case).

We consider now the breakdown of translation invariance. We
will for simplicity not consider internal degrees of freedom, so
that now

$$H = H_0 + V - \mu N$$

where

$$H_0 = \frac{1}{2m} \int d^n x \ \underline{\nabla} \psi^+(x) \cdot \underline{\nabla} \psi(x)$$

$$\varsigma(x) = \psi^+(x) \psi(x)$$

$$N = \int d^n x \ \varsigma(x)$$

$$V = \int d^n x \ d^n y \ V(x-y) \ : \varsigma(x) \varsigma(y) :$$

The local generator of the translation group is

$$\underset{\sim}{\dot{j}}(x) = \frac{1}{2i} \left[\psi^+(x) \nabla \psi(x) - \nabla \psi^+(x) \psi(x) \right]$$

We consider the case T>0 with similar conclusions for the
case T=0.

We must show

$$\underset{x}{\sup} \ \int_{|y-x| \geqslant D} dy \ |(\Omega, [J_k(x), \dot{J}_k(y)] \Omega)| \longrightarrow 0 \quad_{D \nearrow \infty}$$

with

$$J_k(x) = \int d^n y \ h(x-y) \ \dot{j}_k(y)$$

Now

$$i [H, \dot{j}_k(x)] = \frac{\partial S_{k\ell}(x)}{\partial x_\ell} - \psi^+(x) \int d^n y \ \partial_k V(x-y) \varsigma(y) \psi(x)$$

where

$$S_{k\ell}(x) = -\frac{1}{2}\left[\partial_k \psi^+(x)\, \partial_\ell \psi(x) + \partial_\ell \psi^+(x)\, \partial_k \psi(x)\right] +$$

$$+ \frac{\delta_{k\ell}}{4}\left[\nabla^2 \psi^+(x)\, \psi(x) + 2\, \underline{\nabla}\cdot \psi^+(x)\cdot \underline{\nabla}\psi(x) + \psi^+(x)\, \nabla^2 \psi(x)\right]$$

where $\quad \partial_k V(\xi) = \dfrac{\partial V(\xi)}{\partial \xi_k}.$

Now since $\quad S_{k\ell}$ is a local term it will not contribute to the estimate, as $D \nearrow \infty$

$$\left[\dot{j}_k(x),\, [V,\, \dot{j}_k(y)]\right] = 2\left\{-V_{kk}(x-y):\wp(x)\wp(y): +\right.$$

$$\left. + \int d^n z\; V_{kk}(z-y):\wp(z)\wp(y):\;\delta(x-y)\right\}$$

where $\quad V_{kk}(\xi) = \dfrac{d^2 V(\xi)}{d\xi_k^2}$

Thus the only term of importance in the estimate is

$$\sup_x \int\limits_{|y-x|\geq D} dy \quad |V_{kk}(x-y)|\,\left|(\Omega,\,:\wp(x)\wp(y):\,\Omega)\right|$$

Thus if $\quad \sup_{x,y}\left|(\Omega,\,:\wp(x)\wp(y)\hat{)}:\,\Omega)\right| < \infty$

and $\quad \int dx \;|V_{kk}(x)| \;<\infty$

the result follows.

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