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ABSTRACT

Using non perturbative methods we prove conservation of the non-local quantum charge of the Gross-Neveu model, providing an exact S-matrix.

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The Gross-Neveu model[1] has been extensively studied in the last years. It is asymptotically free, displays mass transmutation, and has a well defined $1/N$ expansion[1]. It has been proved that the model has the factorization property in lowest order, providing a calculable S-matrix[2]. Afterwards, this S-matrix was extended to second order in $1/N$ perturbation theory[3]. However, we do not have, up to now, a general proof of the factorization property of this model outside the framework of perturbation theory, in contrast to the case of the non linear $O(n)$ symmetric σ model[4]. In this model we use the same approach of ref. (4) to show that the factorization property is a non-perturbative feature of the model.

In section I we define the model and the classical non local charge. In section II we define the quantum non local charge. In section III we write it in terms of asymptotic fields. In section IV we prove absence of particle production and the factorization equations. Section V is the conclusion.

I. THE MODEL AND THE NON LOCAL CHARGE

The Gross-Neveu model, is defined by the lagrangean density

$$L = i\bar{\psi}_\alpha \gamma^\mu \partial_\mu \psi_\alpha + \frac{g^2}{2}(\bar{\psi}_\alpha \psi_\alpha)^2 \qquad (I.1)$$

and describes $2N$ Majorana fields in $1+1$ space-time dimensions. We choose the following representation for the γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y \qquad (I.2a)$$

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\,\sigma_x \tag{I.2b}$$

$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{I.2c}$$

$$\overline{\psi} = \psi^T\gamma^0$$

The Fourier decomposition ψ_{in} reads (free case)

$$\psi^d_{in}(x) = \int_{-\infty}^{\infty} d\mu(p)\; b^d(p)\; u(p)\; e^{-ip^0x^0+ipx^1} \tag{I.3}$$

$$d\mu(p) = dp\sqrt{\frac{m}{2\pi p^0}} \tag{I.3a}$$

$$p^0 = \sqrt{m^2+p^2}$$

$$\{b^+(p),\, b(p')\} = \delta(p-p') \tag{I.3b}$$

and for the (m) two point Wightman function:

$$<0|\psi^a_{in,\alpha}(0)\;\psi^{b^+}_{in,\beta}(y)|0> = \delta^{ab}\; S_+(y)_{\alpha\beta} =$$

$$= \delta^{ab}\left[\gamma^0(\not{p}+m)\right]_{\alpha\beta}\frac{1}{i}\;\Delta_+(y) \tag{I.4}$$

$$\frac{1}{i}\Delta_+(y) = \int\frac{dp}{2\pi p^0}\; e^{-ip^0y^0+ipy'} = \frac{1}{2\pi}K_0(m\sqrt{-y^2}) \tag{I.5}$$

The model has a conserved Noether current associated to the $0(2N)$ symmetry:

$$J^{ab}_\mu(x) = 2i\;\overline{\psi}^a_{(x)}\;\gamma_\mu\;\psi^b_{(x)} = -\,J^{ba}_\mu(x) \tag{I.6}$$

This current satisfies the so-called integrability condition[5]

$$\partial_\mu J^{ab}_\nu - \partial_\nu J^{ab}_\mu - 2g^2(J^{ac}_\mu J^{cb}_\nu - J^{ac}_\nu J^{cb}_\mu) = 0 \tag{I.7}$$

which allows us to write down the conserved non-local charge

$$Q^{ab} = \int_{-\infty}^{\infty} dy_1 dy_2\;\varepsilon(y_1-y_2)\;J^{ac}_0(t,y_1)\;J^{cb}_0(t,y_2)$$

$$-g^2\int_{-\infty}^{\infty} dy\;J^{ab}_1(t,y) \tag{I.8}$$

II. QUANTUM DEFINITION OF THE NON-LOCAL CHARGE

In field theory, the expression (I.8) is ill defined, due to the divergence of the product of two currents in the first integral, which displays a linear divergence for small $|y_1 - y_2|$. We look for a Wilson expansion[6] for the product of two currents, which can be achieved and put in the form of a theorem:

Theorem: the Wilson expansion in the Gross-Neveu model for the product of two currents is given by

$$[J_\mu(z),J_\nu(0)]^{ab} = [C_1(z^2)z^2 g_{\mu\nu} z^\rho + C_2(z^2)z^2(z_\mu\delta^\rho_\nu + z_\nu\delta^\rho_\mu) + C_3(z^2)z_\mu z_\nu z^\rho]\, J^{ab}_\rho(0) + [D_1(z^2)z^\rho(z_\mu\delta^\sigma_\nu - z_\nu\delta^\sigma_\mu) + D_2(z^2)z^\sigma(z_\mu\delta^\rho_\nu - z_\nu\delta^\rho_\mu) + D_3(z^2)(\delta^\rho_\nu\delta^\sigma_\mu - \delta^\rho_\mu\delta^\sigma_\nu) + \frac{1}{2} C_1(z^2)z^2 g_{\mu\nu} z^\sigma z^\rho + \frac{1}{2} C_2(z^2)z^2 z^\sigma(z_\mu\delta^\rho_\nu + z_\nu\delta^\rho_\mu) + \frac{1}{2} C_3(z^2)z_\mu z_\nu z^\sigma z^\rho]\quad \partial_\sigma J^{ab}_\rho(0) + \mathcal{O}(|z|^{1-0}) \qquad \text{(II.1)}$$

where
$$C_1(z^2) = -\frac{(n-2)}{\pi}\frac{1}{(z^2)^2} + \mathcal{O}(|z|^{-2-0}) \qquad \text{(II.2a)}$$

$$C_2(z^2) = \frac{(n-2)}{\pi}\frac{1}{(z^2)^2} + \mathcal{O}(|z|^{-2-0}) \qquad \text{(II.2b)}$$

$$C_3(z^2) = 0 \qquad \text{(II.2c)}$$

$$D_1(z^2) = \frac{(n-2)}{4\pi}\frac{\ell n(-z^2\mu^2)}{z^2} + \frac{D_3(z^2)}{z^2} + \mathcal{O}(|z|^{-1-0}) \qquad \text{(II.2d)}$$

$$D_2(z^2) = -\frac{(n-2)}{4\pi}\frac{\ell n\,(-\mu^2 z^2)}{z^2} - \frac{D_3}{z^2} + \mathcal{O}(|z|^{-1-0}) \qquad \text{(II.2e)}$$

and μ is related to the fermion mass by

$$\mu = \frac{1}{2} m\, e^\gamma \quad , \quad \gamma = 0.577\ldots \text{ is the Euler-Mascheroni constant.} \qquad \text{(II.3)}$$

The proof of this theorem goes as follows. We first prove a lemma:

$$J^{ab}_\mu(z)\psi^c(0) = -\frac{1}{2\pi}\frac{\not{z}\gamma_\mu}{z^2}(\delta^{ac}\psi^b(0) - \delta^{bc}\psi^a(0)) + \mathcal{O}(|z|^{-0}) \qquad \text{(II.4)}$$

if the current is normalized by

$$\int dz^1\; i[J^{ab}_0\;(z),\psi^c(0)]_{z^0=0} = \delta^{ac}\psi^b(0) - \delta^{bc}\psi^a(0) \qquad \text{(II.5)}$$

The proof is straighforward, using the fact that this current has to be proportional to γ^μ (remembering that $J^{ab}_\mu = 2i\,\bar\psi^a\gamma_\mu\psi^b$) and current conservation.

The most general Wilson expansion has the form:

$$[J_\mu(z),J_\nu(0)]^{ab} = C^\rho_{\mu\nu}(z)\; J^{ab}_\rho(0) + D^{\sigma\rho}_{\mu\nu}(z)\;\partial_\sigma J^{ab}_\rho(0)\,. \qquad \text{(II.6)}$$

Using locality, PT and CP invariance, we have the relations[4]:

$$C^\rho_{\mu\nu}(z) = -C^\rho_{\nu\mu}(-z) = C^\rho_{\nu\mu}(z) \qquad \text{(II.7a)}$$

$$D^{\sigma\rho}_{\mu\nu}(z) = -D^{\sigma\rho}_{\nu\mu}(-z) - [C^\rho_{\nu\mu}(-z)z^\sigma - \frac{1}{2}g^{\sigma\rho}C^\lambda_{\nu\mu}(-z)z_\lambda] \qquad \text{(II.7b)}$$

$$D^{\sigma\rho}_{\mu\nu}(-z) = D^{\sigma\rho}_{\mu\nu}(z) \qquad \text{(II.7c)}$$

and with Lorentz invariance, we have (II.1) as the most general Wilson expansion.

Current conservation implies:

$$\frac{d}{dz^2}\,(z^2 D_1 - D_3) = -\frac{1}{4}C_1\, z^2 \qquad \text{(II.8a)}$$

$$\frac{d}{dz^2}\left(z^2 D_2 + D_3\right) = -\frac{1}{4} C_2 z^2 \qquad \text{(II.8b)}$$

$$z^2 \frac{d}{dz^2}\left(C_1 + C_2 + C_3\right) = -\left(C_1 + C_2 + 2C_3\right) \qquad \text{(II.8c)}$$

$$z^2 \frac{dC_2}{dz^2} = -\frac{1}{2}\left(C_1 + 5C_2\right) \qquad \text{(II.8d)}$$

$$D_1 + D_2 = \frac{1}{4}\left(C_1 + C_2 + C_3\right) z^2 \qquad \text{(II.8e)}$$

The above equations do not determine the coefficients completely. We use now:

$$i\left[\left[J_\mu(z), J_\nu(0)\right]^{ab}, \psi^d(t,0)\right]_{z^0=0} =$$

$$= i\, C^\rho_{\mu\nu}(z)\Big|_{z_0=0} \left[J^{ab}_\rho(0), \psi^d(t,0)\right] + \theta(|z|^{-0}) \qquad \text{(II.9)}$$

For $\mu=0$, $\nu=1$ and using (II.4), we get after some calculation:

$$C_2(z^2) = \frac{(n-2)}{\pi} \frac{1}{(z^2)^2} + \theta(|z|^{-2-0}) \qquad \text{(II.10)}$$

It follows from (II.8d) that C_1, is given by (II.2a) and from (II.8c)

$$C_3 = \frac{\lambda}{(z^2)^2} + \theta(|z|^{-2-0}) \qquad \text{(II.11)}$$

The normalization of the current, (II.5), implies:

$$\int_{-\infty}^{\infty} dz^1\; i\left\{\left[J_0(z), J_\nu(0)\right]^{ab} - \left[J_0(z), J_\nu(0)\right]^{ba}\right\}_{z^0=0} =$$

$$= -2(n-2)\, J^{ab}_\nu(0) \qquad \text{(II.12)}$$

and this relation forces $\lambda=0$, giving (II.2c).

Now, D_1, D_2 follow directly from the current conservation (II.8a) and (II.8b). We do not need the coefficient D_3, for our purposes. Equation (II.3) can be obtained by exactly the same procedure outlined in ref. (4).

We can define the cut-off non local charge:

$$Q^{ab} = \frac{1}{2n} \int_{|y_1-y_2|\geq\delta} dy_1 dy_2\; \varepsilon(y_1-y_2)\left[J_0(t,y_1), J_0(t,y_2)\right]^{ab}$$

$$- \frac{n-2}{2\pi} \ln(\mu\delta) \int_{-\infty}^{\infty} dy\; J^{ab}_1(t,y) \qquad \text{(II.13)}$$

which is readily seen to be finite and conserved in the limit $\delta \to 0$, by using the theorem of this section.

III. ASSYMPTOTIC CHARGE

Because of the conservation of the charge Q^{ab},

$$Q^{ab} = \lim_{\delta\to 0} Q^{ab}_\delta \,, \qquad \text{(III.1)}$$

we can write:

$$Q^{ab} = \lim_{t\to-\infty} Q^{ab}_{in}(t) = \lim_{t\to\infty} Q^{ab}_{out}(t) \,. \qquad \text{(III.2)}$$

It is our purpose to write expressions for the limits in (III.2). We write:

$$Q_{\delta_{in(out)}} = \frac{1}{n}\left[A_{\delta_{in(out)}}(t) + (n-2)\, B_{\delta_{in(out)}}\right] \qquad \text{(III.3)}$$

$$A_{\delta_{in}} = \int_{|y_1-y_2|\geq\delta} dy_1 dy_2\ \varepsilon(y_1-y_2) : J^{ac}_{0_{in}}(t,y_1) J^{cb}_{0_{in}}(t,y_2) : \qquad \text{(III.4)}$$

$$B_{\delta_{in}} = \frac{1}{2}\int_{|y_1-y_2|\geq\delta} dy_1\, dy_2\ \varepsilon(\eta_1-\eta_2) : \{S_+(0,y_1-y_2)(\psi^+_{a_{in}}(t,y_1)\psi^b_{in}(t,y_2)$$

$$- \psi^+_{b_{in}}(t,y_1)\psi_{a_{in}}(t,y_2)) + S_+(0,y_2-y_1)(\psi^{b^+}_{in}(t,y_2)\psi^a_{in}(t,y_1)$$

$$- \psi^{a^+}_{in}(t,y_2)\psi^b_{in}(t,y_1))\} : -\frac{1}{\pi}\,\ell n\,(\mu\delta) \int_{-\infty}^{\infty} dy\ J^{ab}_{1_{in}}(t,y) \qquad \text{(III.5)}$$

and analogously for the out fields.

Using (I.3) up to (I.5), and taking the limits (III.2), we have, after a long calculation:

$$\lim_{t\to+\infty\,(-)} A(t) = -A_{(+)} \qquad \text{(III.6)}$$

$$\lim_{t\to\pm\infty} B(t) = B \qquad \text{(III.7)}$$

where

$$A_{in} = -\int dp_1\, dp_2\ \varepsilon(p_1-p_2) : (b^{a^+}_{in}(p_1)\, b^c_{in}(p_1)$$

$$- b^{c^+}_{in}(p_1)\, b^c_{in}(p_1))(b^{c^+}_{in}(p_2) b^b_{in}(p_2) - b^{b^+}_{in}(p_2) b^c_{in}(p_2)) : \qquad \text{(III.8)}$$

$$B_{in} = \frac{1}{i\pi}\int_{-\infty}^{\infty} dp\ \ell n\,\frac{p^0+p}{m} : (b^{a^+}_{in}(p) b^b_{in}(p) - b^{b^+}_{in}(p) b^a_{in}(p)) : \qquad \text{(III.9)}$$

The out fields have the same expression.

Summing up:

$$Q^{ab}_{in} = \frac{1}{n}\left[A_{in} + (n-2)\, B_{in}\right] \qquad \text{(III.10)}$$

$$Q^{ab}_{out} = \frac{1}{n}\left[-A_{out} + (n-2)\, B_{out}\right] \qquad \text{(III.11)}$$

IV. ABSENCE OF PARTICLE PRODUCTION, AND FERMION-FERMION SCATTERING

Using the fact that

$$Q^{ab^+} = Q^{ab}$$

we will achieve many constraints applying Q^{ab}_{in} on the right and Q^{ab}_{out} on the left, in the amplitude

$$\langle \alpha \text{ out} \,|\, Q^{ab} \,|\, \beta \text{ in}\rangle \quad .$$

Calling θ the rapidity defined by

$$\theta = \ell n\,\frac{p^0 + p}{m} \quad , \qquad \text{(IV.1)}$$

we have, using (II.8) up (II.11)

$$Q^{ab}\,|\theta_1 c_1 \ldots \theta_\ell c_\ell \text{ in}\rangle = |\theta_1 d_1 \ldots \theta_\ell d_\ell \text{ in}\rangle\,(M^{ab}_{in})_{d_1-d_\ell c_1-c_\ell} \qquad \text{(IV.2a)}$$

$$\langle\theta_1 c_1 \dots \theta_\ell c_\ell \text{ out}|Q^{ab} = (M^{ab}_{out})_{c_1-c_\ell d_1-d_\ell} \langle\theta d_1 \dots \theta_\ell d_\ell \text{ out}| \qquad (IV.2b)$$

$$M^{ab}_{\substack{in\\out}} = \mp \frac{1}{n} \sum_{K<j=1}^{\ell} (I^{ac}_k I^{cb}_j - I^{bc}_k I^{ca}_j) + \frac{n-2}{i\pi n} \sum \theta_k I^{ab}_k \qquad (IV.3)$$

$$(I^{ab}_k)_{d_k c_k} = \delta^a_{d_k} \delta^b_{c_k} - \delta^b_{d_k} \delta^a_{c_k} \qquad (IV.4)$$

Now we need an expression (ansatz) for:

$$\langle\alpha \text{ out} \mid \beta \text{ in}\rangle \quad .$$

To achieve it we first prove that we have only elastic scattering.

Consider the amplitude

$$\sum_{c=1}^{2N} \langle\theta'_1 c'_1 \dots \theta'_{2\ell} c'_{2\ell} \text{ out}|Q^2|\theta_1 c \theta_2 c \text{ in}\rangle \qquad (IV.5)$$

Q^2 comutes with the isospin operator J^{ab}, so that the state $\sum_c |\theta_1 c, \theta_2 c \text{ in}\rangle$ is an eigenstate of Q^2. Its eigenvalue can be easily calculated, using (III.10):

$$Q^2 \sum_n |\theta_1 c\ \theta_2 c \text{ in}\rangle = \lambda \sum_c |\theta_1 c\ \theta_2 c \text{ in}\rangle \qquad (IV.6)$$

$$\lambda = 2(n-1)\left(\frac{n-2}{n}\right)^2 \left(1 + \frac{\theta^2}{\pi^2}\right) \qquad (IV.7)$$

$$\theta = \theta_1 - \theta_2$$

Now let us make Q^2 act on the left. We should remember that if the amplitude

$$\langle\theta'_1 c'_1 \dots \theta'_{2\ell} c'_{2\ell} \text{ out}|\theta_1 c_1\ \theta_2 c_2 \text{ in}\rangle \qquad (IV.8)$$

is non zero for some value of the set θ'_i, θ_i, (IV.5) should also be zero, because J^{ab} and Q^2 act irredutibly on states of particles with definite momenta.

The eigenvalue of Q^2, when acting on the left of (IV.5) in zero rapidity ($\theta_1 = \theta_2 = \dots = \theta_{2\ell} = 0$) is given by the value of (IV.7) on the threshold:

$$\lambda = 2(n-1)\left(\frac{n-2}{n}\right)^2 \left(1 + 4\left[\frac{1}{\pi}\ \ell n(\ell + \sqrt{\ell^2 - 1}\,\right]^2\right) \qquad (IV.9)$$

However λ should be an algebraic number, because Q^2 acting on the left is a matrix with racional coefficients and

$$a = \frac{1}{\pi}\ \ell n\ (\ell + \sqrt{\ell^2 - 1}\,)$$

should be an algebraic number, which is impossible by a theorem of number theory[(5),(7)].

So we conclude:

$$\langle\theta'_1 c'_1 \dots \theta'_{2\ell} c'_{2\ell} \text{ out}|\theta_1 c_1, \theta_2 c_2 \text{ in}\rangle = 0 \qquad (IV.10)$$

and as a consequence:

$$\langle\theta'_1 c'_1\ \theta'_2 c'_2 \text{ out}|\theta_1 c_1\ \theta_2 c_2 \text{ in}\rangle =$$

$$= (4\pi)^2\ \delta(\theta'_1 - \theta_1)\,\delta(\theta'_2 - \theta_2)\{\delta^{c'_1 c'_2}\ \delta^{c_1 c_2}\ \sigma_1(\theta)$$

$$+ \delta^{c'_1 c_1}\ \delta^{c'_2 c_2}\ \sigma_2(\theta) + \delta^{c'_1 c_2}\ \delta^{c'_2 c_1}\ \sigma_3(\theta)\} - (\theta_1 \leftrightarrow \theta_2) \qquad (IV.11)$$

The equation:

$$<\theta_1'c_1'\theta_2'c_2' \text{ out}|(Q^{ab}_{out})^+ \; |\theta_1 c_1 \theta_2 c_2 \text{ in}> =$$

$$= <\theta_1'c_1'\theta_2'c_2' \text{ out}|Q^{ab}_{in} \; |\theta_1 c_1 \theta_2 c_2 \text{ in}> \qquad (IV.12)$$

gives a set of equations for σ_1, σ_2, σ_3, which can be solved, giving as result:

$$\sigma_1(\theta) = \frac{2\pi i}{i\pi - \theta} \frac{\sigma_2(\theta)}{n-2} \qquad (IV.13)$$

$$\sigma_3(\theta) = -\frac{2\pi i}{\theta} \frac{\sigma_2(\theta)}{n-2} \qquad (IV.14)$$

V. CONCLUSION

The Gross-Neveu model has the factorization property exactly. When we proved absence of particle production, it was already enough to prove this assertion because of a well known theorem on S-matrix theory[8]. To calculate the exact total S-matrix, we need to know the bound state spectrum. This is already used in the calculation of ref. (2), and what we proved is that their S-matrix is the exact one.

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