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ON THE INFRARED DIVERGENCES OF THE CP^{n-1} MODELS

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ABSTRACT

In this paper we discuss some properties of the two-dimensional SU(n) non-linear sigma models, i.e., the CP^{n-1} models. They are 1/n expandable and ultraviolet renormalizable. Our main result is a proof that the infrared divergences associated with the topological gauge field are cancelled in the case of Green functions of gauge invariant operators.

RESUMO

Neste trabalho, apresentamos algumas propriedades dos modelos sigma não lineares bidimensionais com simetria SU(n), ou seja, os modelos CP^{n-1}. Eles são expansíveis em série de potências de 1/n e renormalizáveis na região ultravioleta. O aspecto mais importante está, porém, na demonstração de que as divergências infravermelhas associadas ao campo de gauge topológico se cancelam no caso de funções de Green de operadores invariantes de gauge.

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I. INTRODUCTION

For more than two decades, non linear sigma models have played an important role in our understanding of strong interactions. Initially, they were proposed by Gell-Mann and Lévy[1] in order to have at one's disposal models incorporating the ideas of PCAC and current algebras. Although non renormalizable, the four-dimensional versions were very useful for the derivation of low energy theorems in the so called phenomenological Lagrangian era[2]. In the seventies the two-dimensional models have gained a very important status for various reasons[3,4]. They are 1/n expandable, exhibit dimensional transmutation and asymptotic freedom. Moreover, while mantaining renormalizability, it is possible generalize these models to encompass local gauge invariance[5,6]. At the classical level such models are integrable having an infinite number of conserved currents, both local and non local[3,4,7].

The simplest extension is the CP^{n-1} model[5] which is the theory of an n-component complex field z, described by the Lagrangian density[8] (our calculations will be done in the Euclidian region):

$$\mathcal{L} = \overline{D_\mu z}\, D_\mu z \tag{I.1}$$

where

$$D_\mu = \partial_\mu + i A_\mu$$

$$A_\mu = i \frac{f}{n} \bar{z} \overleftrightarrow{\partial_\mu} z$$

subject to the constraint $\bar{z} z = \frac{n}{2f}$.

The invariance of $\mathcal{L}$ under the field transformation $z \to e^{i\alpha} z$ is trivially verified. Due to this fact, z itself is not an observable field. The latter must be a gauge invariant object.

Quantically, the dummy field A_μ becomes an independent field. Within the $1/n$ expansion, its propagator develops a pole at zero momentum and consequently, the quanta of the z_i fields are confined(8). This fact, on the other hand, raises some suspicious about the existence of the $1/n$ expansion. Truly, as mentioned in reference (8), infrared divergences are cancelled in the Green functions of gauge invariant operators. A proof of this statement, valid to every order of $1/n$, is the subject of this communication. This result will be probably useful in the formulation of the bound-state problem for this model.

The paper is organised as follows:

In section II we present the Feynman rules adequated to the $1/n$ expansion and discuss the ultraviolet structure of the theory. Section III is dedicated to the proof of the infrared finiteness of the Green functions of gauge invariant operators. Some remarks about possible extensions of our result are made in the Conclusions. We have also added an appendix with a brief derivation of the Feynman rules used in the text.

II. FEYNMAN RULES AND ULTRAVIOLET DIVERGENCES

The Feynman rules adequated to the $1/n$ expansion

were given in reference (8). For completeness, we present an alternative derivation in Appendix A. The momentum-space rules are given in figure (1), where

$$\Delta(p) = \frac{\delta_{ij}}{p^2 + m^2} \quad \text{is the } z \text{ propagator;} \tag{II.1}$$

$$D(p) = [A(p)]^{-1} \quad \text{is the } \sigma \text{ propagator} \tag{II.2}$$

with

$$A(p) = \frac{1}{2\pi} \frac{1}{[p^2(p^2+4m^2)]^{1/2}} \ln \frac{\sqrt{p^2+4m^2} + \sqrt{p^2}}{\sqrt{p^2+4m^2} - \sqrt{p^2}} \tag{II.3}$$

and

$$\Delta_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) D^\lambda(p) \tag{II.4}$$

with

$$D^\lambda(p) = \left[(p^2+4m^2) A(p) - \frac{1}{\pi}\right]^{-1} \tag{II.5}$$

is the $A_\mu = \frac{\lambda_\mu}{\sqrt{n}}$ propagator.

The σ field is a Lagrange multiplier introduced in order to enforce the constraint $\bar{z}z = \frac{n}{2f}$.

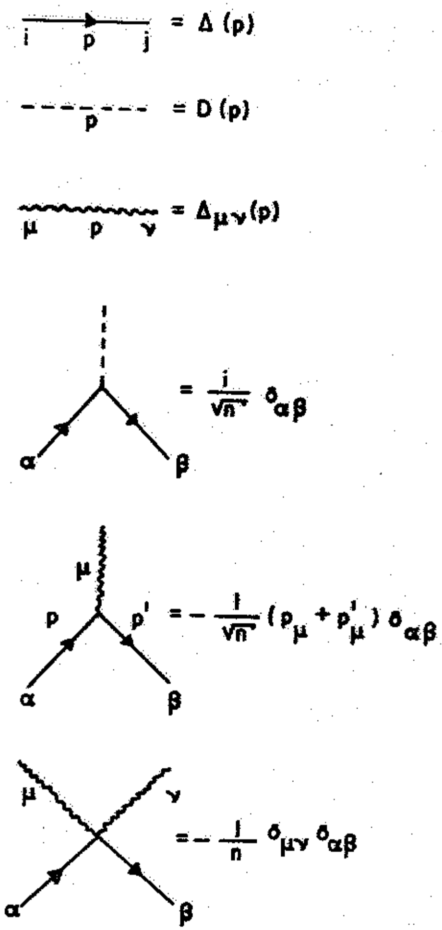

FIGURE 1

The graphs are to be constructed using the above rules, but ommiting diagrams containing the graphs of figure 2 as subgraphs.

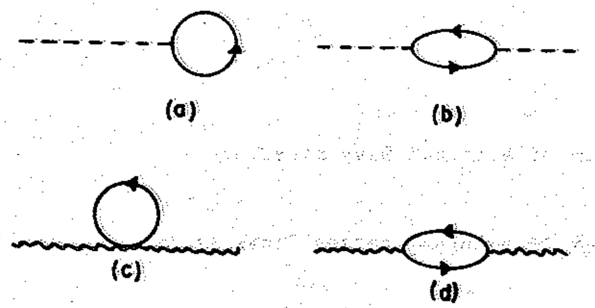

FIGURE 2

Note the pole of the λ_μ propagator at zero momentum. Therefore, in the non relativistic approximation, the quanta of the z_i fields (called partons in reference (8)) interact via a Coulomb like potential. In two dimensions this means confinement.

The σ field, on the other hand, does not have any singularities for real momentum. Thus, there is no particle like interpretation for this field.

Another consequence of the pole of the λ_μ propagator is that the Green functions are, in general, infrared divergent. However, in the sector of gauge invariant objects, these divergences are cancelled, as we discuss in the next section. At present (assuming some kind of infrared regulator), we want to argue that the model is renormalizable. This is done as follows.

The degree of superficial divergence associated with a proper graph γ can be obtained by power counting and is given by:

$$\delta(\gamma) = 2 - N^{\lambda} - 2N^{\sigma} \tag{II.6}$$

where

N^{λ} = # of external wavy lines of γ

N^{σ} = # of external dotted lines of γ.

Observe that $\delta(\gamma)$ does not depend on the number of external lines of the z_i fields. However, as shown by Aref'eva[9], if $N_z > 2$, these divergences will be cancelled. This result follows from the graphical identity of figure (3) which corresponds to the classical constraint $\bar{z}z$ = constant. Figure (4) provides an specific example of how this cancellation works. In that figure, graph (b) has a subgraph with the same divergence as the graph (a). If we contract this subgraph to a point and use the identity of figure (3) we obtain the cancellation of these divergences.

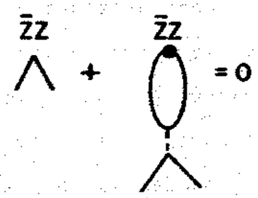

FIGURE 3

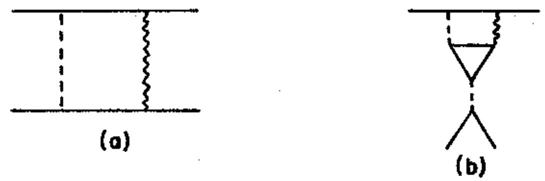

FIGURE 4

From the above discussion, we conclude that we can restrict our analysis to graphs with $N_z \leq 2$. We have then to consider the following cases:

1. $N^{\lambda} = 0$, $N^{\sigma} = 1$. As the σ field is not physical, this kind of divergence will occur only in 1PI subgraphs of Green functions with at least four z_i fields (more precisely: at least two z's and two $\bar{z}$'s). As argued before, these divergences are cancelled.

2. $N^{\lambda} = 0$, $N^{\sigma} = 0$, $N^{z} = 2$. $\delta(\gamma) = 2$. Convergence can be achieved by adding a second degree polynomial on the external momenta of the graph γ. The corresponding counterterm has the form $a\,\bar{z}z + b\,\partial_\mu\bar{z}\partial_\mu z$.

3. $N^{\lambda} = 1$, $N^{\sigma} = 0$, $N^{z} = 2$. $\delta(\gamma) = 1$. The necessary counterterm has the form $-b\frac{i}{\sqrt{n}}\lambda_\mu\bar{z}\overleftrightarrow{\partial}_\mu z$. Because of gauge invariance, the coefficient of the counterterm is the same as in the previous case. This can be readly verified by noting the following facts:

(i) the countertems can be simulated by application of Taylor operators of degree $\delta(\gamma)$ in the external momenta of γ.

(ii) at zero momentum, the insertion of a wavy line in a continuous one has the same effect of a derivation with respect to the momentum going through the latter.

4. $N^\lambda = 2$, $N^\sigma = 0$, $N^z = 0$. Although each graph of this type is logarithmically divergent, the sum of them is finite. This is proved by the same argument used in the previous case. For example, the sum of the graphs of figure (5) calculated for zero external momentum is proportional to:

$$\int d^2q\, \Delta_{\mu\nu}(q) \int d^2k \, \frac{\partial}{\partial k_\beta} \left\{ 2k_\alpha \Delta(k)(2k+q)_\nu \Delta(k+q)(2k+q)_\mu \Delta(k) \right\} = 0 \tag{II.7}$$

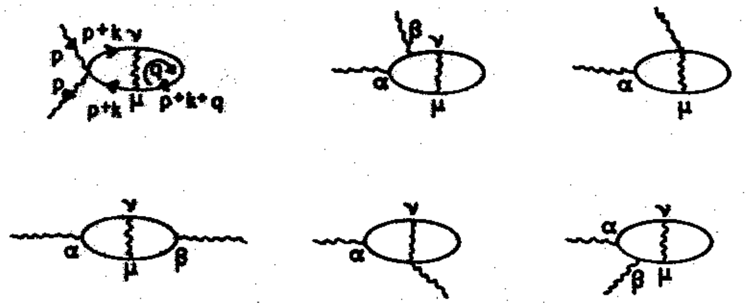

FIGURE 5

Thus, the sum of the graphs of figure 5 is finite.

5. $N^\lambda = 2$, $N^\sigma = 0$, $N^z = 2$. The necessary counterterm is of the form $\frac{b}{n}\bar{z}z\lambda_\mu\lambda_\mu$ and, again, the coefficient turns out to be the same as in the cases 2 and 3 because of gauge invariance.

We conclude that the theory can be made ultraviolet finite by adding to the original Lagrangian the counterterm

$a\,\bar{z}z + b\,\overline{D_\mu z}\,D_\mu z$ with the coefficients a and b fixed by mass and wave function renormalization.

III. INFRARED DIVERGENCES

As mentioned before, the fact that the λ_μ propagator has a pole at zero momentum implies the existence of severe infrared divergences. These appear already in the lowest non trivial order as exemplified by the graph of figure 6.

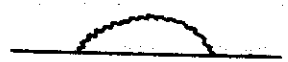

FIGURE 6

Nonetheless, the physics is in the sector of gauge invariant objects and there we can prove the cancellation of the divergences.

To attain this goal we note that in the infrared region any graph is at most logarithmically divergent. Then we prove infrared finiteness by verifying the cancellation of the residues of the pole associated with any internal wavy line. To be more precise we state our result in the form of a theorem.

Theorem : The CP^{n-1} Green functions of local gauge invariant operators are infrared finite in any order of the 1/n-expansion.

By linearity, we need to prove the theorem only for the case of Green functions containing operators of the type:

$$\partial^{(i_1)} F_{\mu_1\nu_1} \cdots \partial^{(i_n)} F_{\mu_n\nu_n} \left[\left(\overline{D^{(n'_1)}_{\rho'_1} z_{\alpha_1}} \right) \left(D^{(n_1)}_{\rho_1} z_{\beta_1} \right) \cdots \left(\overline{D^{(n'_m)}_{\rho'_m} z_{\alpha_m}} \right) \left(D^{(n_m)}_{\rho_m} z_{\beta_m} \right) \right] \quad \text{(III.1)}$$

$$\partial^{(j)} = \partial_{\sigma_1} \cdots \partial_{\sigma_j} \qquad \partial^{0} = I \quad \text{(III.2)}$$

$$D^{(n)}_{\rho} = \left(\partial_{\rho_1} + i A_{\rho_1} \right) \cdots \left(\partial_{\rho_n} + i A_{\rho_n} \right) \qquad D^{0} = I \quad \text{(III.3)}$$

$$A_{\mu} = \frac{\lambda_\mu}{\sqrt{n}} \quad \text{(III.4)}$$

The objects (III.1) constitute a basis in the sector of formally local gauge invariant operators. To prove the convergence, we observe that:

(i) As $A_\mu = \frac{\lambda_\mu}{\sqrt{n}}$ couples to the gauge invariant current $j_\mu = \bar{z}\overleftrightarrow{D}_\mu z$, it is sufficient to prove the convergence for operators without factors of $F_{\mu\nu}$.

(ii) It is always possible to choose the loop momenta so that the set of lines belonging to a given loop contains, at most, two lines joining at a given vertex: one of the lines is associated wiht a z field and the other one with a $\bar{z}$.

It is useful to decompose the operators in (III.1) into a sum of terms of the type

$$\left[\left(I_{\alpha_1} \bar{z}_{\alpha_1} \right) \left(I_{\beta_1} z_{\beta_1} \right) \cdots \left(I_{\alpha_m} \bar{z}_{\alpha_m} \right) \left(I_{\beta_m} z_{\beta_m} \right) \right] \quad \text{(III.5)}$$

where each I_{α_i} (and also each I_{β_i}) is a product of factors, each one being either a derivative ∂_μ or a field λ_μ.

After these considerations, let us examine the possible divergences associated with a given internal wavy line. There are two cases which must be analysed:

1º) At least one of the two ends of the line belong to a loop which does not contain external vertices. This means that one of the ends of the line belongs to a loop, C let us say, which only contains vertices of the type $\bar{z}D_\mu z$.

As remarked before, in the infrared region, the insertion of a wavy line is equivalent to the derivative operation. Therefore, the graphs that differ from each other only by the internal vertex of C in which the wavy line ends, summed up will give a total derivative with respect to the loop momentum through C. After integration this gives zero.

As the infrared divergence is, at most, logarithmical, we conclude that the graphs of the type considered add up to an infrared finite result.

2º) None of the ends of the wavy line belongs to a loop containing only internal vertices. The reasoning is the same as before. Here we also have to consider the possibility that one of the ends of the wavy line is attached to an special vertex of type (III.1). However, it is easily verified that this case gives contributions to the derivative of the momentum factors that, without the wavy line, would appear in the mentioned vertex. We conclude that the sum of the graphs results finite in the infrared region. This completes the proof of the theorem.

IV. CONCLUSIONS

We have shown that the Green functions of local, gauge invariant operators are free of infrared divergences. Although our discussion can not in general be applied to non local objects, there are some instances where the validity of such extension is easily verified. For example, if the non local operators are functions of $\oint_C A_\mu dx^\mu$, where C is some smooth contourn (Wilson loops, the instanton topological charge, etc.), one uses Stoke's theorem obtaining an expression obviously infrared finite.

Another case is that of the open string $\bar{z}(x) \exp i\int_x^y A_\mu dx^\mu z(y)$. Although classically this object can be written in terms of the operators (III.1) (by expanding around the point $x = y$), it seems that quantically there is no simple argument.

We have considered just the case of $\theta = 0$ vacuum. The treatment for the case $\theta \neq 0$ is similar, because the Feynman rules for the latter possibility[8] differ from those given in section II only by the addition of a new vertex proportional to the topological charge $\int \varepsilon_{\mu\nu} F^{\mu\nu} d^2x$. It is easily verified that these contributions do not produce new infrared divergences.

Due to the mass transmutation, the $1/n$-expansion is less singular than the perturbative one. In this context, it is interesting to compare our result with that obtained by a perturbative expansion of the $O(n)$ non linear sigma model[10]. In that case it was found that the infrared finite physical objects are those globally gauge invariant.

APPENDIX

In this appendix we want to give a brief derivation of the Feynman rules of the $1/n$ expansion for the CP^{n-1} model. Using functional techniques this was done in reference (8). Here we proceed as follows. First of all, in order to implement the classical constraint $\bar{z}z = n/2f$, we introduce a Lagrange multiplier field $\sigma(x)$, so that the Lagrangian for the model becomes:

$$\mathcal{L} = \overline{D_\mu z}\, D_\mu z + \sigma\left(\bar{z}z - \frac{n}{2f}\right) \tag{A.1}$$

The field equations are then:

$$D_\mu (D_\mu z) - \left(\bar{z}_\alpha D_\mu D_\mu z_\alpha\right) \frac{2f}{n} z = 0 \tag{A.2}$$

Quantically, the σ field can develop a non zero vacuum expectation value $\langle\sigma\rangle = m^2 \neq 0$. Making a shift $\sigma \to \sigma + m^2$ where the new σ has zero vacuum expectation value, we get (discarding a constant term):

$$\mathcal{L} = \overline{D_\mu z}\, D_\mu z + m^2 \bar{z}z + \sigma\left(\bar{z}z - \frac{n}{2f}\right) \tag{A.3}$$

The condition $\langle\sigma\rangle = 0$ gives

$$\int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+m^2} - \frac{n}{2f} = 0 \tag{A.4}$$

As the integral in (A.4) is logarithmically divergent, we replace it by the regularized expression

$$\int \frac{d^2p}{(2\pi)^2} \left[\frac{1}{(p^2+m^2)} - \frac{1}{(p^2+\Lambda^2)} \right] \tag{A.5}$$

where the Pauli-Villars regulator (Λ) shall tend to infinite at the end of the calculation. Before that, we introduce a renormalized coupling constant $f_r(\mu)$, defined by:

$$\frac{1}{f_r} = \frac{1}{f} - \frac{1}{2\pi} \ln \frac{\Lambda^2}{\mu^2} \tag{A.6}$$

The use of (A.5) and (A.6) in (A.4) results into "mass transmutation" by which a theory containing only dimensionless parameters generates a mass. In the present case it is given by:

$$m^2 = \mu^2 \exp\left(-\frac{2\pi}{f_r}\right) \tag{A.7}$$

Using (A.3), it is easy to compute the leading 1/n contributions to the proper two point functions for the σ and λ_μ fields.

$$\Gamma_\sigma^{(2)}(p) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2+m^2)} \frac{1}{[(k+p)^2+m^2]} \tag{A.8}$$

$$\Gamma^{(2)}_{\lambda,\mu\nu}(p) = 2\,\delta_{\mu\nu} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2+m^2)} - \int \frac{d^2k}{(2\pi)^2} \frac{(p+2k)_\mu (p+2k)_\nu}{(k^2+m^2)[(k+p)^2+m^2]} \tag{A.9}$$

(A.8) and (A.9) come from the graphs of figure 2(b) and 2(c,d) respectively. As it happens in gauge theories,

(A.9) has no inverse. To obtain a propagator we need to fix the gauge what is made by adding $\frac{1}{2\alpha}(\partial_\mu A_\mu)^2$ to (A.3). In the Landau gauge ($\alpha \to 0$), we obtain:

σ propagator: $\mathcal{D}(p) = [A(p)]^{-1}$

with (A.10)

$$A(p) = \frac{1}{2\pi} \frac{1}{[p^2(p^2+4m^2)]^{1/2}} \ln \frac{\sqrt{p^2+4m^2} + \sqrt{p^2}}{\sqrt{p^2+4m^2} - \sqrt{p^2}}$$

λ_μ propagator: $\Delta_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) D^\lambda(p)$

with (A.11)

$$D^\lambda(p) = \left[(p^2+4m^2) A(p) - \frac{1}{\pi} \right]^{-1}$$

The Lagrangian (A.3) and the expressions (A.10) and (A.11) give the Feynman rules listed in the text. The diagrams of figure 2 are to be omitted since they have already been explicitly considered.

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