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GENERALIZED LANGEVIN EQUATION FOR THE EXTENDED CHARGE IN STOCHASTIC ELECTRODYNAMICS

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SUMMARY - We derive a covariant equation for the motion of the extended charge and show how a consistent description is achieved for non relativistic velocities. If the external force is generated by the classical stochastic zero-point electromagnetic field the equation of motion has the form of a Langevin equation with memory. The memory function is due to radiation reaction and is related to the charge density which we have assumed to be spherically symmetric and rigid in the non relativistic limit. Some deviations from similar attempts are obtained. The extention of our results to finite temperatures is discussed .

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1. INTRODUCTION

Stochastic Electrodynamics is a theory which received growing attention in the last decade[1]. The central idea of this theory is to assume that the zero-point radiation is a classical and real field which produces observables effects and seems to have been introduced by Planck and Nernst in the beginning of our century. Its developments, however, was only advanced very recently mainly with the pioneer works of Brafford[2], Marshall[3] and many others[1,4,5,6].

The role of this classical theory, which is able to give a satisfatory description of a small but significant set of microscopic phenomena, is not yet very clear. It is obvious that it is a theory which is more complete than the orthodox classical physics because Stochastic Electrodynamics takes into account the interaction of charged particles with the radiation emitted by all the matter that fulfil the universe[7]. This radiation is identified with the zero-point radiation and is considered random because it is generated by a large number of sources that emits incoherently. The spectral distribution of this background radiation is univocaly fixed by the requirement that it must be isotropic and homogeneous in any inertial reference frame. Planck's constant enter in the theory as a free parameter, which is necessary to fix the intensity of the zero-point radiation and whose numerical value is determined by comparing the theoretical results with the experimental observation. In this way it is possible to derive (on classical grounds) Planck's formula[8,9] for the cavity radiation at temperature T as well as to get a satisfactory microscopic description of some systems as for instance the diamagnetic[10] behaviour of charged particles, the harmonic oscillator[5] and a few others successful results[1,4,5,6].

If we consider this initial success it is quite natural to raise the following question: it would be possible, with Stochastic Electrodynamics, to get a satisfactory classical explanation of the whole set of microscopic phenomena currently described by Quantum Mechanics? To answer this question it will be necessary to extend the Stochastic Electrodynamics calculations to many other phenomena including those involving non linear forces as the hydrogen atom for instance. At this point Stochastic Electrodynamics found its major obstacle and none non linear problem has been solved in a satisfactory way up to now[1]. The main difficulties

are concentrated around the following aspects of the theory: 1) the Lorentz invariant spectral distribution of the zero-point radiation generates a non Markovian stochastic processes and also divergent contribution to the kinetic energy of the free[11] and harmonically bound particle[12]; 2) the dissipative force associated to the non Markovian stochastic process is te radiation reaction force in the Abraham-Lorentz approximation and presents inconsistencies as runaway solutions or violation of causality[13,14].

Those two characteristics implies that the Fokker-Planck type equation for the probability distribution[1] (apparently the simplest way to study phenomena involving non linear forces) can only be obtained in an approximate way which is certainly inconsistent because the starting point, the so called Brafford-Marshall equation[1], presents the same shortcomings of the Abraham-Lorentz equation. Therefore the problem is to establish the equation of motion to be used in Stochastic Electrodynamics. This will be the goal of our paper.

The establishment of a consistent equation of motion, at least in the non relativistic limit, is clearly the starting point in order to answer the question raised above. The answer, no matter which, will be quite important to our understanding of Nature. If affirmative one can say that Quantum Mechanics is the "stationary" limit of Stochastic Electrodynamics[12]. If negative we shall be able to identify very clearly what are the ingredients <u>really non classical</u> of our microscopic world. Even this last alternative, much less ambitious, certainly will advance to a more profound comprehension of Quantum Theories.

The problem of getting a consistent equation of motion for a charged particle is very old in Classical Electrodynamics. The consensus is that this equation can only be obtained for extended charges[13-17]. To admit that microscopic charges are extended particles not only permits the obtention of an equation of motion without the inconsistencies of the Abraham-Lorentz one, but also permits to justify why Stochastic Electrodynamics gives finite results to the kinetic energy of the free and harmonically bound particle for instance, despite of the fact that the spectral distribution of zero-point radiation is not integrable.

We shall describe, initially, how to get a covariant equation of motion for a localized distribution of charge and matter by using the method proposed earlier

by Dixon[17]. Afterwards we shall discuss very briefly how the non relativistic limit leads to a consistent equation for a spherically symmetric (monopolar) charge distribution whose deformations are negligible[13,14]. Small deviations from previous works[11] are obtained and we show that they are really negligible in the case of the free particle. The so called dipole approximation[18], very frequently used in Stochastic Electrodynamics, is justified in the case of the extended charge in non relativistic motion. However those approximations are possible only if the mean square charged radius is much larger than a critical value which is approximately one over ten of the Compton wavelength for particles with the elementary charge. In passing we comment what is the order of magnitude of the charged radius of some microscopic particles, as the proton and the electron for instance, assuming that Relativistic Quantum Mechanics gives a realistic description of those corpuscles.

The equation of motion obtained takes a form quite similar to the generalized Langevin equation of non Markovian theories of Brownian motion[19,20]. The random force is generated by the interaction with the fluctuating electric field of zero-point radiation and have a temporal correlation function which depends on the spectral distribution of background radiation and is also a function of the Fourier transform of the charge density. The dissipative force is the radiation reaction force and have a memory kernel[19,20] like the generalized Langevin equations.

The effects of thermal radiation are discussed in the final part of our paper. We show that the expected dissipation due to the motion through the thermal radiation with Planck's spectrum (not Lorentz invariant) can be obtained only if we include the interaction with the Lorentz magnetic force which we have neglected before. We achieve a dissipative average force proportional to the velocity[21] (Stokes's law) only if we consider the combined effect of the radiation reaction force and the Lorentz magnetic force (treated perturbatively in our analysis).

Based on the results we propose that the non relativistic equation of motion for a charged particle in Stochastic Electrodynamics must be formulated for the extended charge and the effect of the fluctuating magnetic field should be included, even in the case of non relativistic motion, if we want to take into account all the dissipative effects which appear for non-zero temperatures.

2. COVARIANT EQUATION OF MOTION

The covariant equation of motion for an extended charge was obtained before by Nodvik[16] and Dixon[17]. However, for the reader convenience, we present here the outline of Dixon's derivation[17] which seems more simple as was stressed by Kaup[13].

We shall consider the particle to be a localized distribution of charge and matter which evolves in space-time according to its internal dynamics which will not be specified in detail. We only assume that the charge distribution is stabilized by attractive non electromagnetic forces.

The electromagnetic fields $F^{\mu\nu}(x)$ and the charge current $J^{\mu}(x)$ are functions of the space-time coordinates $x^{\mu} = (x_o, \vec{x})$ which obey Maxwell's equations

$$\partial_{\nu} F^{\mu\nu}(x) = 4\pi J^{\mu}(x) \quad , \tag{2.1}$$

$$\partial_{\lambda} F_{\mu\nu} + \partial_{\nu} F_{\lambda\mu} + \partial_{\mu} F_{\nu\lambda} = 0 \tag{2.2}$$

and the electromagnetic energy momentum density tensor is defined in the usual way[13]

$$T_e^{\mu\nu}(x) = \frac{-1}{4\pi} (F^{\mu\lambda} F_\lambda^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}) \quad . \qquad (2.3)$$

Local conservation of energy and momentum together with Maxwell's equations implies the following equation for $T_m^{\mu\nu}$ the mechanical energy momentum density tensor of non electromagnetic origin:

$$\partial_\nu T_m^{\mu\nu}(x) = - \partial_\nu T_e^{\mu\nu}(x) = - F^{\mu\nu}(x) J_\nu(x) \quad . \qquad (2.4)$$

We denote by $z^\mu(\tau)$ the space time coordinates of an arbitrary material point of the extended charge with respect to some inertial reference frame; $\dot{z}^\mu(\tau) \equiv \equiv dz^\mu/d\tau \equiv \gamma(1,\vec{\dot{z}})$ is the four velocity of this material point, τ is the proper time and $\gamma = (1-|\vec{\dot{z}}|^2)^{-1/2}$ is the Lorentz factor since we are taking the velocity of hight c in such units that $c=1$.

The instantaneous mechanical four momentum of the particle is given by

$$p_m^\mu = \int_\Sigma d^3x \;\; T_m^{\mu o}(x) \quad , \qquad (2.5)$$

in relation to the system in which $\vec{\dot{z}}=0$ instantaneously and the equation which defines the integration hyper-

surface Σ is

$$\dot{z}.(x-z) \equiv \gamma\left[x_o - z_o - \vec{\dot{z}}.(\vec{x}-\vec{z})\right] \equiv 0 \quad . \qquad (2.6)$$

We can easily convert (2.5) into a explicitly covariant form by writing

$$p_m^\mu(\tau) = \int_\Sigma d\sigma_\nu \; T_m^{\mu\nu}(x) \quad , \qquad (2.7)$$

where the element of hypersurface and its normal is written as the four vector $d\sigma_\mu = \gamma d^3x \; \dot{z}_\mu(\tau)$. In this way $p_m^\mu(\tau)$ is understood to be the instantaneous mechanical four momentum of the extended particle with respect to an inertial reference frame in which the material point z^μ has an arbitrary instantaneous velocity $\vec{\dot{z}}$.

The covariant equation of motion can be obtained by taking the following limit

$$\frac{d\, p_m^\mu}{d\tau} = \lim_{\Delta\tau\to 0} \left\{ \left[\int_{\Sigma(\tau+\Delta\tau)} d\sigma_\nu \, T_m^{\mu\nu}(x) - \int_{\Sigma(\tau)} d\sigma_\nu \, T_m^{\mu\nu}(x) \right] (\Delta\tau)^{-1} \right\} . \quad (2.8)$$

Denoting by V the four dimensional volume delimited by the hypersurfaces $\Sigma(\tau+\Delta\tau)$ and $\Sigma(\tau)$, and using Gauss' theorem we write (2.8) as a volume integral, namely

$$\dot{p}^{\mu}_{m} = \lim_{\Delta\tau\to 0} \left[\frac{1}{\Delta\tau} \int_{V} d^4x \, \partial_{\nu} \, T^{\mu\nu}_{m}(x)\right] \tag{2.9}$$

since we are assuming that $T^{\mu\nu}_{m}(x)$ is associated to a localized distribution of matter.

The generic volume element d^4x, delimited by $\Sigma(\tau)$ and $\Sigma(\tau+\Delta\tau)$, can be written up to first order in $\Delta\tau$ as:

$$d^4x = d\sigma_{\mu}(\tau) \left[x^{\mu}(\tau+\Delta\tau) - x^{\mu}(\tau)\right] =$$
$$= d\sigma_{\mu} \frac{dx^{\mu}}{d\tau} \Delta\tau + 0(\Delta\tau^2) \quad , \tag{2.10}$$

where $x^{\mu}(\tau+\Delta\tau)$ belongs to $\Sigma(\tau+\Delta\tau)$ and $x^{\mu}(\tau)$ belongs to $\Sigma(\tau)$ defined in (2.6). Denoting by $\ddot{z}^{\mu} \equiv d\dot{z}^{\mu}/d\tau$ the four acceleration of the arbitrary material point z^{μ} is easy to show that from $d[(x-z).\dot{z}]/d\tau \equiv 0$ we get

$$\frac{dx^{\mu}}{d\tau} = \dot{z}^{\mu} \left[1-(x-z).\ddot{z}\right] + a^{\mu} \quad , \tag{2.11}$$

where a^{μ} is an arbitrary four vector such that: $a^{\mu} \dot{z}_{\mu} = 0$.

The results (2.4), (2.10) and (2.11) are the requirements to put (2.9) in the following form in the limit $\Delta\tau \to 0$

$$\dot{p}^{\mu}_{m} = - \gamma(\dot{\vec{z}}) \int_{\Sigma} d^3x \, F^{\mu\nu}(x) \, J_{\nu}(x) \left[1-(x-z).\ddot{z}\right] . \tag{2.12}$$

This equation is a particular case of a more general formula derived previously by Dixon[17] and discussed by Kaup[13] and by França, Marques and da Silva[14]. It is important to mention that expression (2.12) is fully consistent with special relativity since its derivation does not involve the assumption that the particle is rigid. Less general and more cumbersome approaches have being discussed by others[16] under the prescription that the shape of the charge distribution is fixed in a sequence of inertial frames in which the charge center is instantaneously at rest. As far as we know the approach indicated by Dixon[17] and outlined here is the more clear and simple.

3. NON RELATIVISTIC LIMIT

The equation of motion derived above will be analized in the situation in which the charged particle interacts with the external world only electromagnetically. In this way the electromagnetic field $F^{\mu\nu}(x)$ will be decomposed in two terms:

$$F^{\mu\nu}(x) = F^{\mu\nu}_{self}(x) + F^{\mu\nu}_{ext}(x) \quad , \qquad (3.1)$$

where $F^{\mu\nu}_{self}(x)$ are the fields generated by the current $J^{\mu}(x)$. Therefore the equation (2.12) becomes non linear and very difficult to be solved in the general case. However the non relativistic limit $|\dot{\vec{z}}| << 1$, discussed before by Kaup[13] and more detailed by França, Marques and da Silva[14], presents remarkable simplifications if some assumptions, only accountable in the case of non relativistic motion, are introduced.

In the following we shall assume that the deformations suffered by the particle are so small that the corpuscle can be considered rigid to a good approximation. The charge distribution $\rho(\vec{x}-\vec{z})$ will be taken spherically symmetric and $\vec{z}$ will be identified with the charge and mass center. The magnetic Lorentz

force will be considered very small as compared with the electric one and we shall also assume that the particle does not rotate and that all the torques are completely negligible.

With these assumptions the current distribution can be simply written as

$$J^{\mu}(x) \simeq (1,\dot{\vec{z}}(t))\, \rho(\vec{x}-\vec{z}(t))$$

$$\equiv e(1,\dot{\vec{z}}) \int \frac{d^3q}{(2\pi)^3}\, \tilde{\rho}(q^2)\, \exp\left[i\vec{q}.(\vec{x}-\vec{z})\right] \quad , \qquad (3.2)$$

where $\tilde{\rho}(0) = 1$ and e is the total charge.

The same hypothesis permits us to write the mechanical four momentum as $p^{\mu}_{m} \simeq m_o(1,\dot{\vec{z}})$ where m_o is the mechanical rest mass of non electromagnetic origin.

All these simplifications allow us to write down the following equation[14] for the spatial components of (2.12)

$$m_o\, \ddot{\vec{z}}(t) = \vec{F}_{self} + \vec{F} \equiv$$

$$\equiv \int d^3x \left[\vec{E}_{self}(\vec{x},t) + \vec{E}(\vec{x},t)\right] \rho(\vec{x}-\vec{z}) \left[1+(\vec{x}-\vec{z}).\ddot{\vec{z}}\right], \quad (3.3)$$

where $\vec{E}_{self}$ and $\vec{E}$ are the self and external electric fields which generates the self force $\vec{F}_{self}$ and the external force $\vec{F}$ respectively.

The calculation of the self force $\vec{F}_{self}$ was discussed with great detail in reference 14 and therefore will not be repeated here. The result can also be written in the form

$$\vec{F}_{self} \simeq \frac{m_e}{3}\dddot{\vec{z}}(t) - \int_0^\infty d\tau\, \alpha(\tau)\dot{\vec{z}}(t-\tau) + 0(\dot{\vec{z}}^2) \quad , \quad (3.4)$$

where m_e is the electromagnetic mass which is given by

$$m_e \equiv \frac{1}{2}\int d^3x\, d^3x' \, \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{e^2}{\pi}\int_0^\infty dk\, \tilde{\rho}^2(k^2) \quad (3.5)$$

and the function $\alpha(\tau)$, the memory kernel which we have mentioned before, is defined in terms of the Fourier transform of the charge distribution as

$$\alpha(\tau) \equiv \frac{4}{3}\frac{e^2}{\pi}\int_0^\infty dk\, k^2\tilde{\rho}^2(k^2)\cos(k\tau) \quad . \quad (3.6)$$

It is not difficult to see that $\alpha(\tau)$ has the following

properties:

$$\int_0^\infty d\tau\, \alpha(\tau) = 0 \quad , \quad (3.7a)$$

$$\int_0^\infty d\tau\, \tau\, \alpha(\tau) = -\frac{4}{3}m_e \quad , \quad (3.7b)$$

$$\int_0^\infty d\tau\, \tau^2\, \alpha(\tau) = -\frac{4}{3}e^2 \quad (3.7c)$$

and

$$\int_0^\infty d\tau\, \tau^3\, \alpha(\tau) = 0(e^2 r) \quad , \quad (3.7d)$$

where r is the mean square radius of the charge distribution.

The approximate Abraham-Lorentz equation can be easily obtained from (3.3) and (3.4) by making a Taylor expansion of $\dot{\vec{z}}(t-\tau)$ and using the properties collected in (3.7). The result is

$$m\,\ddot{\vec{z}}(t) \simeq \frac{2}{3}e^2\,\dddot{\vec{z}}(t) + \vec{F} \quad , \quad (3.8)$$

where $m = m_o + m_e$ is the total observable mass of the particle.

This last equation exhibits runaway solutions which can be eliminated at the price of violating causality[22]. This means that equation (3.8) is an approximation which can be used but with care. However the original non relativistic limit, namely (3.3) and (3.4), does not have runaway solutions nor violation of causality if

$$m > \frac{4}{3} m_e \quad , \tag{3.9}$$

as was shown before by Kaup[13], França[14] and more recently by de la Peña, Jiménez and Montemayor[15]. This condition implies that the charge radius of the particle cannot be taken arbitrarily small. If we admit, only to fix idea, that the charge distribution is Gaussian with mean square charge radius r therefore

$$\tilde{\rho}(k^2) = \exp(-\frac{r^2k^2}{6}) \tag{3.10}$$

and condition (3.9) is equivalent to

$$m^2r^2 > \frac{4}{3\pi} e^4 \quad . \tag{3.11}$$

This restriction on the radius of the corpuscle

is also important for the calculation of the external force $\vec{F}$ generated by the interaction with the electric field of zero-point radiation as we shall see below.

Following the notation of Boyer[4,5] the zero-point electric field will be written as a superposition of transverse plane waves

$$\vec{E} = Re \sum_{\lambda=1}^{2} \int d^3k \, \vec{\epsilon}(\vec{k},\lambda) h(\omega) \exp\left[i\vec{k}.\vec{x} - i\omega t + i\sigma(\vec{k},\lambda)\right] \tag{3.12}$$

where $\sigma(\vec{k},\lambda)$ are random phases[4-6], $\vec{\epsilon}(\vec{k},\lambda)$ are the polarization vectors, $\omega = |\vec{k}|$ and the function $h^2(\omega) = \omega/2\pi^2$ (our units are such that $\hbar=c=1$).

With this notation is easy to show that (3.2), (3.3) and (3.12) gives the following expression for the external force

$$\vec{F} \equiv \int d^3x \, \vec{E}(\vec{x},t) \rho(\vec{x}-\vec{z}) \left[1 + (\vec{x}-\vec{z}).\ddot{\vec{z}}\right]$$

$$\simeq Re \, e \sum_{\lambda=1}^{2} \int d^3k \, \vec{\epsilon}(\vec{k},\lambda) h(\omega) \exp\left[i\vec{k}.\vec{z} - i\omega t + i\sigma(\vec{k},\lambda)\right] \tilde{\rho}(k^2) \quad , \tag{3.13}$$

where we have neglected the term proportional to the acceleration $\ddot{\vec{z}}$. We can disregard this term only if the charge radius is sufficiently small as compared with the inverse of the acceleration namely

$$|\ddot{\vec{z}}|\ r \ll 1 \quad . \tag{3.14}$$

This restriction introduces considerable simplification in the equation of motion (3.3) but would be in conflict with the causality condition (3.11), for the Gaussian charge distribution, or (3.9) in general. We shall return to this point latter on.

Expression (3.13) for the random external force can be further simplified in the case of non relativistic motion. If we choose, without loss of generality, the origin of the coordinate system and the origin of time in such a way that $\vec{z}(t=0) = 0$ then

$$\exp\left[-i\omega t+i\vec{k}.\vec{z}(t)\right] =$$

$$= \exp\left\{-i\omega t\left[1 - \frac{\vec{k}}{\omega} . \int_0^t \frac{dt'}{t}\ \dot{\vec{z}}(t')\right]\right\} \simeq \exp(-i\omega t) \tag{3.15}$$

because $|\vec{k}|/\omega = 1$ and $|\dot{\vec{z}}| \ll 1$. The above approximation implies that the random force can be taken as a function of time only and is usually called dipole approximation[18] in Stochastic Electrodynamics. We must

note that the assumption of non relativistic motion $|\dot{\vec{z}}| \ll 1$, required to justify the dipole approximation, will be discussed latter and, as we shall see, will also restrict the charge radius of the particle.

In summary all these approximations means that the external random force can be written as

$$\vec{F}(t) = \mathcal{R}e\ e \sum_{\lambda=1}^{2} \int d^3k\ \vec{\epsilon}(\vec{k},\lambda)h(\omega)\tilde{\rho}(k^2)\exp\left[-i\omega t+i\theta(\vec{k},\lambda)\right] \ , \tag{3.16}$$

while the equation of motion takes the form

$$\left(m_o - \frac{m_e}{3}\right)\ddot{\vec{z}}(t) = -\int_0^\infty d\tau\ \alpha(\tau)\dot{\vec{z}}(t-\tau) + \vec{F}(t) \tag{3.17}$$

which is quite similar to the generalized Langevin equation from non Markovian theories of Brownian motion[19,20].

4. STATISTICAL PROPERTIES OF THE FREE PARTICLE

In recent work de la Peña[11] discussed with great detail the statistical properties of the random motion of an extended charge which obey equation (3.17). The only point in which our analyses differs from that of de la Peña[11] is the fact that he introduced a cut off in the high frequency contributions for the random force (3.16). In our case the form factor $\tilde{\rho}(k^2)$ which appear in (3.16) gives a natural attenuation of the high frequencies and, as we shall see, assures finite result to the ensemble average $\langle\dot{\vec{z}}^2\rangle$ of the square equilibrium velocity.

For what follows we need to calculate ensemble averages of some physical quantities. These averages are simply averages in the random phases $\theta(\vec{k},\lambda)$ which appear in the zero-point electric field (3.12). This is a standard procedure[4-6] and we only give here, without proof, a useful formula for the average of the product of the components of two random vectors.

Consider for instance a vector $\vec{A}(t)$ which can be written as

$$\vec{A}(t) = \mathcal{R}e \sum_{\lambda=1}^{2} \int d^3k\ \vec{a}(\vec{k},\lambda)\ \exp\left[-i\omega t+i\theta(\vec{k},\lambda)\right] \quad (4.1)$$

and a vector $\vec{B}(t)$ with similar definition ($\vec{a}(\vec{k},\lambda)$ replaced by $\vec{b}(\vec{k},\lambda)$). The ensemble average of the product $A_i(t)B_j(t')$ and denoted by $\langle A_i(t)B_j(t')\rangle$

will be given by the following expression[23]

$$\langle A_i(t)B_j(t')\rangle = \frac{1}{2}\ \mathcal{R}e \sum_{\lambda=1}^{2} \int d^3k\ a_i(\vec{k},\lambda)b_j^*(\vec{k},\lambda)\exp\left[i\omega(t-t')\right]. \quad (4.2)$$

With the help of this formula it is easy to obtain the correlation function for the external random force at zero temperature. The result follows from (3.16) and is given by

$$\langle F_i(t)F_j(0)\rangle = \frac{2}{3}\frac{e^2}{\pi}\delta_{ij}\int_0^\infty d\omega\ \omega^3\ \tilde{\rho}^2(\omega^2)\cos(\omega t) \quad . \quad (4.3)$$

Here we just want to mention that the usual fluctuation-dissipation relation[19,20], namely the proportionality between $\langle\dot{\vec{z}}\rangle\alpha(t)$ and $\langle\vec{F}(t).\vec{F}(0)\rangle$ where $\dot{\vec{z}}$ is the equilibrium fluctuating velocity, is not valid in our approximation of Stochastic Electrodynamics as can be seen from (4.3) and (3.6).

We shall discuss now a few properties of the stationary solution of the equation of motion (3.17). The reader interested in the analysis of how the stationary regime is reached should look at the paper by de la Peña[11] where this point is discussed in detail.

The stationary velocity which is solution of

(3.17) can be written as

$$\dot{\vec{z}}(t) = \vec{v} + \dot{\vec{z}}_f(t) \quad , \qquad (4.4)$$

where $\dot{\vec{z}}_f(t)$ is the fluctuating part with zero mean value and $\vec{v}$ is a constant vector ($|\vec{v}| \ll 1$). It is simple to verify that

$$\dot{\vec{z}}_f(t) = \mathcal{R}e\; e \sum_{\lambda=1}^{2} \int d^3k \; \frac{\vec{\epsilon}(\vec{k},\lambda)\, h(\omega)\, \tilde{\rho}(\omega^2) \exp\left[-i\omega t + i\sigma(k,\lambda)\right]}{\tilde{\alpha}(\omega) - i\omega\left(m_o - \frac{m_e}{3}\right)} \qquad (4.5)$$

with

$$\tilde{\alpha}(\omega) \equiv \int_0^\infty d\tau \; \alpha(\tau)\, \exp(i\omega\tau) =$$

$$\simeq -\, i\omega\, \frac{4}{3}\, m_e + \frac{2}{3}\, \omega^2 e^2 + 0(e^2 \omega^3 r) \quad , \qquad (4.6)$$

where we have used the properties (3.7) of the memory function $\alpha(\tau)$.

If we want to make a careful analysis we must verify under which conditions $\langle \dot{\vec{z}}_f^{\,2} \rangle \ll 1$ in order to check the consistency of the non relativistic treatment. The ensemble average $\langle \dot{\vec{z}}_f^{\,2} \rangle$ is calculated by using (4.2) and (4.5) and the result is

$$\langle \dot{\vec{z}}_f^{\,2} \rangle = \frac{2}{\pi}\, e^2 \int_0^\infty d\omega \; \frac{\omega\, \tilde{\rho}^2(\omega^2)}{\left| m_o - \frac{m_e}{3} + i\, \frac{\tilde{\alpha}(\omega)}{\omega} \right|^2}$$

$$\simeq \frac{2}{\pi}\, \frac{e^2}{m^2} \int_0^\infty d\omega \; \frac{\omega\, \tilde{\rho}^2(\omega^2)}{1 + \frac{4e^4\omega^2}{9m^2}} \quad , \qquad (4.7)$$

if we use the expansion (4.6).

The explicit calculation of $\langle \dot{\vec{z}}_f^{\,2} \rangle$ depends on the details of the charge distribution. Since we are only interested in the qualitative aspects of the motion we present the result for a Gaussian charge distribution whose mean square radius is r . In this case

$$\langle \dot{\vec{z}}_f^{\,2} \rangle \simeq \frac{2}{\pi}\, \frac{e^2}{m^2} \int_0^\infty d\omega \; \omega \left(1 - \frac{4}{9}\, \frac{e^4\omega^2}{m^2}\right) \exp\left(-\, \frac{r^2\omega^2}{3}\right) \qquad (4.8)$$

or more explicitly

$$\langle \dot{\vec{z}}_f \rangle = \frac{3e^2}{\pi m^2 r^2} \left(1 - \frac{4}{3}\, \frac{e^4}{m^2 r^2} + \dots \right) \quad . \qquad (4.9)$$

Therefore the motion is non relativistic if

$$m^2 r^2 \gg \frac{3}{\pi}\, e^2 \qquad (4.10)$$

which is a result that does not contradict the causality condition (3.11) obtained previously.

The average acceleration given by the fluctuating zero-point electric field is such that

$$\langle \ddot{z}^2 \rangle \simeq \frac{2}{\pi} \frac{e^2}{m^2} \int_0^\infty d\omega\, \omega^3 \left(1 - \frac{4e^4\omega^2}{9m^2}\right) \exp\left(-\frac{r^2\omega^2}{3}\right)$$

$$\simeq \frac{9}{\pi} \frac{e^2}{m^2 r^4} \simeq \frac{3\langle \dot{\vec{z}}_f^2 \rangle}{r^2} \qquad (4.11)$$

for the gaussian charge distribution. This means that if $\langle \dot{\vec{z}}_f^2 \rangle \ll 1$ then $\langle \ddot{\vec{z}}^2 \rangle r^2 \ll 1$ which is a result consistent with the assumption (3.14) mode above.

The conclusion we reach is that the equation of motion (3.17), as well as the approximate expression for the random force (3.16), are consistent non relativistic limits of the covariant equation of motion (2.12). The motion of the extended charged particle is causal, does not exhibits violation of energy conservation by self acceleration (runaway) and stay non relativistic in the stationary limit if the mean square charged radius is much larger than a critical radius of order e/m. This condition is verified for instance by the spin 0 nuclei which are spherically symmetric and have charge radii

which are much larger than the critical radius we have mentioned before. The experimental results[24] for the pion mean square charged radius gives $r_\pi = 0.8 \pm 0.1$ fermi and therefore large than the critical radius which in this case is $e/m \simeq 0.1$ fermi. In recent work Moniz and Sharp[25] obtained the generalization of the classical Abraham-Lorentz equation starting from non relativistic Quantum Theory for the spin zero particle. The equation of motion they arrive, as the classical limit of Heisenberg equations with self electromagnetic interaction, has the form (3.17) except for the term $m_e \ddot{\vec{z}}/3$. Their expression for the memory kernel $\alpha(\tau)$ is analogous to (3.6) as if the quantum particle was extended but with a charged radius of order of the Compton wavelengh and therefore larger than e/m.

Our analysis cannot be applied to the proton and the electron since these particles have a more complicated electromagnetic structure because they have spin and intrinsic magnetic dipole. Despite of this it is interesting to remember the existing qualitative estimations for the charge radius of these particles.

One of the simplest one is due to Moller[26] and is based on the fact that the spin vector $\vec{S}$ of an extended object with mass density $\rho_m(\vec{x})$, total mass m and a maximum radius r obey the following obvious innequality

$$|\vec{S}| = \left|\int d^3x\, \vec{x} \times \dot{\vec{x}}\, \rho_m(\vec{x}')\right| \lesssim rm \quad . \qquad (4.12)$$

For a spin 1/2 particle we get

$$r^2 \geq \frac{\vec{S}^2}{m^2} = \frac{3}{4m^2} \quad . \qquad (4.13)$$

It is quite interesting to note the similarity between this result and that obtained by Foldy[27] in its interpretation of Darwin's term[28] which appear in the non relativistic limit of Dirac's equation for the electron. More explicitely, Foldy[27] showed that Darwin's term is the deviation from the pointlike interaction with a potential $\phi(\vec{x})$ namely:

$$\int d^3x\, \rho(\vec{x}-\vec{z})\phi(\vec{x}) - e\phi(\vec{z}) \simeq \frac{er^2}{6} \nabla^2\phi(\vec{z}) + \ldots \quad , \qquad (4.14)$$

where r^2 is given by the equality[27] in (4.13). The result (4.13) was confirmed later by Yennie[29], Gourdin[30] and França[31], which have shown that this radius correspond to a real extension of the particle and not a

pseudo-extension generated by zitterbewegung as is often believed[28].

In summary the known microscopic particles like the proton and electron are extended charge distributions quite localized whose mean charge radius is larger than its Compton wave lentgh. Our analysis of Stochastic Electrodynamics, validy only for spin zero corpuscles, shows that for radius of order e/m the motion in the zero-point radiation becomes relativistic and for radius of order e^2/m the theory violates causality.

5. EFFECTS OF THERMAL RADIATION AND CONCLUSION

The inclusion of thermal effects in Stochastic Electrodynamics is done in a standard way[1,5]. It is sufficient to add Planck's spectral distribution to the zero-point distribution since the first can be derived in the context of Stochastic Electrodynamics. In this way we substitute $h^2(\omega) = \frac{\omega}{2\pi^2}$ in (3.12) by

$$h^2(\omega,T) = \frac{\omega}{2\pi^2} \coth\left(\frac{\omega}{2kT}\right) \quad . \qquad (5.1)$$

Here we are assuming that the temperature is not very high and does not induces relativistic effects that is

$$kT \ll m \quad . \tag{5.2}$$

The fluctuations in the velocity are affected by the thermal radiation. The mean square fluctuating velocity is given by the generalization of (4.8) that is

$$\langle \dot{\vec{z}}_f^2 \rangle \simeq \frac{4}{\pi} \frac{e^2}{m^2} \int_0^\infty d\omega\, \omega \left[\frac{1}{2} + \frac{1}{\exp\left(\frac{\omega}{kT}\right) - 1}\right] \exp\left(-\frac{\omega^2 r^2}{3}\right) \tag{5.3}$$

for a gaussian charge distribution. For low temperatures we can replace $\exp\left(-\frac{\omega^2 r^2}{3}\right)$ by 1 when multiplied by $\left[\exp\left(\frac{\omega}{kT}\right) - 1\right]^{-1}$. With this approximation we get

$$\langle \dot{\vec{z}}_f^2 \rangle \simeq \frac{3e^2}{\pi} \left[\frac{1}{m^2 r^2} + \frac{2\pi^2}{9} \left(\frac{kT}{m}\right)^2\right] \quad , \tag{5.4}$$

that is the thermal correction for the mean square velocity is small for particles such that $m^2 r^2 \simeq 1$.

The modification in the spectral distribution introduced by the thermal radiation has also some

implications for our equation of motion (3.17). This is because the thermal fluctuations must be associated to some dissipation which is not included in the approximate formulas (3.17) and (3.16). In fact, when $T \neq 0$, the particle is moving in a radiation field whose spectral distribution is not Lorentz invariant, therefore we expect the appearance of a dissipative force proportional to the velocity, the Thomson cross section σ and also proportional to the density $u(T)$ of thermal electromagnetic energy[21].

We shall see in what follows how such a dissipative force show up as a combined effect[21] of the radiation reaction force and the Lorentz magnetic force which is not included in (3.16) and (3.17). The discussion will be at the level of the Abraham-Lorentz equation (3.8) for a point charge. This approximation will not bring problems to the calculation of the dissipative force we are talking about.

Formula (5.1) is validy only in the reference frame S in which the thermal radiation is isotropic (cavity reference frame). If we look the radiation spectral distribution from an inertial frame S' which is moving with velocity $\vec{v}$ with respect to S the spectral distribution will be given by

$$h'^2(\omega,T) \simeq h^2(\omega,T) - \left[h^2(\omega,T) - \omega \frac{\partial}{\partial\omega} h^2(\omega,T)\right] \frac{\vec{v}.\vec{k}}{\omega} \quad , \qquad (5.5)$$

in the limit $|\vec{v}| << 1$ of low velocity[32].

The fluctuating motion of the charged particle, as seen from frame S' , will be with a stationary velocity $\dot{\vec{z}}'_f$ which is solution of the Abraham-Lorentz equation (3.8) where the external force $\vec{F}$ is simply $e\vec{E}'$ where $\vec{E}'$ is the electric field[32] as seen from S' . In this way we get

$$\dot{\vec{z}}'_f = Re\ ie \sum_{\lambda=1}^{2} \int \frac{d^3k}{\omega} \frac{\vec{\varepsilon}(\vec{k},\lambda)h'(\omega,T)\exp[-i\omega t + i\theta(\vec{k},\lambda)]}{[m + i\frac{2}{3}e^2\omega]} \quad , \qquad (5.6)$$

where $h'(\omega,T)$ is given by (5.5).

The average Lorentz magnetic force in the frame S' is

$$\vec{f} \equiv e <\dot{\vec{z}}'_f \times \vec{B}'> \quad , \qquad (5.7)$$

where $\vec{B}'$ is the random magnetic field in S' . The dipole approximation of $\vec{B}'$ is written as[32]

$$\vec{B}' = Re \sum_{\lambda=1}^{2} \int d^3k \frac{\vec{k}\times\vec{\varepsilon}(k,\lambda)}{\omega} h'(\omega,T)\exp\left[-i\omega t+i\theta(\vec{k},\lambda)\right] \quad . \qquad (5.8)$$

The force $\vec{f}$ can be calculated from its definition (5.7) with the help of (4.2). The result is different from zero because of the radiation reaction term in (3.8) and it is not difficult to show that (5.5), (5.6), (5.7), (5.8) and (4.2) leads to

$$\vec{f} = -\xi(T)\vec{v} \qquad (5.9)$$

where

$$\xi(T) = \frac{32}{135}\pi^3 \frac{e^4(kT)^4}{m^2} \equiv \frac{4}{3}u(T)\sigma \qquad (5.10)$$

and $\sigma = \frac{8\pi}{3}\frac{e^4}{m^2}$ is the Thomson cross section.

The physical content of the above result (5.9) is that if a particle moves in the electromagnetic field of thermal radiation with average velocity $\vec{v}$, with respect to the reference frame in which thermal radiation is isotropic, therefore the charged particle will experiences an average force $\vec{f} = -\frac{4}{3}u\sigma\vec{v}$ due to radiation losses[21]. This effect only appears if we include also the magnetic Lorentz force in the equation of motion. This is basically the reason we have to propose the inclusion of the Lorentz magnetic force[33] in (3.17) getting

$$\left(m_o - \frac{m_e}{3}\right) \ddot{\vec{z}}(t) = - \int_0^\infty d\tau\ \alpha(\tau) \dot{\vec{z}}(t-\tau) + $$

$$+ \int d^3x\ \rho(\vec{x}) \left[\vec{E}(\vec{x},t) + \dot{\vec{z}}(t) \times \vec{B}(\vec{x},t)\right] \qquad (5.11)$$

as the generalized Langevin equation for the non relativistic motion of the extended charge in Stochastic Electrodynamics. This is, in our opinion, the simplest equation compatible with causality, non relativistic motion and which, probably, includes fluctuation and dissipation in a consistent way. However many approximations were made in order to arrive (5.11). The most drastic approximation used was to assume that the charge distribution was rigid in the non relativistic limit. A more realistic calculation should include not only the deformations but also the rotations of the particle in order to study its effects in the motion of the center of mass and in the dissipation of kinetic energy.

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