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THE K-HARMONICS METHODS AND THE BOUND-STATE SPECTRUM OF ONE-DIMENSIONAL THREE-BODY SYSTEM

by

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SUMMARY

The symmetry properties of one-dimensional hyperspherical harmonics components have been investigated. For a system of three identical particles moving in one-dimension it is shown how to construct solutions of definite parity and definite transformation properties under permutation of any particle pair. General qualitative features of the spectrum of the one-dimensional system are deduced for particles satisfying Bose-Einstein, Fermi-Dirac and Boltzmann statistics.

I. INTRODUCTION

The hyperspherical harmonics[1,2] (or K-harmonics) method has recently been extensively used[3,4] for solving the time-independent Schrödinger equation of few-body systems. This method allows, in principle, for a systematic treatment of the few-body problem but it has the drawback that the high dimensionality of the hyperangle makes its visualisation rather difficult. In order to gain a better insight of the method, there has been a few applications of it to one-dimensional systems. Moszkowski and Strobel[5], for instance, used the K-harmonics method to determine the ground-state energy of one-dimensional many-boson systems with attractive two-body δ-interaction. Amado and Coelho[6] considered the case of one-dimensional three-body system, interacting via a general two-body potential, and obtained an infinite set of coupled ordinary differential equations (see equations (9) and (10) of reference 6) which were then solved in the particular case of three identical particles interacting via an attactive δ-potential.

In both of the above mentioned works, the two-body interaction was chosen to be the attractive δ-potential because, in this case, the one-dimensional problem of N equal mass particles is exactly solvable[7] and the accuracy of the K-harmonics solution can be easily tested.

In this paper we investigate the symmetry properties, under the permutation group, of the K-harmonics (bound-state) solutions for the one-dimensional system of identical particles interacting via an attractive two-body potential $V(|x_i-x_j|)$. As the one-dimensional problem of N identical particles interacting via an attractive δ-potential has a single[7] bound-state solution which is totally symmetric under the permutation of any pair of particles, this particular case will not be considered here. We shall restrict ourselves to the three-particle system for simplicity. The three-dimensional problem of three interacting particles is considerably more involved[2,3] and we believe that the solution given here is of considerable pedagogical value.

In Section II we make a brief exposition of the hyperspherical harmonics method for the case of three identical particles; the time independent Schrödinger equation is reduced to an infinite set of coupled ordinary differential equations. In Section III we discuss the effect of the action of the permutation group on the K-harmonics components and construct functions, of definite parity, that provide a basis for irreducible representations[8] of the S_3 symmetry group. In Section IV these functions are used for constructing eigenfunctions of definite parity and permutation properties. Concluding the paper, we present some very general properties exhibited by the bound-state spectrum of a one-dimensional system of three identical interacting particles obeying Bose-Einstein, Fermi-Dirac and Boltzmann statistics.

II. THE HYPERSPHERICAL METHOD FOR ONE-DIMENSIONAL THREE-BODY SYSTEM

The Schrödinger equation for a system of three identical particles moving in one dimension, interacting via an attractive two-body potential $V(|x_i-x_j|)$, is

$$\left[-\sum_{i=1}^{3}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2}+V(x_1,x_2,x_3)\right]\psi(x_1,x_2,x_3) = E\,\psi(x_1,x_2,x_3) \quad , \qquad (1)$$

where

$$V(x_1,x_2,x_3) = V(|x_1-x_2|) + V(|x_2-x_3|) + V(|x_1-x_3|) \quad . \qquad (2)$$

In terms of the Jacobi coordinates

$$\eta = \frac{1}{\sqrt{2}}(x_1-x_2) \qquad (3a)$$

$$\xi = \sqrt{\frac{2}{3}}\left(\frac{x_1+x_2}{2}-x_3\right) \qquad (3b)$$

$$R = \frac{x_1+x_2+x_3}{3} \quad , \qquad (3c)$$

the Schrödinger equation (1), with the center of mass removed, can be written as

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)+V(\xi,\eta)\right]\psi(\xi,\eta) = E\,\psi(\xi,\eta) \qquad (4)$$

where

$$V(\xi,\eta) = V(\sqrt{2}|\eta|) + V\left(\left|\sqrt{\frac{3}{2}}\,\xi+\frac{1}{\sqrt{2}}\,\eta\right|\right) + V\left(\left|\sqrt{\frac{3}{2}}\,\xi-\frac{1}{\sqrt{2}}\,\eta\right|\right) \quad . \qquad (5)$$

Now, introducing the "hyperspherical coordinates", the hyperradius ρ and the hyperangle θ

$$\eta = \rho\cos\theta \quad , \quad \xi = \rho\sin\theta \quad , \quad 0 \le \theta \le 2\pi \quad , \qquad (6)$$

the Schrödinger equation (4) can be written as

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right\}\psi(\rho,\theta)+V(\rho,\theta)\psi(\rho,\theta)=E\,\psi(\rho,\theta)\ , \qquad (7)$$

where

$$V(\rho,\theta)=V(\sqrt{2}\,\rho|\cos\theta|)+V(\sqrt{2}\,\rho|\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta|)+$$

$$+V(\sqrt{2}\,\rho|\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta|)\ . \qquad (8)$$

The hyperspherical (or K-harmonics) method for solving the equation (7) consists of expanding $\psi(\rho,\theta)$ in terms of a complete set of angular eigenfunctions. Following reference 6, we use the set $\left\{e^{iK\theta}/(2\pi)^{1/2}\right\}$, K integer, so that

$$\psi(\rho,\theta) = \sum_{K=-\infty}^{\infty} R_K(\rho)\frac{e^{iK\theta}}{(2\pi)^{1/2}}\ . \qquad (9)$$

Substitution of the expansion given by equation (9) in the time independent Schrödinger equation (7), obtains the following infinite set of coupled ordinary differential equations

$$-\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{K^2}{\rho^2}\right)R_K(\rho)+\sum_{K'=-\infty}^{\infty}\langle K|V|K'\rangle R_{K'}(\rho)=E\,R_K(\rho)\ , \qquad (10)$$

where

$$\langle K|V|K'\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(K'-K)\theta}\,V(\rho,\theta)\,d\theta\ . \qquad (11)$$

III. THE SYMMETRY PROPERTIES OF THE K-HARMONICS SOLUTIONS

As is well known[8], the invariance of the hamiltonian under any symmetry group implies the existence of eigenfunctions exhibiting the group transformation properties. We shall, then, explore the symmetry invariances of the three-body hamiltonian H for obtaining eigenfunctions with definite symmetry properties labelled by the K-harmonics components (equation (9)). Under the action of the parity operator, (x_1,x_2,x_3) go onto $(-x_1,-x_2,-x_3)$, (ξ,η) go onto $(-\xi,-\eta)$ (see (3a) and (3b)) and (ρ,θ) go onto $(\rho,\pi+\theta)$ (see (6)) so that the potential V of the three-body system is invariant under parity transformation (see equations (2), (5) and (8)). Thus, there exist eigenfunctions of H that have definite parity and we shall now determine the set of K-values that enter the hyperspherical harmonic expansion (equation (9)) for an eigenfunction with definite parity.

Due to parity invariance of the potential, the matrix elements $\langle K|V|K'\rangle$, given in equation (11), have the following properties

$$\langle K|V|K'\rangle \equiv 0 \quad \text{for} \quad (K'-K) \ \text{odd}\ , \qquad (12a)$$

$$\langle K|V|K'\rangle \equiv \langle K'|V|K\rangle \equiv \langle -K|V|-K'\rangle\ . \qquad (12b)$$

Selection rule (12a) decouples K-even and K-odd equations in the system (10) and we are left with two infinite sets of coupled ordinary differential equations: one for even K and one for odd K. So, the eigenfunctions of H having positive (negative) parity will contain only even(odd) K components when expanded in terms of the hyperspherical harmonics (equation (9)).

Now, using property (12b), the differential equations for components $R_{-K}(\rho)$ and $R_K(\rho)$ (equation (10)) are

$$-\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{d}{d\rho}\,\rho\,\frac{d}{d\rho} - \frac{K^2}{\rho^2}\right) R_K(\rho) + \sum_{K'=-\infty}^{\infty} \langle K|V|K'\rangle\, R_{K'}(\rho) = E\, R_K(\rho)$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{d}{d\rho}\,\rho\,\frac{d}{d\rho} - \frac{K^2}{\rho^2}\right) R_{-K}(\rho) + \sum_{K'=-\infty}^{\infty} \langle K|V|K'\rangle\, R_{-K'}(\rho) = E\, R_{-K}(\rho) \ ,$$

so that,

$$R_{-K}(\rho) = \pm R_K(\rho) \quad . \qquad (13)$$

The invariance of the potential V under the group S_3 (group of all permutations of three objects) is evident from expression (2). We shall now obtain the transformation properties of variables (ρ,θ) and (ξ,η) under the group S_3.

The permutation of any pair (i,j) of particles leaves ρ invariant. As for the angle θ, under the action of the permutation operator P_{ij}, it transforms as follows

$$P_{12}: \qquad \theta \rightarrow \pi - \theta \qquad (14a)$$

$$P_{13}: \qquad \theta \rightarrow 5\pi/3 - \theta \qquad (14b)$$

$$P_{23}: \qquad \theta \rightarrow \pi/3 - \theta \quad . \qquad (14c)$$

Using equations (14a), (14b) and (14c) in equation (6) we obtain the well known[(2)] transformation properties of ξ and η

$$P_{12}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \ , \qquad (15a)$$

$$P_{13}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \ , \qquad (15b)$$

and

$$P_{23}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \ . \qquad (15c)$$

In terms of variables ρ and θ, the invariance of the potential V under the action of the permutation operator P_{ij} is expressed as

$$V(\rho,\theta) = V(\rho,5\pi/3-\theta) = V(\rho,\pi/3-\theta) = V(\rho,\pi-\theta) \quad . \qquad (16)$$

(Relation (16) is obtained by using (14a), (14b) and (14c) in expression (8) for $V(\rho,\theta)$).

Using the invariance property (16), the matrix elements $\langle K|V|K'\rangle$, given by (11), can be rewritten as

$$\langle K|V|K'\rangle = \frac{1}{2\pi}\int_0^{\pi/3} d\theta\; e^{i(K'-K)\theta}\, V(\rho,\theta)\left[1 + e^{-i(K'-K)\pi}\right] \times$$

$$\times\left[1 + e^{-i(K'-K)\pi/3} + e^{-i(K'-K)2\pi/3}\right] \ ,$$

from which it follows that (remember that parity invariance requires K'-K even)

$$\langle K|V|K'\rangle = \begin{cases} \frac{6}{2\pi}\int_0^{\pi/3} d\theta\, e^{i(K'-K)\theta} V(\rho,\theta) & \text{for } K'-K=6n \text{ , n integer from } (-\infty \text{ to } +\infty) \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

Therefore each of the two infinite sets (K even and K odd) of coupled ordinary differential equations (10) splits into three infinite sets of K values

$$\begin{aligned} &K \text{ even} : \quad 6n \text{ , } 2+6n \text{ , } 4+6n \\ &K \text{ odd} \quad : \quad 1+6n \text{ , } 3+6n \text{ , } 5+6n \text{ ,} \end{aligned} \quad (18)$$

with n , integer, running from $-\infty$ to $+\infty$.

Notice that

$$2 + 6n \equiv -4 + 6(n+1) \quad ,$$

and (18a)

$$1 + 6n \equiv -5 + 6(n+1) \quad ,$$

i.e., the elements belonging to the sets (4+6n) and (5+6n) are minus the elements of the sets (2+6n) and (1+6n) respectively.

The invariance of the potential V under the action of the permutation group S_3 implies the invariance of the hamiltonian H under S_3 so that there are eigenfunctions of H that have definite transformation properties under permutation of particles, besides having definite parity. We shall show that these eigenfunctions can be obtained by including in the

K-harmonics expansion (11) the appropriate set of K-values (equation (18)).

Equations (15a), (15b) and (15c) show that the vectors ξ and η provide a basis for the two dimensional irreducible (mixed) representation of the permutation group S_3. An object that transforms as ξ will carry the index MS (mixed symmetric) as it is symmetric under the permutation of particles 1 and 2 (see equation (15a)) and mix with the corresponding η under permutation of particles 1 and 3 or 2 and 3 (see equations (15b) and (15c)). Analogously, an object that transforms like η will carry the index MA (mixed antisymmetric) as it is antisymmetric under the permutation of particles 1 and 2 (see equation (15a)) and mix with the corresponding ξ under permutation of any other pair of particles (see equations (15b) and (15c)). An object totally symmetric (antisymmetric) will carry the index S (A).

Using equations (14a), (14b) and (14c), the transformation properties, under S_3 , of $\cos(K\theta)$ and $\sin(K\theta)$ are easily obtained. We find that for each set of K-values given in (18) the functions $\cos(K\theta)$ and $\sin(K\theta)$ exhibit definite symmetry properties under permutation of particle pairs. Thus, $\cos(6n\theta)$ ($\sin(6n\theta)$) is totally symmetric (antisymmetric) providing a one-dimensional symmetric (antisymmetric) irreducible representation of S_3 , with positive parity. The function $\cos[(2+6n)\theta]$ is MS and $\sin[(2+6n)\theta]$ is MA thus providing a basis for the two-dimensional mixed irreducible representation of S_3 (also with the positive parity). Another positive parity mixed irreducible representation is provided by $\cos[(4+6n)\theta]$ (MS) and $\sin[(4+6n)\theta]$ (MA). As for the negative parity irreducible representations, $\sin[(3+6n)\theta]$ ($\cos[(3+6n)\theta]$) provide the one-dimensional totally symmetric (antisymmetric) irreducible representation. The odd parity two-dimensional mixed irreducible representations of S_3 are provided by $\sin[(1+6n)\theta]$ (MS) and $\cos[(1+6n)\theta]$ (MA) and also by $\sin[(5+6n)\theta]$ (MS) and $\cos[(5+6n)\theta]$ (MA). These properties of $\sin(K\theta)$ and $\cos(K\theta)$ allows for the construction of twelve functions (six with positive parity and six with negative parity) which will be used in the construction of the eigenfunctions (expansion (11)) in the next section. With parity explicitly indicated by (+) or (−) superscript, these functions are

$$\phi_S^{(+)}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{6n}(\rho)\cos(6n\theta) \quad , \tag{19}$$

$$\phi_A^{(+)}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{6n}(\rho)\sin(6n\theta) \quad , \tag{20}$$

$$\phi_S^{(-)}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{3+6n}(\rho)\sin[(3+6n)\theta] \quad , \tag{21}$$

$$\phi_A^{(-)}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{3+6n}(\rho)\cos[(3+6n)\theta] \quad , \tag{22}$$

$$\phi_{MS}^{(+)}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{2+6n}(\rho)\cos[(2+6n)\theta] \quad , \tag{23}$$

$$\phi_{MA}^{(+)}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{2+6n}(\rho)\sin[(2+6n)\theta] \quad , \tag{24}$$

and

$$\phi'^{(+)}_{MS}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{4+6n}(\rho)\cos[(4+6n)\theta] \quad , \tag{23a}$$

$$\phi'^{(+)}_{MA}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{4+6n}(\rho)\sin[(4+6n)\theta] \quad , \tag{24a}$$

$$\phi^{(-)}_{MS}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{1+6n}(\rho)\sin[(1+6n)\theta] \quad , \tag{25}$$

$$\phi^{(-)}_{MA}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{1+6n}(\rho)\cos[(1+6n)\theta] \quad , \tag{26}$$

$$\phi'^{(-)}_{MS}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{5+6n}(\rho)\sin[(5+6n)\theta] \quad , \tag{25a}$$

$$\phi'^{(-)}_{MA}(\rho,\theta) = \sum_{n=-\infty}^{\infty} R_{5+6n}(\rho)\cos[(5+6n)\theta] \quad . \tag{26a}$$

Notice that not all of the above mixed symmetry functions are independent. Using (13) and (18a) in (23a), (24a), (25a) and (26a), we find that

$$\phi'^{(+)}_{MS}(\rho,\theta) = \pm\phi^{(+)}_{MS}(\rho,\theta) \quad , \quad \phi'^{(+)}_{MA}(\rho,\theta) = \mp\phi^{(+)}_{MA}(\rho,\theta) \tag{27a}$$

and

$$\phi'^{(-)}_{MS}(\rho,\theta) = \mp\phi^{(-)}_{MS}(\rho,\theta) \quad , \quad \phi'^{(-)}_{MA}(\rho,\theta) = \pm\phi^{(-)}_{MA}(\rho,\theta) \quad . \tag{27b}$$

Functions (19) to (22) will be used for constructing the totally symmetric and totally antisymmetric eigenfunctions and functions (23), (24), (25), (26) will be used for constructing the eigenfunctions of mixed symmetry. This will be done in the next section.

We conclude this section with a comment about the usefulness of mixed symmetry eigenfunctions. Systems of identical fermions (bosons) must be described by totally anti-symmetric (symmetric) wave functions. If the particles have only one degree of freedom then only the totally symmetric (for bosons) or totally antisymmetric (for fermions) eigenfunctions are admissible. However, if the particles have an extra degree of freedom, for instance, spin, using the well known[8,9] prescription of multiplying mixed representations, we can obtain total eigenfunctions that are totally symmetric or antisymmetric.

IV. CONSTRUCTION OF EIGENFUNCTIONS WITH DEFINITE PARITY AND PERMUTATION PROPERTIES

We shall first construct the eigenfunctions that are totally symmetric or totally antisymmetric. As we have seen (equations (19) to (22)), these eigenfunctions are associated with $K=6n$ and $K=3+6n$ sets (positive and negative parity respectively). For these K sets, expansion (11) can be written as

$$\psi^{(+)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \sum_{n=0}^{\infty} \left[R_{6n}(\rho)e^{i6n\theta} + R_{-6n}(\rho)e^{-i6n\theta}\right] \qquad (28)$$

and

$$\psi^{(-)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \sum_{n=0}^{\infty} \left[R_{3+6n}(\rho)e^{i(3+6n)\theta} + R_{-3-6n}(\rho)e^{-i(3+6n)\theta}\right] , \qquad (29)$$

respectively.

Introducing now the notation

$$R_{6n}(\rho) = R^{E}_{6n}(\rho) \quad \text{if} \quad R_{6n}(\rho) = R_{-6n}(\rho)$$
$$R_{6n}(\rho) = R^{O}_{6n}(\rho) \quad \text{if} \quad R_{6n}(\rho) = -R_{-6n}(\rho) \quad , \quad n\neq 0 \qquad (30)$$

the eigenfunctions $\psi_S^{(+)}(\rho,\theta)$ and $\psi_A^{(+)}(\rho,\theta)$ are obtained from (28) by comparison with (19) and (20)

$$\psi_S^{(+)}(\rho,\theta) = \frac{2}{(2\pi)^{1/2}} \sum_{n=0}^{\infty} R^{E}_{6n}(\rho)\cos(6n\theta) \equiv \frac{1}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R^{E}_{6n}(\rho)\cos(6n\theta) \qquad (31)$$

$$\psi_A^{(+)}(\rho,\theta) = \frac{2i}{(2\pi)^{1/2}} \sum_{n=1}^{\infty} R^{O}_{6n}(\rho)\sin(6n\theta) \equiv \frac{i}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R^{O}_{6n}(\theta)\sin(6n\theta) . \qquad (32)$$

Analogously, introducing the eigenvectors $R^{E}_{3+6n}(\rho)$ and $R^{O}_{3+6n}(\rho)$,

$$R^{E}_{3+6n}(\rho) = R^{E}_{-3-6n}(\rho)$$
$$R^{O}_{3+6n}(\rho) = -R^{O}_{-3-6n}(\rho) , \qquad (33)$$

the eigenfunctions $\psi_S^{(-)}(\rho,\theta)$ and $\psi_A^{(-)}(\rho,\theta)$ are immediatly obtained from (29) by comparison with (21) and (22)

$$\psi_S^{(-)}(\rho,\theta) = \frac{i}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R^{O}_{3+6n}(\rho)\sin[(3+6n)\theta] \qquad (34)$$

$$\psi_A^{(-)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R^E_{3+6n}(\rho) \cos[(3+6n)\theta] \quad . \tag{35}$$

As we have seen (equations (23) to (26a)), the mixed symmetry eigenfunctions are associated with $K=2+6n$ and $K=4+6n$ sets, for positive parity, and with $K=1+6n$ and $K=5+6n$ sets, for negative parity. Expansion (11) for the K sets are, respectively,

$$\psi_M^{(+)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R_{2+6n}(\rho)\left\{\cos[(2+6n)\theta] + i\sin[(2+6n)\theta]\right\}, \tag{36}$$

$$\psi_{M'}^{(+)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R_{4+6n}(\rho)\left\{\cos[(4+6n)\theta] + i\sin[(4+6n)\theta]\right\}, \tag{36a}$$

and

$$\psi_M^{(-)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R_{1+6n}(\rho)\left\{\cos[(1+6n)\theta] + i\sin[(1+6n)\theta]\right\}, \tag{37}$$

$$\psi_{M'}^{(-)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} R_{5+6n}(\rho)\left\{\cos[(5+6n)\theta] + i\sin[(5+6n)\theta]\right\}, \tag{37a}$$

which, in terms of the functions $\phi_{MS}^{(\pm)}$, $\phi_{MA}^{(\pm)}$ (equations (23), (24), (25), (26)), using (27a) and (27b), are given by

$$\psi_M^{(+)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}}\left\{\phi_{MS}^{(+)}(\rho,\theta) + i\,\phi_{MA}^{(+)}(\rho,\theta)\right\} \tag{38}$$

$$\psi_{M'}^{(+)}(\rho,\theta) = \pm\frac{1}{(2\pi)^{1/2}}\left\{\phi_{MS}^{(+)}(\rho,\theta) - i\,\phi_{MA}^{(+)}(\rho,\theta)\right\} = \pm\left[\psi_M^{(+)}(\rho,\theta)\right]^*, \tag{38a}$$

and

$$\psi_M^{(-)}(\rho,\theta) = \frac{i}{(2\pi)^{1/2}}\left\{\phi_{MS}^{(-)}(\rho,\theta) - i\,\phi_{MA}^{(-)}(\rho,\theta)\right\} \tag{39}$$

$$\psi_{M'}^{(-)}(\rho,\theta) = \mp\frac{i}{(2\pi)^{1/2}}\left\{\phi_{MS}^{(-)}(\rho,\theta) + i\,\phi_{MA}^{(-)}(\rho,\theta)\right\} = \pm\left[\psi_M^{(-)}(\rho,\theta)\right]^* \quad . \tag{39a}$$

In order to exhibit the mixed symmetry character of the eigenfunctions (38) to (39a), the simple and elegant method devised by Simonov[2] will be used. The following complex conjugate vectors are introduced

$$Z = \sin\theta + i\cos\theta \qquad \text{and} \qquad Z^* = \sin\theta - i\cos\theta \quad . \tag{40}$$

Their transformation properties under P_{ij} can be obtained from those of ξ and η (equations (15a), (15b), (15c) since $Z = (\xi+i\eta)/\rho$ (equation (6)). The transformations are

$$P_{12}Z = Z^* \quad , \quad P_{13}Z = e^{-i2\pi/3}Z^* \quad , \quad P_{23}Z = e^{i2\pi/3}Z^* \tag{41a}$$

and

$$P_{12}Z^* = Z \quad , \quad P_{13}Z^* = e^{i2\pi/3}Z \quad , \quad P_{23}Z^* = e^{-i2\pi/3}Z \quad . \tag{41b}$$

Likewise, for any given mixed symmetry basis (MS, MA), complex conjugate vectors of the form (40),

$$M_Z = MS + iMA \qquad \text{and} \qquad M_{Z^*} = MS - iMA \quad , \tag{40a}$$

can be introduced (they transform, of course, like Z and Z^*, thus the indexes).

So, using the mixed symmetry vectors given by equations (23) and (24), for positive parity, and by (25) and (26), for negative parity, the following complex conjugate vectors are introduced

$$\phi_Z^{(+)}(\rho,\theta) = \phi_{MS}^{(+)}(\rho,\theta) + i\,\phi_{MA}^{(+)}(\rho,\theta) \quad , \quad \phi_{Z*}^{(+)}(\rho,\theta) = \phi_{MS}^{(+)}(\rho,\theta) - i\,\phi_{MA}^{(+)}(\rho,\theta) \tag{42}$$

and

$$\phi_Z^{(-)}(\rho,\phi) = \phi_{MS}^{(-)}(\rho,\theta) + i\,\phi_{MA}^{(-)}(\rho,\theta) \quad , \quad \phi_{Z*}^{(-)}(\rho,\theta) = \phi_{MS}^{(-)}(\rho,\theta) - i\,\phi_{MA}^{(-)}(\rho,\theta) . \tag{43}$$

The mixed symmetry character of eigenfunctions $\psi_M^{(\pm)}(\rho,\theta)$ (equations (38) and (39)) is now evident. Using $\phi_Z^{(+)}(\rho,\theta)$ (42) and $\phi_{Z*}^{(-)}(\rho,\theta)$ (43), the eigenfunctions $\psi_M^{(\pm)}(\rho,\theta)$ (equations (38) and (39)) are written as

$$\psi_M^{(+)}(\rho,\theta) = \frac{1}{(2\pi)^{1/2}}\,\phi_Z^{(+)}(\rho,\theta) \tag{44}$$

and

$$\psi_M^{(-)}(\rho,\theta) = \frac{i}{(2\pi)^{1/2}}\,\phi_{Z*}^{(-)}(\rho,\theta) \quad , \tag{45}$$

respectively.

Thus, $\psi_M^{(+)}(\rho,\theta)$ (44), and its complex conjugate, transform under S_3 as Z and Z^*, respectively (equations (41a) and (41b)). As for the eigenfunction $\psi_M^{(-)}(\rho,\theta)$ (45), and its complex conjugate, they transform like Z^* and Z respectively (equations (41b) and (41a).

We now show how the eigenfunctions $\psi_M^{(\pm)}(\rho,\theta)$ and $(\psi_M^{\pm}(\rho,\theta))^*$ can be used to obtain totally symmetric or totally antisymmetric functions. As mentioned in the end of section III, the particles must have an extra degree of freedom. We shall assume that the particles have spin $\frac{1}{2}$ and the eigenvector corresponding to spin up (down) will be denoted by u(d). Spin eigenfunctions of the three particle system that under the permutation group S_3, transform as Z and Z^* will be obtained through that procedure of combining the MS and MA spin functions as in (40a). The three-particle mixed symmetry functions[9] have total spin $\frac{1}{2}$ and superscript $\pm\frac{1}{2}$ will be used to indicate the Z-component of the total spin. The spin up functions are

$$\chi_{MS}^{1/2} = \frac{1}{\sqrt{6}}\,[udu + duu - 2uud]$$

$$\chi_{MA}^{1/2} = \frac{1}{\sqrt{2}}\,[udu - duu] \quad ,$$

and the spin down are

$$\chi_{MS}^{-1/2} = \frac{1}{\sqrt{6}}\,[dud + udd - 2ddu]$$

$$\chi_{MA}^{-1/2} = \frac{1}{\sqrt{2}}\,[dud - udd] \quad .$$

And the corresponding complex conjugate vectors, that transform as Z and Z^* ((41a), (41b)), are

$$\chi_Z^{1/2} = \chi_{MS}^{1/2} + i\,\chi_{MA}^{1/2} \quad , \quad \chi_{Z*}^{1/2} = \chi_{MS}^{1/2} - i\,\chi_{MA}^{1/2} \tag{46}$$

and

$$\chi_Z^{-1/2} = \chi_{MS}^{-1/2} + i\,\chi_{MA}^{-1/2} \quad , \quad \chi_{Z*}^{-1/2} = \chi_{MS}^{-1/2} - i\,\chi_{MA}^{-1/2} \quad . \qquad (47)$$

The totally symmetric and totally antisymmetric wave functions are now obtained from the appropriate combination of $\psi_M^{(\pm)}(\rho,\theta)$ (equations (44) and (45)), and their complex conjugate, with $\chi_Z^{\pm 1/2}(\rho,\theta)$, and their complex conjugate, (equations (46) and (47)). The positive parity functions are

$$\psi_S^{(+)}(\rho,\theta;\pm\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\left[\psi_M^{(+)}(\rho,\theta)\;\chi_{Z*}^{\pm 1/2} + \left(\psi_M^{(+)}(\rho,\theta)\right)^*\;\chi_Z^{\pm 1/2}\right] , \qquad (48)$$

$$\psi_A^{(+)}(\rho,\theta;\pm\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\left[\psi_M^{(+)}(\rho,\theta)\;\chi_{Z*}^{\pm 1/2} - \left(\psi_M^{(+)}(\rho,\theta)\right)^*\;\chi_Z^{\pm 1/2}\right] , \qquad (49)$$

and the negative parity are

$$\psi_S^{(-)}(\rho,\theta;\pm\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\left[\left(\psi_M^{(-)}(\rho,\theta)\right)^*\;\chi_{Z*}^{\pm 1/2} + \psi_M^{(-)}(\rho,\theta)\;\chi_Z^{\pm 1/2}\right] , \qquad (50)$$

$$\psi_A^{(-)}(\rho,\theta;\pm\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\left[\left(\psi_M^{(-)}(\rho,\theta)\right)^*\;\chi_{Z*}^{\pm 1/2} - \psi_M^{(-)}(\rho,\theta)\;\chi_Z^{\pm 1/2}\right] .. \qquad (51)$$

Of course, if the three-fermion system has no other degree of freedom besides spin, only the above totally antisymmetric eigenfunctions $\psi_A^{(\pm)}(\rho,\theta;\pm\frac{1}{2})$ (equations (49) and (51)) are admissible.

V. CONCLUSIONS

We now analyse some qualitative features of the bound-state spectrum of one-dimensional three-particle system described by a hamiltonian invariant under parity and under the group S_3. We shall consider the cases of bosons of spin zero, fermions of spin $\frac{1}{2}$ and also the case of distinguishable particles (Boltzmann statistics).

For a system of bosons of zero spin, the only admissible eigenfunctions are the totally symmetric ones, $\psi_S^{(\pm)}(\rho,\theta)$ (equations (31) and (34)). Due to the absence of centrifugal barrier for $K=0$, the ground-state eigenfunction will be $\psi_S^{(+)}(\rho,\theta)$. There may be an excited state described by $\psi_S^{(-)}(\rho,\theta)$, for which the lowest component is $K=3$, thus having a considerable centrifugal barrier. Depending on the deepness of the potential, there may be a whole band of states of type $\psi_S^{(+)}(\rho,\theta)$ and $\psi_S^{(-)}(\rho,\theta)$, but no mixed symmetry type is allowed. In the case of the δ-potential, there is only one-bound-state, $\psi_S^{(+)}(\rho,\theta)$.

Now, for a system of spin $\frac{1}{2}$ fermions, the eigenfunction in spin-space will have mixed symmetry character due to Pauli principle (two states for accomodating three particles). Thus the spatial eigenfunction will be either $\psi_M^{(+)}(\rho,\theta)$ or $\psi_M^{(-)}(\rho,\theta)$ (equations (36) and (37)), the corresponding total eigenfunction being, respectively, $\psi_A^{(+)}(\rho,\theta;\pm\frac{1}{2})$ or $\psi_A^{(-)}(\rho,\theta;\pm\frac{1}{2})$ (equations (49) and (51)). Due to the centrifugal barrier we expect the eigenfunction $\psi_M^{(-)}(\rho,\theta)$ (which contains $K = -1,5$ components) to have an eigenvalue smaller than that of the eigenfunction $\psi_M^{(+)}(\rho,\theta)$ (which contains $K = 2,-4$ components).

Finally, for a system of distinguishable particles, (Boltzmann statistics), all symmetry type states are allowed.

The ground-state will be described by $\psi_S^{(+)}(\rho,\theta)$ (equation (31)) due to the absence of centrifugal barrier for K=0 . The other symmetry states, due to the centrifugal barrier, should occur in the following order: $\psi_M^{(-)}(\rho,\theta)$ (equation (37)), $\psi_M^{(+)}(\rho,\theta)$ (equation (36)), $\psi_{S,A}^{(-)}$ (degenerate) (equation (33), (34)), $\psi_A^{(+)}(\rho,\theta)$ (equation (32)) (assuming that the potential is attractive enough to bind all of them). Depending on the deepness of the potential, there may be whole bands of excited states for each type of symmetry state and the above order of occurence of states refer to the heads of the bands.

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