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GENERAL STATISTICS AND SECOND QUANTIZATION

by

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"It is just barely conceivable that the baby has been let out with the bathwater"

ABSTRACT

In order to obtain the commutation relations for the creation and annihilation operators for the para-particles in Quantum Mechanics, a somewhat new second quantization method is proposed in this paper. We show that for bosons and fermions the usual bi-linear commutation relations are valid and that for the para-particles the commutation relations have a multi-linear matricial form. It is also shown that from a symmetric group point of view, it is hard to accept the paraboson and parafermion concepts in Quantum Mechanics.

1. INTRODUCTION

In a preceding paper[1] we have shown, using the irreducible representations of the symmetric group in Hilbert space, that boson and fermion states and also para-states are compatible with the postulates of Quantum Mechanics and with the Principle of Indistinguishability. Our analysis that gives support, within the framework of Quantum Mechanics, to the mathematical existence of para-states, justifies, in a certain sense, the general statistics proposed a long time ago by Gentile in a thermodynamic context[2] . We have improperly named para-bosons and para-fermions our intermediate para-states, only to be in agreement with Green's terminology[3]. Nevertheless, there are substantial differences between the two concepts as it is seen in the sequel. Throughout this work we use the term para-state in a broader and more precise sense than usually employed. A better name would be general-states or gentileon-states and the para-particles that could be represented by these para-states would be named gentileons.

In a few words, we have shown that an isolated system consisting of N identical particles with total energy E has a $N!$ degenerate energy spectrum, due to the permutations $P_i (i = 1, 2, \ldots, N!)$ of the labels $1, 2, 3, \ldots, N$ of the particles in their configuration space $\varepsilon^{(N)}$. Our analysis has been performed considering the eigenfuctions of the energy operator $\hat{H}(1, 2, \ldots, N)$, but it is easy to see that similar

results could be obtained by taking into account any hermitean operator $\hat{f}(1, 2, \ldots, N)$.

The energy eigenfunctions $\{e_i\}$ $(i = 1, 2, \ldots, N!)$, where $e_1 = u(1,2, \ldots, N)$ and $e_2, e_3, \ldots, e_{N!}$ are obtained from e_1 by permuting the labels $1, 2, \ldots, N$, constitute a $N!$ dimensional basis of a Hilbert space that was indicated by $\mathcal{L}_2(\varepsilon^{(N)})$. This $\mathcal{L}_2(\varepsilon^{(N)})$ is decomposed into irreducible sub-spaces $h^{(\alpha)}$, that are the underlying sub-spaces of the representations of the symmetric group $S^{(N)}$ in $\mathcal{L}_2(\varepsilon^{(N)})$ corresponding to the different partitions (α) of the number $\underline{N}$. There are two one-dimensional sub-spaces that corespond to $(\alpha) = (N)$ and $(\alpha) = (1^N)$ and the wavefunctions associated to them are, respectively, Y_S , which is totally symmetric, and Y_A , which is totally anti-symmetric under permutations. The remaining sub-spaces $h^{(\alpha)}$ have dimensions going from 2^2 up to $(N-1)^2$ with attached wavefunctions indicated by the column vectors

$$Y(\alpha) = \frac{1}{\sqrt{\tau}} \begin{pmatrix} Y_1(\alpha) \\ Y_2(\alpha) \\ \vdots \\ Y_\tau(\alpha) \end{pmatrix} \tag{1.1}$$

where $\tau = (f^{(\alpha)})^2$ is the $h^{(\alpha)}$ dimension and $Y_i(\alpha)$ $(i = 1, 2, \ldots, \tau)$ (which constitute the basis of $h^{(\alpha)}$, are given by a linear combination of the unitary vectors $\{e_j\}$

($j = 1, 2, \ldots, N!$) .

By applying a permutation P to the particle labels in $\varepsilon^{(N)}$, the vector $Y(\alpha)$ becomes $P\,Y(\alpha) = X(\alpha) = T_\alpha\, Y(\alpha)$, where T_α is a unitary matrix with τ^2 components. For the one-dimensional sub-spaces we have $Y_S = Y_S$ and $Y_A = -Y_A$ so that the concept of totally symmetric and totally anti-symmetric wavefunctions subsists. For the multidimensional $h^{(\alpha)}$ these concepts are meaningless because the permutation operation $P\,Y(\alpha)$ implies in a rotation of $Y(\alpha)$, defined by a matrix T_α with τ^2 components (there are τ^2 numbers, put into a matrix form, associated to the permutation P, instead of only one number that can be $+1$ or -1).

Since T_α is a unitary matrix (orthogonal in the real case), it was also shown that the function

$$\phi(\alpha) = \sum_{i=1}^{\tau} |Y_i(\alpha)|^2 \qquad (1.2)$$

is permutation invariant and has been defined as the probability density function.

According to Okayama[4], to take into account the correct dimensions of the irreducible sub-spaces, it is necessary an extension of the definition of the multi-dimensional wavefunctions. However, with this in mind, he used a somewhat cumbersome matricial form for the $Y(\alpha)$, in which we do not see clearly all symmetries involved. Thus, we are convinced now that the

original and main results of our preceding paper[1] are: (1) we have obtained a more compact and precise representation of $h^{(\alpha)}$, showing that the $Y(\alpha)$ can be put into a vectorial form, with τ orthonormal components $Y_i(\alpha)$ and (2) we have shown that the function $\phi(\alpha)$, defined by equation (1.2) is a permutation invariant. Thus, if para-particles (gentileons) exist, they must be represented by the state vectors $Y(\alpha)$ of the multi-dimensional sub-spaces. The number of columns of the first row of the Young shape associated to each irreducible sub-space $h^{(\alpha)}$ will determine the possible maximal occupation number $\underline{d}$ of the gentileon. This maximal order $\underline{d}$ will be named statistical order $\underline{d}$ of the gentileon. If, for instance, we have a system composed by $\underline{N}$ gentileons of order $\underline{d}$, with $d \geqslant N$, the only possible state vector that could represent the system should be a bosonic function. So, the gentilionic behaviour should be masked when the above condition is fulfilled. A natural extension of these statements allows us to infer that all particles are gentileons. The fermions correspond to $\underline{d} = 1$, whereas the bosons correspond to $\underline{d} = \infty$.

The present paper, which follows the same algebraic-geometric reasoning adopted before[1], is arranged as : in Section 2 we study some geometric properties of the $Y(\alpha)$ states, by analysing carefully a system formed by 3 gentileons of order $\underline{d} = 2$, and by extending to the N-particles systems several conclusions. In Section 3 it is developed a somewhat different process of second quantization and multi-linear commutation relations for the creation and annihilation operators, and their hermitean conjugates are obtained. The usual bi-linear commutation and anti-commutation relations for bosons and fermions are particular cases of our general expressions. In Section 4, our results are also compared with para-statistics theory proposed by Green[3] and the substantial differences which have been found between them are exhibited.

2. ROTATIONS IN THE HILBERT SPACE

In this section we analyse some transformation properties of the wavefunctions of a system of N non-interacting particles. A geometrical interpretation of the transformations, based on the representation of the symmetric group, is given in terms of the basis vectors of a Hilbert space. We will restrict ourselves to the detailed study of the simplest non trivial 3 particles case. The generalizations of the essential results which can be extracted from this simple case appear at the end of the section.

Thus, if the system is composed by 3 particles, according to Cattani and Fernandes[1], the Hilbert space has dimension 6, comprising two one-dimensional sub-spaces and one four-dimensional sub-space. It was shown that, if we indicate by.

$$e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_6 \end{pmatrix}$$

the vectors of the basis of $\mathcal{L}_2(\varepsilon^{(3)})$, the uni-dimensional wavefunctions Y_S and Y_A are given by

$$Y_S = (1/\sqrt{6}) \sum_{i=1}^{6} e_i$$

and $\Upsilon_A = (1/\sqrt{6})(e_1 - e_2 - e_3 + e_4 + e_5 + e_6)$ and the 4 dimensional Υ is given by

$$\Upsilon = \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \\ \Upsilon_4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/2 & -1/2 & 0 & -1/2 \\ 1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & 1/2\sqrt{3} & -1/\sqrt{3} & -1/2\sqrt{3} \\ -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} \\ 1/2 & 0 & -1/2 & -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} \qquad (2.1)$$

non normalized to one.

These wavefunctions can be put into a compact form Ψ :

$$\Psi = \begin{pmatrix} \Upsilon_S \\ \\ \Upsilon \\ \\ \\ \Upsilon_A \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/2 & 0 & 1/2 & -1/2 & 0 & -1/2 \\ 1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & 1/2\sqrt{3} & -1/\sqrt{3} & -1/2\sqrt{3} \\ -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} \\ 1/2 & 0 & -1/2 & -1/2 & 0 & 1/2 \\ 1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = U\,e \qquad (2.2)$$

where U is a unitary matrix which determines the structure of the functions Υ_S, Υ and Υ_A .

Now, if we assume that the particles do not interact and we indicate by α , β and γ the states allowed for them, we see that the basis vectors $\underline{e}$ can be written as

$$e\,(\alpha\gamma\beta) = \begin{pmatrix} e_1(\alpha\beta\gamma) \\ e_2(\alpha\beta\gamma) \\ e_3(\alpha\beta\gamma) \\ e_4(\alpha\beta\gamma) \\ e_5(\alpha\beta\gamma) \\ e_6(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} \alpha(1) & \beta(2) & \gamma(3) \\ \alpha(1) & \beta(3) & \gamma(2) \\ \alpha(2) & \beta(1) & \gamma(3) \\ \alpha(2) & \beta(3) & \gamma(1) \\ \alpha(3) & \beta(1) & \gamma(2) \\ \alpha(3) & \beta(2) & \gamma(1) \end{pmatrix}$$

If, instead of the order $\alpha\beta\gamma$ we have, for instance, $\alpha\gamma\beta$ the basis vector is given by

$$e(\alpha\gamma\beta) = \begin{pmatrix} e_1(\alpha\gamma\beta) \\ e_2(\alpha\gamma\beta) \\ e_3(\alpha\gamma\beta) \\ e_4(\alpha\gamma\beta) \\ e_5(\alpha\gamma\beta) \\ e_6(\alpha\gamma\beta) \end{pmatrix} = \begin{pmatrix} \alpha(1) & \gamma(2) & \beta(3) \\ \alpha(1) & \gamma(3) & \beta(2) \\ \alpha(2) & \gamma(1) & \beta(3) \\ \alpha(2) & \gamma(3) & \beta(1) \\ \alpha(3) & \gamma(1) & \beta(2) \\ \alpha(3) & \gamma(2) & \beta(1) \end{pmatrix}$$

Since $e_1(\alpha\gamma\beta) = e_2(\alpha\beta\gamma)$, $e_2(\alpha\gamma\beta) = e_1(\alpha\beta\gamma)$, $e_3(\alpha\gamma\beta) = e_4(\alpha\beta\gamma)$, $e_4(\alpha\gamma\beta) = e_3(\alpha\beta\gamma)$, $e_5(\alpha\gamma\beta) = e_6(\alpha\gamma\beta)$ and $e_6(\alpha\gamma\beta) = e_5(\alpha\beta\gamma)$, the basis transformation $e(\alpha\beta\gamma) \rightarrow e(\alpha\gamma\beta)$ can be described by the matricial relation

$$e(\alpha\gamma\beta) = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} e_1(\alpha\beta\gamma) \\ e_2(\alpha\beta\gamma) \\ e_3(\alpha\beta\gamma) \\ e_4(\alpha\beta\gamma) \\ e_5(\alpha\beta\gamma) \\ e_6(\alpha\beta\gamma) \end{pmatrix} = P\begin{pmatrix} \alpha\beta\gamma \\ \alpha\gamma\beta \end{pmatrix} e(\alpha\beta\gamma)$$

where $P\begin{pmatrix} \alpha\beta\gamma \\ \alpha\gamma\beta \end{pmatrix}$ is a unitary matrix.

Thus, for a generic transformation $(\alpha\beta\gamma) \rightarrow (ijk)$ where the indices i, j and k can assume the values α, β and γ in an arbitrary order, we write :

$$e(ijk) = P\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} e(\alpha\beta\gamma) \qquad (2.3)$$

The equation (2.3) means the following: when the indices $(\alpha\beta\gamma)$ are permuted in an arbitrary way, the basis $\underline{e}$ is rotated in the Hilbert space $\mathcal{L}_2(\varepsilon^{(3)})$.

If we choose the order $(\alpha\beta\gamma)$, the wavefunction $\Psi(\alpha\beta\gamma)$ is, according to equation (2.2), given by :

$$\Psi(\alpha\beta\gamma) = \begin{pmatrix} Y_S(\alpha\beta\gamma) \\ Y(\alpha\beta\gamma) \\ Y_A(\alpha\beta\gamma) \end{pmatrix} = U\, e(\alpha\beta\gamma)$$

and, in a generic order (ijk) we have, using equations (2.2) and (2.3) :

$$\Psi(ijk) = U\, e(ijk) = U\, P\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} e(\alpha\beta\gamma) =$$

$$= U\, P\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} U^+\, U\, e(\alpha\beta\gamma) = M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} \Psi(\alpha\beta\gamma).$$

That is,

$$\Psi(ijk) = M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} \Psi(\alpha\beta\gamma) \tag{2.4}$$

where the matrix $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix}$, defined as $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} = U\, P\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} U^+$ depends on the structure matrix U .

In appendix 1 we show explicitly whe $\underline{M}$ matrices for all possible values of i, j and k . These 6 x 6 matrices have the general form

$$M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} = \left(\begin{array}{c|cccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & & 0 \\ 0 & & & & & 0 \\ 0 & & G\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} & & & 0 \\ 0 & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \pm 1 \end{array}\right) \tag{2.5}$$

where $G\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix}$ is a unitary 4 x 4 matrix associated to the para-states.

So, when the indices $(\alpha\beta\gamma)$ are permuted, assuming the (ijk) values, the wavefunctions Y_S, Y_A and Y undergo rotations according to the following relations :

$$Y_S(ijk) = Y_S(\alpha\beta\gamma),\ Y_A(ijk) = \pm Y_A(\alpha\beta\gamma)$$

and $Y(ijk) = G\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} Y(\alpha\beta\gamma)$. Of course, due to the unitarity of the matrices $\underline{M}$, the functions $|Y_S|^2$, $|Y_A|^2$ and $|Y|^2 = \sum_{i=1}^{4} |Y_i|^2$ are invariants under permutations. This assures that the physical interpretations of the wavefunctions are unaltered by the unobservables transformations $(\alpha\beta\gamma) \to (ijk)$.

There are no restrictions in the occupation numbers of the states $(\alpha\beta\gamma)$ in the basis vectors $e(\alpha\beta\gamma)$, i. e., one, two or three particles can occupy the same state in $e(\alpha\beta\gamma)$. However, if there are three particles in the same state, we have $Y_A = Y = 0$ and $Y_S \neq 0$. For two particles in the same state, $Y_S \neq 0$, $Y_A = Y_2 = Y_4 = 0$ and $Y_1 \neq 0$ and $Y_3 \neq 0$. Here we see that Y_2 and Y_4 have a fermionic behaviour and

Y_1 and Y_3 a bosonic behaviour at least when the number of particles that occupy the same state is smaller than d=2.

When two particles occupy the same state the basis vector $\underline{e}$ has only three independent components. Thus, if we put two particles in the same state α , for instance, $\alpha = \beta$ and one in the state γ , $\Psi(\alpha\alpha\gamma)$ can be written in a compact form :

$$\Psi(\alpha\alpha\gamma) = \begin{pmatrix} Y_S \\ Y_1 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} e_1(\alpha\alpha\gamma) \\ e_2(\alpha\alpha\gamma) \\ e_4(\alpha\alpha\gamma) \end{pmatrix} = u\ e(\alpha\alpha\gamma) \quad (2.6)$$

remembering that $Y_2 = Y_4 = Y_A = 0$.

If, instead of $(\alpha\alpha\gamma)$ we have $(\alpha\gamma\alpha)$ it is easy to see that $\Psi(\alpha\gamma\alpha) = u\ e(\alpha\gamma\alpha) = u\,P\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix}\ e(\alpha\alpha\gamma) = u\ P\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix}\ u^+\ \Psi(\alpha\alpha\gamma) = m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix}\ \Psi(\alpha\alpha\gamma)$, since $\underline{u}$ is a unitary matrix. That is, $\Psi(\alpha\gamma\alpha) = m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix}\ \Psi(\alpha\alpha\gamma)$, where $m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} = u\ P\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix}\ u^+$. Similarly, for $(\gamma\alpha\alpha)$, $\Psi(\gamma\alpha\alpha) = m\begin{pmatrix}\alpha\alpha\gamma\\ \gamma\alpha\alpha\end{pmatrix}\ \Psi(\alpha\alpha\gamma)$.

Since

$$e(\alpha\gamma\alpha) = \begin{pmatrix} e_2(\alpha\gamma\alpha) \\ e_1(\alpha\gamma\alpha) \\ e_3(\alpha\gamma\alpha) \end{pmatrix} \quad \text{and} \quad e(\gamma\alpha\alpha) = \begin{pmatrix} e_5(\gamma\alpha\alpha) \\ e_3(\gamma\alpha\alpha) \\ e_1(\gamma\alpha\alpha) \end{pmatrix}$$

one can verify that $m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} = m\begin{pmatrix}\alpha\alpha\gamma\\ \gamma\alpha\alpha\end{pmatrix} = I$ is the identity matrix. Consequently, $Y_S(\alpha\alpha\gamma) = Y_S(\alpha\gamma\alpha) = Y_S(\gamma\alpha\alpha)$ and $Y(\alpha\alpha\gamma) = Y(\gamma\alpha\alpha) = Y(\gamma\alpha\alpha)$.

As was pointed out before, the basis vectors $\underline{e}$ do not present any restrictions on the occupation numbers. The restrictions only appear in the structure of Y_S, Y and Y_A .

The totally symmetric function Y_S and the totally antisymmetric Y_A represent, as well known, bosons and fermions. The multidimensional hybrid function Y must represent, in our formalism, a system composed by 3 gentileons of order 2 .

For N non interacting particles, we can also show that the basis vectors $\underline{e}$ and the wavefunctions transformations are generically given by the matrix relations :

$$e(\ldots\ ijk\ \ldots) = P\begin{pmatrix}\ldots & \alpha\beta\gamma & \ldots\\ \ldots & ijk & \ldots\end{pmatrix}\ e(\ldots\ \alpha\beta\gamma\ \ldots)$$

and

$$\Psi(\ldots\ ijk\ \ldots) = M\begin{pmatrix}\ldots & \alpha\beta\gamma & \ldots\\ \ldots & ijk & \ldots\end{pmatrix}\ \Psi(\ldots\ \alpha\beta\gamma\ \ldots)$$

where the matrix $\underline{P}$ can be easily computed, but the matrix $\underline{M}$ requires an extremely laborious calculations, since it depends on the structure of the multidimensional manifolds.

3. SECOND QUANTIZATION

In the second quantization that we propose, the creation a_k^* and annihilation a_k operators act, row by row, on the column vector $\underline{e}$.

Indicating by $e_0 = e(000)$ the "basis-vector vacuum state", the creation operators are defined by the following relations :

$$a_\alpha^* \; a_\beta^* \; a_\gamma^* \; e(000) = a_\alpha^* \; a_\beta^* \; e(00\gamma) = a_\alpha^* \; e(0\beta\gamma) = e(\alpha\beta\gamma)$$

$$a_\alpha^* \; e(0\alpha\beta) = \sqrt{2} \; e(\alpha\alpha\beta)$$

$$a_\alpha^* \; a_\alpha^* \; a_\alpha^* \; e(000) = a_\alpha^* \; a_\alpha^* \; e(00\alpha) = \sqrt{2} \; a_\alpha^* \; e(0\alpha\alpha) = \sqrt{2} \; \sqrt{3} \; e(\alpha\alpha\alpha) \qquad (3.1)$$

$$a_\alpha^* \; e(\beta 0\gamma) = P\begin{pmatrix}0\beta\gamma\\ \beta 0\gamma\end{pmatrix} \; a_\alpha^* \; e(0\beta\gamma) = P\begin{pmatrix}0\beta\gamma\\ \beta 0\gamma\end{pmatrix} \; e(\alpha\beta\gamma)$$

$$a_\alpha^* \; e(\beta\alpha 0) = P\begin{pmatrix}0\alpha\beta\\ \beta\alpha 0\end{pmatrix} \; a_\alpha^* \; e(0\alpha\beta) = P\begin{pmatrix}0\alpha\beta\\ \beta\alpha 0\end{pmatrix} \; \sqrt{2} \; e(\alpha\alpha\beta) \; .$$

Similarly, the annihilation operators are defined by :

$$a_\alpha \; e(\alpha\beta\gamma) = e(0\beta\gamma)$$

$$a_\alpha \; e(0\alpha\gamma) = e(00\gamma)$$

$$a_\alpha \; e(\alpha\alpha\beta) = \sqrt{2} \; e(0\alpha\beta)$$

$$a_\alpha \; e(\alpha 0\gamma) = P\begin{pmatrix}0\alpha\gamma\\ \alpha 0\gamma\end{pmatrix} \; a_\alpha \; e(0\alpha\gamma) = P\begin{pmatrix}0\alpha\gamma\\ \alpha 0\gamma\end{pmatrix} \; e(00\gamma)$$

$$a_\alpha \; e(\gamma\alpha\alpha) = P\begin{pmatrix}\alpha\gamma\alpha\\ \gamma\alpha\alpha\end{pmatrix} \; a_\alpha \; e(\alpha\gamma\alpha) = \sqrt{2} \; P\begin{pmatrix}\alpha\gamma\alpha\\ \gamma\alpha\alpha\end{pmatrix} \; e(0\gamma\alpha)$$

Since, by equation (2.3) we have, $e(ijk) = P\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} e(\alpha\beta\gamma)$, it is straightforward to show that

$$\begin{aligned} a_i^* \; a_j^* \; a_k^* \; e &= P\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} \; a_\alpha^* \; a_\beta^* \; a_\gamma^* \; e \\ a_i \; a_j \; a_k \; e &= P\begin{pmatrix}kji\\ \gamma\beta\alpha\end{pmatrix} \; a_\alpha \; a_\beta \; a_\gamma \; e \\ a_\beta^* \; a_\gamma \; a_\alpha \; e &= P\begin{pmatrix}\alpha\gamma\beta\\ \gamma\alpha\beta\end{pmatrix} \; a_\beta^* \; a_\alpha \; a_\gamma \; e \\ a_\alpha \; a_\beta^* \; a_\gamma \; e &= P\begin{pmatrix}\gamma\alpha\beta\\ \gamma\beta\alpha\end{pmatrix} \; a_\beta^* \; a_\alpha \; a_\gamma \; e \\ a_\alpha^* \; a_\beta \; a_\gamma^* \; e &= P\begin{pmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\end{pmatrix} \; a_\gamma^* \; a_\beta \; a_\alpha^* \; e \end{aligned} \qquad (3.3)$$

and others derived relations.

The creation and annihilation operators have been defined in such a way that $a_\alpha^* \; a_\alpha \; e = \bar{N}_\alpha \; e = N_\alpha \; e$, where N_α is the occupation number of the α state in the basis vector $\underline{e}$. It can also be verified that $a_\alpha \; a_\alpha^* = 1 + a_\alpha^* \; a_\alpha$, or, equivalently $\left[a_\alpha^* \, , \, a_\alpha^*\right]_- = 1$.

The equations (3.1), (3.2) and (3.3) define a composition law for the creation and annihilation operators when they act on the basis vector $\underline{e}$. We see that the permutation of the indices α and β , for instance, depends on the third element γ . This generalized result comprises several equivalent relations deduced in Grassmann algebras and belongs to more complicated structures defined in a third order algebraic system[5].

Let us obtain now the algebraic relations for a_k^* and a_k

when they act on the wavefunctions Y_S, Y and Y_A . Thus, using equations (2.2), (2.4) and (3.1) we get for $\alpha \neq \beta \neq \gamma \neq \alpha$:

$$\Psi(ijk) = U\, e(ijk) = a_i^* \; a_j^* \; a_k^* \; U\, e(000) = a_i^* \; a_j^* \; a_k^* \; \Psi(000)$$

and

$$\Psi(ijk) = M\begin{pmatrix}\alpha\beta\gamma \\ ijk\end{pmatrix} \; \Psi(\alpha\beta\gamma) = M\begin{pmatrix}\alpha\beta\gamma \\ ijk\end{pmatrix} \; a_\alpha^* \; a_\beta^* \; a_\gamma^* \; \Psi(000)$$

which permit us to conclude that :

$$a_i^* \; a_j^* \; a_k^* = M\begin{pmatrix}\alpha\beta\gamma \\ ijk\end{pmatrix} \; a_\alpha^* \; a_\beta^* \; a_\gamma^*$$

Similarly, it can be shown that

$$\begin{aligned}
a_i \; a_j \; a_k \; \Psi &= M\begin{bmatrix}kji \\ \gamma\beta\alpha\end{bmatrix} \; a_\alpha \; a_\beta \; a_\gamma \; \Psi \\
a_\beta^* \; a_\gamma \; a_\alpha \; \Psi &= M\begin{bmatrix}\alpha\gamma\beta \\ \gamma\alpha\beta\end{bmatrix} \; a_\beta^* \; a_\alpha \; a_\gamma \; \Psi \\
a_\alpha \; a_\beta^* \; a_\gamma \; \Psi &= M\begin{bmatrix}\gamma\alpha\beta \\ \gamma\beta\alpha\end{bmatrix} \; a_\beta^* \; a_\alpha \; a_\gamma \; \Psi \\
a_\alpha^* \; a_\beta \; a_\gamma^* \; \Psi &= M\begin{bmatrix}\alpha\beta\gamma \\ \alpha\beta\gamma\end{bmatrix} \; a_\gamma^* \; a_\beta \; a_\alpha^* \; \Psi
\end{aligned} \tag{3.4}$$

and so on.

We must note that the operator $a_\alpha^* \; a_\alpha = \hat{N}_\alpha$ gives the occupation number of the α state, that is, $a_\alpha^* \; a_\alpha \Psi = N_\alpha \Psi$.

When $\alpha = \beta \neq \gamma$, for instance, using the results of

section 2 and following the reasoning delineated above, we get

$$\begin{aligned}
a_\gamma^* \; a_\alpha^* \; a_\alpha^* \; \Psi &= m\begin{pmatrix}\alpha\alpha\gamma \\ \gamma\alpha\alpha\end{pmatrix} \; a_\alpha^* \; a_\alpha^* \; a_\gamma^* \; \Psi = m\begin{pmatrix}\alpha\gamma\alpha \\ \gamma\alpha\alpha\end{pmatrix} \; a_\alpha^* \; a_\gamma^* \; a_\alpha^* \; \Psi \\
a_\gamma \; a_\alpha \; a_\alpha \; \Psi &= m\begin{pmatrix}\alpha\alpha\gamma \\ \gamma\alpha\alpha\end{pmatrix} \; a_\alpha \; a_\alpha \; a_\gamma \; \Psi = m\begin{pmatrix}\alpha\alpha\gamma \\ \alpha\gamma\alpha\end{pmatrix} \; a_\alpha \; a_\gamma \; a_\alpha \; \Psi \\
a_\alpha \; a_\gamma^* \; a_\alpha \; \Psi &= m\begin{pmatrix}\alpha\alpha\gamma \\ \alpha\gamma\alpha\end{pmatrix} \; a_\gamma^* \; a_\alpha \; a_\alpha \Psi = m\begin{pmatrix}\gamma\alpha\alpha \\ \alpha\gamma\alpha\end{pmatrix} \; a_\alpha \; a_\alpha \; a_\gamma^* \; \Psi \\
a_\alpha^* \; a_\gamma \; a_\alpha^* \; \Psi &= m\begin{pmatrix}\alpha\gamma\alpha \\ \gamma\alpha\alpha\end{pmatrix} \; a_\gamma \; a_\alpha^* \; a_\alpha^* \Psi = m\begin{pmatrix}\alpha\gamma\alpha \\ \alpha\alpha\gamma\end{pmatrix} \; a_\alpha^* \; a_\alpha^* \; a_\gamma \; \Psi
\end{aligned} \tag{3.5}$$

When the operator $a_\alpha \; a_\alpha^*$ acts row by row on Y_S, Y and Y_A , results,

$a_\alpha \; a_\alpha^* = 1 + a_\alpha^* \; a_\alpha$ for Y_S

$a_\alpha \; a_\alpha^* = 1 + a_\alpha^* \; a_\alpha$ for Y_1 and Y_3 if $N_\alpha = 0,1$

$a_\alpha \; a_\alpha^* = d - a_\alpha^* \; a_\alpha$ for Y_1 and Y_3 if $N_\alpha \geqslant 2$, where $d = 2$ is the order of the gentileon statistics (3.6)

$a_\alpha \; a_\alpha^* = 1 - a_\alpha^* \; a_\alpha$ for Y_2 and Y_4

$a_\alpha \; a_\alpha^* = d_F - a_\alpha^* \; a_\alpha$ for Y_A, where $d_F = 1$ is the order of the Fermi statistics.

By taking into account the matrices $\underline{M}$ and $\underline{m}$ (given in appendix 1), we analyse now the action, row by row, of the creation and annihilation operators on Ψ :

a) Bosons and Fermions

When a_k and a_k^* act on Y_S and Y_A, we see that the following usual commutation relations are satisfied :

$$
\begin{aligned}
&[a_\alpha^* \ , \ a_\gamma^*]_\pm = [a_\alpha \ , \ a_\gamma]_\pm = 0 \\
&[a_\alpha \ , \ a_\gamma^*]_\pm = \delta_{\alpha\gamma} \quad \text{and, consequently,} \\
&[\hat{N}_\alpha \ , \ a_\gamma^*]_\pm = \delta_{\alpha\gamma} \ a_\gamma^* \\
&[\hat{N}_\alpha \ , \ a_\gamma]_\pm = -\delta_{\alpha\gamma} \ a_\gamma
\end{aligned}
\tag{3.7}
$$

where the (-) sign corresponds to Y_S (bosons) and the (+) sign to Y_A (fermions)

b) Para-particles

On the other hand, when a_k and a_k^* act on the para-state Y, we get, for $\alpha \neq \beta \neq \gamma \neq \alpha$:

$$
\begin{aligned}
a_i^* \ a_j^* \ a_k^* \ Y &= G\begin{pmatrix}\alpha\beta\gamma \\ ijk\end{pmatrix} \ a_\alpha^* \ a_\beta^* \ a_\gamma^* \ Y \\
a_i \ a_j \ a_k \ Y &= G\begin{pmatrix}kji \\ \gamma\beta\alpha\end{pmatrix} \ a_\alpha \ a_\beta \ a_\gamma \ Y \\
a_\beta^* \ a_\gamma \ a_\alpha \ Y &= G\begin{pmatrix}\alpha\gamma\beta \\ \gamma\alpha\beta\end{pmatrix} \ a_\beta^* \ a_\alpha \ a_\gamma \ Y \\
a_\alpha \ a_\beta^* \ a_\gamma \ Y &= G\begin{pmatrix}\gamma\alpha\beta \\ \gamma\alpha\beta\end{pmatrix} \ a_\beta^* \ a_\alpha \ a_\gamma \ Y \\
a_\alpha^* \ a_\beta \ a_\gamma^* \ Y &= G\begin{pmatrix}\alpha\beta\gamma \\ \alpha\gamma\beta\end{pmatrix} \ a_\gamma^* \ a_\beta \ a_\alpha^* \ Y \text{, and so on...}
\end{aligned}
\tag{3.8}
$$

When $\alpha = \beta \neq \gamma$, we have, for $i = 2$ and 4 :

$$
a_\alpha^* \ a_\alpha^* \ Y_i = a_\alpha \ a_\alpha \ Y_i = 0
$$

and for $i = 1$ and 3 :

$$
\begin{aligned}
a_\gamma^* \ a_\alpha^* \ a_\alpha^* \ Y_i &= a_\alpha^* \ a_\alpha^* \ a_\gamma^* \ Y_i = a_\alpha^* \ a_\gamma^* \ a_\alpha^* \ Y_i \\
a_\gamma \ a_\alpha \ a_\alpha \ Y_i &= a_\alpha \ a_\alpha \ a_\gamma \ Y_i = a_\alpha \ a_\gamma \ a_\alpha \ Y_i \\
a_\alpha \ a_\gamma^* \ a_\alpha \ Y_i &= a_\gamma^* \ a_\alpha \ a_\alpha \ Y_i = a_\alpha \ a_\alpha \ a_\gamma^* \ Y_i \\
a_\alpha^* \ a_\gamma \ a_\alpha^* \ Y_i &= a_\gamma \ a_\alpha^* \ a_\alpha^* \ Y_i = a_\alpha^* \ a_\alpha^* \ a_\gamma \ Y_i
\end{aligned}
\tag{3.9}
$$

Finally, when $\alpha = \beta = \gamma$, we have

$$
a_\alpha^* \ a_\alpha^* \ a_\alpha^* \ Y_i = a_\alpha \ a_\alpha \ a_\alpha \ Y_i = 0 \text{ for all } Y_i \ .
$$

It is very important to remark that for the components Y_2 and Y_4 we have $[\hat{N}_\alpha \ , \ a_\alpha^*]_+ = \delta_{\alpha\gamma} \ a_\gamma^*$ and $[\hat{N}_\alpha \ , \ a_\gamma]_+ = -\delta_{\alpha\gamma} \ a_\gamma$ and that for the components Y_1 and Y_3, $[\hat{N}_\alpha \ , \ a_\gamma^*]_- = \delta_{\alpha\gamma} \ a_\gamma^*$ and $[\hat{N}_\alpha \ , \ a_\gamma]_- = -\delta_{\alpha\gamma} \ a_\gamma$, when $N_\alpha \leqslant 2$. This means that two components of Y, namely Y_2 and Y_4 have a fermionic behaviour, whereas the two components Y_1 and Y_3 have a bosonic behaviour when $N_\alpha \leqslant d = 2$. It is commonly accepted(6) as a natural requirement, that these relations remain valid, in general, in the theory of free para-particles. Here we show that this assumption is actually correct and consistent.

For the N-particles case, following the above reasoning and taking into account the general results of section 2, we obtain, for bosons and fermions, the usual commutation relations. However, for the para-states, the commutation relations have multilinear matricial forms governed by the $G\begin{pmatrix}\dots & \alpha\beta\gamma & \dots\\ \dots & ijk & \dots\end{pmatrix}$ matrices. We are not aware of the exact form of these matrices, since they depend on the structure of the irreducible manifolds. Nevertheless, at least in priciple, they can be calculated.

Summarizing, we can say that for the one-dimensional sub-spaces, where the permutation of particles changes the state-vectors only by a numerical factor (± 1), the commutation relations are very simple. They are bi-linear single valued relations with the properties :

(1) The commutation relation between any two operators does not depend on the position of the remaining ones.

(2) For Y_A there are only commutation relations whereas for Y_S we have only anti-commutation relations.

For the multi-dimensional sub-spaces, where the symmetry properties of Y under permutations of particles are defined by a matrix, the commutation relations have multi-linear matricial forms obeying :

(1') The commutation relation between two operators depends on the position of the remaining ones.

(2') It depends on the particular row Y_i where it is applied.

4. CONCLUSIONS AND COMMENTS

As it is easy to see, our second quantization procedure is a natural extension of the usual second quantization method adopted in the literature(7). The well known boson and fermion commutation relations are obtained as two particular cases of our general expressions when we restrict ourselves to the one-dimensional sub-spaces. In our approach, the symmetry properties of the multi-dimensional sub-spaces, induced by the group of permutations, are preserved and the occupation numbers $N_\alpha=2, 3, 4,\dots$ arise as a natural consequence of the symmetries contained in the wavefunctions Ψ.

This work and the preceding one(1) about para-bosons and para-fermions in quantum mechanics, are based on several concepts and results, as those derived from the classical spectral theory of partial differential equations, that presuppose a classical heritage like the action-at-a-distance(8) which is, evidently, present in our wavefunctions and commutation relations. We are not aware, at the moment, of a method for generalizing our multilinear matricial commutation relations in order to apply them to the study of relativistic phenomena, which is the main purpose of Quantum Field Theory. In elaborating a Quantum Field Theory we try to define, in a consistent way, a set of field operators which are completely characterized by defining all possible algebraic relations between them. A more rigorous formulation can be given in terms of bounded operators and their algebras, but this is beyond the scope of the present work[9]. In our concern here, we must expect that the formalism of Quantum Field Theory could give the exact occupation numbers of the particles, which are the crucial observables and that, in the non relativistic limit, our wavefunctions and commutation relations are reproduced. As we have seen in the example of $N=3$, the gentileons of order $\underline{d}=2$ are represented by the 4-dimensional state vector Y with the rotation properties in the Hilbert space exhibited in Section 2 and with the corresponding commutation relations shown in Section 3.

As one can verify from equations (3.8) and (3.9), the commutation relations for the components Y_i are tri-linear. It is worthy to note that for Y_2 and Y_4 we have the commutation relations $[\hat{N}_\alpha, a^*_\gamma]_+ = \delta_{\alpha\gamma}\, a^*_\gamma$ and $[\hat{N}_\alpha, a_\gamma]_+ = -\delta_{\alpha\gamma}\, a_\gamma$, which are peculiar to fermions. On the other hand, for Y_1 and Y_3, when $N_\alpha \leq \underline{d} = 2$, the relations $[\hat{N}_\alpha, a^*_\gamma]_- = \delta_{\alpha\gamma}\, \alpha^*_\gamma$ and $[\hat{N}_\alpha, a_\gamma]_- = -\delta_{\alpha\gamma}\, \alpha_\gamma$ are satisfied. This means that Y_1 and Y_3, when $N_\alpha \leq 2$, show a bosonic character. However, when $N_\alpha=3$, $Y_1=Y_3=0$, i. e., it is impossible to accomodate more than 2 particles in the same state. Thus, the components Y_1 and Y_3 do not have a genuine bosonic behaviour. To sum up we can say that, at least in the non relativistic Quantum Mechanical limit, a gentileon of order $\underline{d}=2$ does not have a pure fermionic or bosonic behaviour. It is a fermion-boson hybrid. The same considerations remain also valid when the

system is composed of N particles. If a convenient basis is chosen for the representation space, as we have done above, the state vectors for the gentileons will be constituted by τ - vectors, where τ is the dimension of the $h^{(\alpha)}$, showing always the hybrid character, analogous to the case $N=3$.

Let us consider now the generalized method of field quantization developed by Green[3], that is anlysed with great detail in an excellent book by Ohnuki and Kamefuchi[10]. Founded in the idea that the hypothesis of complete symmetry or complete anti-symmetry of the state vector of a system of particles is stronger than the assumption of the physical identity of the particles[6], Green suggested the tri-linear commutation relations

$$\left[a^*_\lambda \; a_\mu - \sigma\, a_\mu\, a^*_\lambda \,,\, a_\nu\right]_- = -2\, \delta_{\lambda\nu}\, a_\mu \tag{4.1}$$

$$\left[a_\lambda \; a_\mu - \sigma\, a_\mu\, a_\lambda \,,\, a_\nu\right]_- = 0$$

for the para-fields. The parameter σ (σ = 1 for fermions and σ = -1 for bosons) will characterize the two possible para-statistics.

To solve the system (4.1) he has also suggested a decomposition of the para-fields by the now well established "Green's anzatz"

$$a_\nu = \sum_{\alpha=1}^{p} b_\nu^{(\alpha)} \tag{4.2}$$

where p is called the order of the parastatistics and the $b_{\nu}^{(\alpha)}$ are fermion fields for $\sigma = 1$ and boson fields for $\sigma = -1$. Obviously, when in equation (4.2) we put $p = 1$ and substitute into (4.1), we get the usual commutation relations for fermions and bosons.

Despite the fact that Green's formalism have solely used the quantum field theoretic framework, there is no reason why parastatistics cannot be applied to the corresponding quantum mechanical description, at least in the non relativistic limit. There are some differences between the quantum mechanical approach and that of the associated field theory. However, when the number N of particles is constant, some resemblances could be expected. Now, if we try to translate Green's field theoretic results into our quantum mechanical language, we fall in with several difficulties from the onset. The first one is the decoupling into two kinds of para-statistics implied by the two-valuedness of σ. The second is the decomposition of the parafield into usual fields which belong to one-dimensional representations of the group of permutations. The third is the problematic interpretation of the order parameter $\underline{p}$. The difficulties pointed out above seem to disguise the true character of the symmetries involved in the multi-dimensional state vector. Thus, it could be infered that, from a symmetric group point of view, it is hard to accept the parabosons and parafermions concepts in Quantum Mechanics. The only entities which could have a consistent interpretation would be the

gentileons.

Now, let us return to our quantum mechanical discourse. Another point to be emphasized is that we have enhanced the symmetric rather than the unitary group. The implications of switching to such aspect of group theory can be stated clearly rigth now. The point is that this switch involves a surrender of the cluster property[11] in quantum mechanics. But this comes as no surprise since, in our context, the operators are symmetric functions of the relevant arguments of all particles. The complete symmetry of the operators with respect to permutations has as another fundamental consequence, the generalized commutation relations given by Eq. (3.3) and (3.4). These relations require that the description of a system of identical particles (cluster 1) depends on the existence of another system of similar particles (cluster 2) even when there is no interaction between the two clusters. It is possible that this result is traceable to an unjustifiable extension of the mathematical apparatus of the symmetric group to quantum mechanics. As an alternative for the group of permutations, we could, tentatively, to study one of its sub-groups, the group of displacements in configuration space[12], for example.

As a final remark, we must note that the introduction of our general statistics formalism, allows us some hope of representing several quantum systems in terms of the multi-dimensional state vectors. Thus, for instance, if quarks are gentileons of order d=2 there is a possibility for describing the hadrons with our non-relativistic approach where the hypothesis of additional quantum numbers (color, charm, etc) are unnecessary. The baryons should be described by the state vectors Y whereas the mesons should be represented by the bosonic state vectors Y_S. Another interesting application of the above reasoning could be the interpretation of quasi-particles which appear as colletive excitations in several condensed matter problems. Thus, it is a long road until accomplishing a through study of the symmetries in configuration space of Quantum Mechanics and its physical implications.

APPENDIX 1 - THE MATRICES $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix}$

The matrices $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix}$, that are defined by $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} = U\,P\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} U^{+}$, where U is the structural matrix defined by the equation (2.2), are the following :

1) $\begin{pmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1/2 & \sqrt{3}/2 & 0 & 0 & 0\\ 0 & \sqrt{3}/2 & 1/2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1/2 & \sqrt{3}/2 & 0\\ 0 & 0 & 0 & \sqrt{3}/2 & 1/2 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

2) $\begin{pmatrix}\alpha\beta\gamma\\ \gamma\beta\alpha\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \gamma\beta\alpha\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 & 0\\ 0 & -\sqrt{3}/2 & 1/2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 & 0\\ 0 & 0 & 0 & -\sqrt{3}/2 & 1/2 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

3) $\begin{pmatrix}\alpha\beta\gamma\\ \beta\alpha\gamma\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \beta\alpha\gamma\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 & 0\\ 0 & \sqrt{3}/2 & -1/2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 & 0\\ 0 & 0 & 0 & \sqrt{3}/2 & -1/2 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4) $\begin{pmatrix}\alpha\beta\gamma\\ \beta\alpha\gamma\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \beta\alpha\gamma\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

5) $\begin{pmatrix}\alpha\beta\gamma\\ \gamma\alpha\beta\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \gamma\alpha\beta\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1/2 & \sqrt{3}/2 & 0 & 0 & 0\\ 0 & -\sqrt{3}/2 & -1/2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1/2 & \sqrt{3}/2 & 0\\ 0 & 0 & 0 & -\sqrt{3}/2 & -1/2 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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